Qualifying Exam Problems: Algebra
(September 9, 2014)

1. (10 points) Let
\[
A = \begin{pmatrix}
-11 & 9 \\
-30 & 22
\end{pmatrix}
\]
Find \(A^{2014}\).

2. Let \(n \geq 2\) be an integer, let \(M_{n,n}(\mathbb{R})\) be the set of all \(n\)-by-\(n\) matrices with real entries, let \(B \in M_{n,n}(\mathbb{R})\) and let \(f_B : M_{n,n}(\mathbb{R}) \rightarrow M_{n,n}(\mathbb{R})\) be given by
\[
f_B(A) = AB - BA
\]
for each \(A \in M_{n,n}(\mathbb{R})\).

(a) (2 points) Show that \(f_B\) is a linear map.

(b) (3 points) If \(B\) has distinct eigenvalues, show that \(\dim \ker(f_B) \geq n\).

(c) (5 points) If \(n = 2\) and \(B\) is not diagonalizable, find \(\dim \ker(f_B)\).

3. (a) (1 point) Let \(n \geq 2\) be an integer, let \(A, B \in M_{n,n}(\mathbb{R})\) and let \(\lambda \in \mathbb{C}\). If \(A\) is invertible, prove that \(\lambda \cdot I_n - AB\) is invertible if and only if \(\lambda \cdot A^{-1} - B\) is invertible.

(b) (2 points) Let \(n \geq 2\) be an integer, let \(A, B \in M_{n,n}(\mathbb{R})\) and let \(\lambda \in \mathbb{C}\). If \(A\) is invertible, prove that \(\det(\lambda \cdot I_n - AB) = \det(\lambda \cdot I_n - BA)\).

(c) (3 points) Let \(n \geq 2\) be an integer, and let \(A, B \in M_{n,n}(\mathbb{R})\). Show that \(\lambda \in \mathbb{C}\) is an eigenvalue of \(AB\) if and only if it is an eigenvalue of \(BA\).

(d) (4 points) Let \(C \in M_{2,3}(\mathbb{R})\) and \(D \in M_{3,2}(\mathbb{R})\) such that
\[
DC = \begin{pmatrix}
2 & -1 & 2 \\
0 & 0 & 3 \\
0 & 0 & 5
\end{pmatrix}
\]
Find \(\det(CD)\).

4. (10 points) Let \(Z \subset G\) be the center of a group \(G\) and suppose that \(G/Z\) is cyclic. Prove that \(G\) is Abelian.

5. (a) (4 points) Determine the minimal polynomial for \(\alpha = \sqrt{3} + \sqrt{5}\) over the field \(\mathbb{Q}\).

(b) (2 points) Determine the minimal polynomial for \(\alpha = \sqrt{3} + \sqrt{5}\) over the field \(\mathbb{Q}(\sqrt{5})\).

(c) (2 points) Determine the minimal polynomial for \(\alpha = \sqrt{3} + \sqrt{5}\) over the field \(\mathbb{Q}(\sqrt{10})\).

(d) (2 points) Determine the minimal polynomial for \(\alpha = \sqrt{3} + \sqrt{5}\) over the field \(\mathbb{Q}(\sqrt{15})\).

6. (a) (2 points) Let \(G\) be a group of prime order \(p\). Show that the order of \(\text{Aut}(G)\), the automorphism group of \(G\), is \(p - 1\).

(b) (2 points) Let \(G\) be a group and let \(N \subset G\) be a normal subgroup. Show that conjugation induces a homomorphism \(\phi : G \rightarrow \text{Aut}(N)\).

(c) (3 points) Show that a group \(G\) of order 15 is cyclic.

(d) (3 points) Show that if the order of a group \(G\) is 255, then \(G\) is cyclic. Hint: Find the number of Sylow 17-subgroups and use the results of parts (a), (b), and (c).