Part I

1. Let $0 < b < a$. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + b \cos(\theta))^2}.$$  

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a, b, c, d > 0$. Show that $A$ has an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, with $x, y > 0$.

3. Prove that every group of order $p^m$ can be generated by $m$ elements. Here $p$ is a prime.

4. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$.

   (a) For which $x$ does the series converge absolutely?
   (b) On which intervals does it converge uniformly?
   (c) Is $f$ continuous wherever the series converges?
   (d) Is $f$ bounded?

5. Suppose $f$ is analytic on $D := \{z \in \mathbb{C} \mid |z| < 1\}$, continuous on its closure, $\bar{D}$, and real-valued on the boundary of $D$. Show that $f$ is constant on $\bar{D}$.

6. Let $M^{n,n}$ denote the vector space of $n \times n$ real matrices. Consider the linear transformation $L : M^{n,n} \to M^{n,n}$ defined by $L(A) = A + A^T$ (here $A^T$ denotes the transpose of the matrix $A$).

   (a) Let $n = 2$. Find bases for the kernel, $Ker(L)$, and the range, $Ran(L)$, of $L$.
   (b) For all $n \geq 2$, find the dimensions of $Ker(L)$ and $Ran(L)$.

Part II

1. Consider the vector field $\mathbf{F}(x, y, z) = (yz + x^4)\hat{i} + (x(1 + z) + e^y)\hat{j} + (xy + \sin(z))\hat{k}$. Let $C$ be a circle of radius $R$ lying in the plane $2x + y + 3z = 6$. What are the possible values of the line integral $\int_C \mathbf{F} \cdot dr$?
2. Let $C = C^0([0, 1])$ be the ring of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. For $a \in [0, 1]$, define $I_a = \{ f \in C \mid f(a) = 0 \}$.

(a) Show that $I_a$ is a maximal ideal of $C$.

(b) Show that every maximal ideal of $C$ is of the form $I_a$ for some $a \in [0, 1]$.

(c) Show that part (b) fails if the closed interval $[0, 1]$ is replaced by the open interval $(0, 1)$.

3. Show that $2e^{-z} - z + 3$ has exactly one root in the right half-plane $\{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \}$.

4. Let $a, b, c, d$ be real numbers, not all zero. Find the eigenvalues of the following $4 \times 4$ matrix and describe the eigenspace decomposition of $\mathbb{R}^4$:

$$
\begin{pmatrix}
aa & ab & ac & ad \\
ba & bb & bc & bd \\
ca & cb & cc & cd \\
da & db & dc & dd
\end{pmatrix}
$$

5. Let $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be continuous, and define $F(t) := \int_0^t f(s, t)ds$. Prove (carefully) that $F$ is continuous on $[0, 1]$.

6. Let $F$ be a finite field, $f(x) \in F[x]$ be a polynomial with coefficients in $F$, and $F \subset E$ be a field extension (not necessarily finite). Show that if $E$ contains one root of $f(x)$ then it contains every root of $f(x)$. 