# PUTNAM PRACTICE SET 27

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*Problem 1.* Players $A$ and $B$ play the following game: each player (starting with player $A$) take turns in writing a real number in one of the (still) empty entries of a 2020-by-2020 matrix. Player $A$ wins if the determinant of the matrix at the end is nonzero, while player $B$ wins if the determinant of the matrix is zero. Who has a winning strategy?

*Problem 2.* Let $n \in \mathbb{N}$. We start with a finite sequence

$$a_1, a_2, \dots, a_n$$

of positive integers and then at each step, we perform the following operation: if for some indices $1 \le i < j \le n$, we have that $a_i$ does not divide $a_j$, then we replace $a_i$ by $\gcd(a_i, a_j)$ and also, replace $a_j$ by $\operatorname{lcm}[a_i, a_j]$.

Prove that after finitely many steps we can no longer perform any new operation. Furthermore, show that the final sequence we obtain is the same one regardless of the order of the above operations that we performed.

*Problem 3.* Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if} \quad x \le e \\ xf(\ln(x)) & \text{if} \quad x > e \end{cases}$$

Determine whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

*Problem 4.* Prove that there exists a positive constant $c$ with the property that in any finite nontrivial group $G$, there exists a suitable subset $S \subseteq G$ satisfying the following two properties:

- $|S| \le c \log(|G|)$; and
- for each $x \in G$, there exist distinct elements $x_1, \dots, x_k \in S$ such that $x = x_1 \cdot x_2 \cdot \dots \cdot x_k$.