Applied Mathematics Qualifying Exam
University of British Columbia
August 30, 2008

Part I

1. Suppose $f$ is a continuous real-valued function on $[0, 1]$. Show that

$$\int_0^1 f(x)x^2 \, dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.

2. Consider the vector field $\mathbf{F}(x, y, z) = (\sin(x) + y^2)i + (\cos(y) + z^2)j + (e^{-z} + x^2)k$.

   (a) Compute curl $\mathbf{F}$ and div $\mathbf{F}$.
   (b) Is $\mathbf{F}$ conservative? Justify your answer.
   (c) Can $\mathbf{F}$ be written as the curl of another vector field $\mathbf{G}$? Justify your answer.
   (d) Let $C$ be the curve of intersection of the cylinder $x^2 + y^2 = 2x$ and plane $z = x$ oriented counterclockwise as viewed from above. Denote by $S$ the part of the plane $z = x$ that is bounded by $C$ and oriented upward.
      i. Parametrize $C$.
      ii. Parametrize $S$.
      iii. State Stokes’ Theorem and use it to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
         [Hint: The following might be useful: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.]

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is $f$ differentiable at the origin? Prove or give a counter-example.

4. Consider the $n \times n$ matrix with a 7 in every entry of the first $p$ rows and 4 in every entry of the last $n - p$ rows. Find its eigenvalues and eigenvectors.

5. Recall that matrices $A$ and $B$ are called similar provided that there exists an invertible matrix $P$ such that $A = PBP^{-1}$. Also recall that $\det$ and $\text{tr}$ are preserved under the similarity transformation $B \to PBP^{-1}$. For $a$ and $\epsilon$ real, define the matrix:

$$A_\epsilon = \begin{pmatrix} a & \epsilon \\ 0 & a \end{pmatrix}.$$

   (a) Show that the family of matrices $\mathcal{F} = \{A_\epsilon : \epsilon \neq 0\} \cup \{A_\epsilon^T : \epsilon \neq 0\}$ are all similar to one another. Note: the superscript $^T$ denotes matrix transpose.
   (b) Show that the following classes of real $2 \times 2$ matrices are each a distance 0 away from the family $\mathcal{F}$:
      – the class of matrices with one eigenvalue with geometric multiplicity two,
      – the class of matrices with distinct real eigenvalues,
      – the class of matrices with non-real complex eigenvalues,

   where distance is defined using the max norm ($||A||_{\text{max}} = \max\{|a_{ij}|\}$).
6. Let $L$ be a linear transformation from polynomials of degree less than or equal to two to the set of $2 \times 2$ matrices ($L : \mathcal{P}^2(\mathbb{R}, \mathbb{R}) \to \mathcal{M}(2, 2)$) given by

$$L(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 + a_2 & a_0 + a_1 \\ a_0 + a_2 & a_0 + a_1 \end{pmatrix}$$

(a) Verify that this transformation is linear.
(b) Find the matrix that represents the linear transformation $L$ with respect to the bases

$$\mathcal{V} = \{1, x, x^2\}$$

$$\mathcal{W} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

which are bases for $\mathcal{P}^2(\mathbb{R}, \mathbb{R})$ and $\mathcal{M}(2, 2)$ respectively.
(c) Find bases for the nullspace $\mathcal{N}(L)$ and range $\mathcal{R}(L)$. 
Part II

1. Find the Laurent series expansion of \( f(z) = \frac{4}{(1 + z)(3 - z)} \) around \( z_0 = 0 \) in the annulus \( 1 < |z| < 3 \).

2. Let \( f(z) = \left( \frac{\sin(3z)}{z^2} - \frac{3}{z} \right) \cdot \left( \frac{z + 1}{z + 2} \right) \cdot \exp \left( \frac{1}{z - 5} \right) \).

   (a) Find and classify all singularities of \( f \).
   (b) Evaluate \( I = \int_{\Gamma} f(z)dz \) where \( \Gamma \) is the positively oriented triangular loop with vertices at \( v_1 = -1 - i, v_2 = 1 - i, \) and \( v_3 = i \).

3. Compute the integral
   \[ \int_0^{2\pi} \frac{1}{2 + \cos(x)} dx. \]
   [Convert to an integral on the unit circle via a substitution, then use residue theory.]

4. Glycolysis is the enzymatic process by which glucose is broken down for the purpose of extracting energy. Consider the following simplified model of glycolysis proposed by Schnakenberg (1979):
   \[ \frac{dx}{dt} = x^2 y - x \]
   \[ \frac{dy}{dt} = a - x^2 y. \]

   As the parameter \( a \) varies from \(-\infty\) to \(\infty\), the structure of solutions near the steady state of the system changes. Determine the sequence of steady state classifications for increasing values of \( a \) (e.g. stable node-to-saddle-to-unstable node).

5. The voltage across the capacitor in an RLC circuit being driven by an oscillatory potential is given by the solution of the equation:
   \[ LC \frac{d^2v}{dt^2} + RC \frac{dv}{dt} + v = V_0 \cos(\omega t) \]
   where \( L, R, C, V_0 \) and \( \omega \), all non-negative, are physical parameters determined by the circuit components.

   (a) Calculate the so-called natural frequency of the circuit, when both \( R = 0 \) and \( V_0 = 0 \)?
   (b) Provide expressions for the change of variables, \( t \to \tau \) and \( v \to y \), that simplifies the equation to: \( y'' + ay' + by = \cos(t) \) where \( ' \) denotes derivative with respect to \( \tau \).
   (c) Calculate the general solution.
   (d) Show that the solution converges to \( v(t) = A \cos(t - \delta) \) as \( t \to \infty \) where \( A = 1/\sqrt{a^2 + (b - 1)^2} \) and \( \delta \) satisfies \( \cos(\delta) = (b - 1)A \).
   (e) Returning to the original variables and parameters, what forcing frequency \( \omega \) maximizes the amplitude of the response?
6. Consider the physical problem of a long metal cylinder with annular cross-section. The temperature in Kelvin on the interior of the metal is described by the equation

\[ u_t = D \left( u_{rr} + \frac{1}{r} u_r \right) \]

where \( r \) is the radial coordinate measured from the middle of the cylinder and we have assumed that \( u \) does not vary along the height of the cylinder. The inner and outer surfaces of the cylinder, located at \( r = a \) and \( r = b \) respectively, are treated such that the following boundary conditions apply:

\[ -Du_r(a, t) = \alpha, \quad -Du_r(b, t) = \beta \]

where \( \alpha \) and \( \beta \) are both positive. Initially, \( u \) is given by \( u(r, 0) = f(r) \).

(a) Provide a physical interpretation of the boundary conditions. What are the units on \( \alpha \) and \( \beta \)?

(b) What condition on \( \alpha \) and \( \beta \) must be satisfied for a steady state solution to exist? Assume it is satisfied and calculate the steady state.

(c) If you were to solve the time-dependent problem by an eigenfunction decomposition, what equation would the eigenfunctions satisfy? You do not need to solve the equation.