Part I

1. (a) Let
   \[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \]
   and verify that for all real numbers \( x_1, y_1, x_2, y_2 \) we have
   \[ (x_1 I + y_1 J)(x_2 I + y_2 J) = (x_1 x_2 - y_1 y_2)I + (x_1 y_2 + x_2 y_1)J. \]

   (b) Find \( A^n \) for any integer \( n \), if
   \[ A = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}. \]

2. Prove that the equation \( XY - YX = I_n \) has no solution (where \( X, Y \) are unknown real \( n \times n \)-matrices, and \( I_n \) is the identity matrix).

3. A nonzero matrix \( A \) is called nilpotent if there exists a positive integer \( n \) such that \( A^n = 0 \). Two matrices \( A \) and \( B \) are called similar if they can be obtained from one another by a change of basis.
   
   (a) Prove that if \( A \) is nilpotent and \( B \) is similar to \( A \), then \( B \) is also nilpotent.
   
   (b) Find a set of representatives of all equivalence classes of nilpotent \( 3 \times 3 \)-matrices with complex entries, where we declare two matrices equivalent if they are similar. (You may want to solve this question for \( 2 \times 2 \)-matrices first).

4. Define the Fourier transform pair to be:
   \[ \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx \quad \text{and} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx} \, dx. \]

   (a) Use contour integration to calculate the Fourier transform \( \hat{f}(k) \) for
   \[ f(x) = \frac{1}{(x^2 + a^2)^2}, \]
   where \( a \in \mathbb{R} \) is a constant.

   (b) Calculate the inverse Fourier transform \( f(x) \) for
   \[ \hat{f}(k) = \frac{1}{(k^2 + a^2)^2}, \]
   where \( a \in \mathbb{R} \) is a constant.
5. Use a keyhole-shaped contour to evaluate the integral.

\[ I = \int_0^{\infty} \frac{dx}{\sqrt{x}(x^2 + 1)} \]

6. Let $D$ be the circle of radius 4 centred at the point $(0,5)$ in the $x-y$ plane. Find a function $\phi(x,y)$ that satisfies the following restrictions:

- $\phi$ is harmonic in the upper half-plane exterior to $D$;
- $\phi = 1$ on $D$;
- $\phi = 0$ on the $x$-axis.

Hint: Consider a conformal map of the form $w = \frac{z + a}{z + b}$.
Part II

1. Suppose $f$ is a continuous function on $\mathbb{R}$ such that $|f(x) - f(y)| \geq |x - y|$ for all $x$ and $y$. Show that the range of $f$ is all of $\mathbb{R}$.

2. For every $a \in \mathbb{R}$, determine whether the integral

$$\int \int_D (x^4 + y^2)^a \, dA$$

is finite, where $D$ is the square $\{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function and assume that $f$ has a local minimum at 0. Prove that there is a disc centered on the $y$-axis which lies above the graph of $f$ and touches the graph of $f$ at $(0, f(0))$.

4. (a) What is the smallest integer $n$ such that there exists a non-abelian group of order $n$?

(b) Give an example of a number $n > 1000$ and not a prime, such that there exists only one group of order $n$ up to isomorphism. How many subgroups does the group in your example have?

5. Let $E$ be the splitting field of the polynomial $(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.

(a) Find the degree $[E : \mathbb{Q}]$.

(b) Find an element $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$.

(c) Find the Galois group $\text{Gal}(E/\mathbb{Q})$.

6. Let $I = \{f \in \mathbb{C}[x, y] \mid f(1, 1) = 0\}$. Prove that $I$ is a maximal ideal in the ring $\mathbb{C}[x, y]$. Find a minimal set of generators for $I$. 