PUTNAM PRACTICE SET 33

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Problem 1. For which positive integers n is there an n-by-n matrix A with integer entries with the property that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Problem 2. Let $a, b \in \mathbb{N}$. Prove that for each $\epsilon > 0$, we can find positive integers m and n with the property that

$$0 < \left|a\sqrt{m} - b\sqrt{n}\right| < \epsilon.$$

Problem 3. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with $g(0) \neq 0$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function with the property that both functions

$$\frac{f}{g} \text{ and } f \cdot g$$

are differentiable at $x = 0$, then does this imply that also f must be differentiable at $x = 0$?

Problem 4. Let p be an odd prime number. Prove that there exist at least $\frac{p+1}{2}$ distinct integers $n \in \{0, 1, 2, \ldots, p-1\}$ with the property that p doesn't divide the integer:

$$\sum_{k=0}^{p-1} k! \cdot n^k.$$

(As always, we use the convention that $0! = 1$.)