Branching random walks

2 ingredients:
- offspring law $p(\cdot)$: $\sum_{k=1}^{\infty} p(k) = 1$
- assume $\sum_{k=1}^{\infty} kp(k) > 1$, $\sum_{k=1}^{\infty} k^2 p(k) < \infty$ (often $p(0) = 0$)
- displacement law: $RV \times \text{Var}(X) > 0$

Start with one particle at 0,
- particles produce offspring according to $p(\cdot)$ (and die)
- offspring takes displacements according to $X$
(all particles behave independently).

Continuous analogue: BRM

\[
\begin{array}{c}
\text{space} \\
\text{time}
\end{array}
\]
Tree-indexed RW

- GW-process according to \( p(\cdot) \):
  \[ T \]
- Label edges with iid RV distributed as \( X \)

\[ D_u = \text{vertices in generation } u \}
\[ \Gamma_{D_u} = \text{Galton-Watson process} \]

\[ S_v = \sum_{x \in [0,v]} X_e \quad \text{position of particle} \]

Recall \( \lim_{m \to 0} m : = \sum_{k=0}^{\infty} kp(k) > 1 \)
\[ \Rightarrow P\left[ T \text{ infinite} \right] > 0 \]

\( m \) "reproduction number"

If \( p(0) > 0 \), can look \[ P^* \left[ \cdot \right] = \left[ P \cdot \Gamma_{D_u} > 0, \forall u \right] \]

Now:

\( (S_v)_{v \in D_u} \) are RV which have not
Move general model: independent particles produce "offspring + displacements" at once according to some point process.

Example:

\[
\begin{array}{c}
\text{each with prob. } \frac{1}{2} \\
\text{(dependence between siblings!)}
\end{array}
\]

Q: \[
\sup_{v \in \Delta_n} S_v
\]

Assume \[
\mathbb{E}[e^{\lambda X}] < \infty \text{ for some } \lambda > 0
\]

Define \( I(\gamma) = \sup_{\lambda} \left\{ \lambda \gamma - \log \mathbb{E}[e^{\lambda X}] \right\} \)

e.g. rate func. for \( \gamma > \mathbb{E}[X] \),

\[
\frac{1}{n} \log \mathbb{P}[S_n \geq n \gamma] \to -I(\gamma)
\]

Indeed,

\[
\mathbb{P}[S_n \geq n \gamma] \leq \mathbb{E}[e^{\lambda S_n}] e^{-\lambda n \gamma}
\]

\[
= e^{-n I(\gamma)}
\]

\( \lambda = \gamma \) optimal.

Define
\[ x^* = \sup \{ s \geq [E[X] : I(s) \leq \log m] \} \]

**Theorem 1** - Beggins, Hammersley, Kingman

\[
\frac{M_n}{n} \rightarrow x^* \quad P^* - a.s.
\]

**Exercises:**

(i) \( X \sim N(0,1) \) Compute \( x^* \)

(ii) \( p(3) = 1, \ P[X=0] = \frac{1}{2} = P[X=1] \) Compute \( x^* \)

Do we have \( P[M_n = n, \forall n] > 0 ? \)

(iii) Same as (ii) if \( p(2) = 1 \).

**Intuition:**
At time \( u \), have \( \approx m^n \) particles.
For each \( u \in \Delta_n \), \( P[S_u \geq n\gamma] \approx e^{-uI(y)} \)

\[
e^{-uI(y)} e^{u \log m n} = 1 \Rightarrow y = x^*.
\]

**Proof**

(i) First moment method.
\[ P[M_u \geq n_y] \leq E[ \sum_{v \in D_u} I_{S_v \geq n_y}] \]
\[ = E[1_{D_u} \prod_{v \in D_u} \mathbb{1}_{S_v \geq n_y}] \leq e^{-n I(y)} \]

Hence if
\[ I(y) > \log m, \]
\[ \sum_u P[M_u \geq n_u] < \infty \]
\[ \Rightarrow \operatorname{esssup} \frac{M_u}{n} \leq y. \]

(ii) "Embedded treee"

Assume \( y < x^* \)
Choose \( \varepsilon > 0 \) s.t. \( I(y) - 2\varepsilon < \log m \)
\[ P[S_k \geq k\gamma] \geq e^{-k(I(y) - \varepsilon)} \]

for \( k = k_0(\varepsilon) \)

"embedded treee":
- keep \( v \in D_k \) if \( \frac{S_v}{k} \geq \gamma \)
- delete \( v \) otherwise
- go to level \( 2k \)

\[ \rightarrow \text{embedded GW-treee} \sim cT \]
What is $\hat{m}$?

$$\hat{m} = m^k \Pr[S_k \geq k\gamma]$$

$$> e^{k \log \hat{m} \cdot e^{-k(1/\gamma) - \varepsilon}} \geq e^k$$

$$> 1 \text{ for } k \text{ large enough.}$$

$$\Pr[\liminf_{n \to \infty} \frac{M_n}{n} \geq \gamma] > 0.$$ 

0-1 Law for inherited properties

Call a property $A$ of trees inherited if each finite tree has $A$, and if $T$ has $A$, then all descendant trees of the children of the root have it.

Then $\Pr^* [T \text{ has } A] \in (0, 1].$

Proof

$$\Pr[T \text{ has } A] = \mathbb{E} \left[ \Pr[T \text{ has } A \mid D_{n}] \right]$$

$$\leq \mathbb{E} \left[ \Pr[T^{(n)} \text{ has } A, ..., T^{(D_{n})} \text{ has } A \mid D_{n}] \right]$$

inherited

$$= \mathbb{E} \left[ \Pr[T \text{ has } A] \mid D_{n} \right]$$
Hence
\[ \Pr[T \text{ has } A] \leq f(\Pr[T \text{ has } A]) \]
where
\[ f(s) = \sum_{k=0}^{\infty} s^k p(k) = \mathbb{E}[s^{1/\lambda}] \]

On the other hand,
\[ \Pr[T \text{ has } A] \geq q \]
\[ q = \Pr[\lim_{u \to \infty} |D_u| = 0] \]
\[ \implies \Pr[T \text{ has } A] \in [q, 1] \]
\[ \implies \Pr^*[T \text{ has } A] \in [0, 1] \]

Look at the foll. prop. A:
\[ A = \exists T \text{ finite} \cup \exists \liminf \frac{M_n}{n} \leq \frac{1}{2} \]