PUTNAM PRACTICE SET 11

PROF. DRAGOS GHILOCA

Problem 1. Let $p > 3$ be a prime number. Prove that at least one of the numbers from the following list:
$$\frac{3}{p^2}, \frac{4}{p^2}, \frac{5}{p^2}, \ldots, \frac{p-2}{p^2}$$
can be written as a sum $\frac{1}{x} + \frac{1}{y}$ for some positive integers $x$ and $y$.

Problem 2. If $r > s > 0$ and $a > b > c > 0$, prove that
$$a^rb^s + b^rc^s + c^ra^s \geq a^sb^r + b^sc^r + c^ra^s.$$  

Problem 3. Find all $f \in \mathbb{C}[x]$ with the property that for each $x \in \mathbb{C}$, we have
$$f(x)f(2x^2) = f(2x^3 + x).$$

Problem 4. Let $n \in \mathbb{N}$ and let $S_n = \{1, \ldots, n\}$. Assume the set $M \subseteq S_n \times S_n$ satisfies the following properties:
- if $(j, k) \in M$ then $1 \leq j < k \leq n$; and
- if $(j, k) \in M$ then for each $i \in S_n$, we have that $(k, i) \notin M$.
What is the largest possible cardinality of the set $M$?