PUTNAM PRACTICE SET 24

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Problem 1. Find a polynomial $P(x,y) \in \mathbb{R}[x,y]$ with the property that for each real number r, we have

$$P\left([r],[2r]\right) = 0,$$

where $[x]$ is always the integer part of the real number x (i.e., the largest integer less than or equal to x).

Problem 2. Show that the curve in the cartesian plane given by the equation:

$$x^3 + 3xy + y^3 = 1$$

contains exactly one set of three points A, B and C which are the vertices of an equilateral triangle.

Problem 3. Let $\{a_n\}_{n\ge 0}$ be a sequence of integers satisfying the two properties:

$$a_i = i \text{ for } i = 0,1,\dots,2020 \text{ and } a_n = a_{n-1} + a_{n-2020} \text{ for } n \ge 2021.$$

Show that for each positive integer M, there exists some integer $k > M + 2020$ such that each one of the integers $a_k, \dots, a_{k+2018}$ are divisible by M.

Problem 4. Let n be an odd positive integer and let $\theta \in \mathbb{R}$ such that θ/π is an irrational number. For each $k = 1,\dots,n$, we let

$$a_k = \tan\left(\theta + \frac{k\pi}{n}\right).$$

Compute $\frac{a_1+a_2+\cdots+a_n}{a_1\cdot a_2\cdot\cdots\cdot a_n}$.