1. Let $0 < b < a$. Use contour integration to evaluate the integral
\[ \int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2}. \]

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a, b, c, d > 0$. Show that $A$ has an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, with $x, y > 0$.

3. Consider the following initial-boundary-value problem for $u(x,t)$:
\[
\begin{cases}
  u_t + u_{xxxx} = 0, & 0 < x < \pi, \ t > 0 \\
  u_x(0, t) = u_{xxx}(0, t) = u_x(\pi, t) = u_{xxx}(\pi, t) = 0, & t > 0 \\
  u(x, 0) = \cos^2(x), & 0 < x < \pi
\end{cases}
\]

(a) The solution tends to a steady-state, $v(x) = \lim_{t \to \infty} u(x,t)$. Find $v(x)$.
(b) Find the solution $u(x,t)$.
(c) How much time does it take for $u(x,t)$ to get within $10^{-2}$ of the steady-state for all $x \in (0, \pi)$?

4. Consider
\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2x}. \]

(a) For which $x$ does the series converge absolutely?
(b) On which intervals does it converge uniformly?
(c) Is $f$ continuous wherever the series converges?
(d) Is $f$ bounded?

5. Suppose $f(z)$ is analytic on a connected region $\Omega \subset \mathbb{C}$. Show that $|f(z)|^2$ is harmonic on $\Omega$ if and only if $f$ is constant.

6. Let $V$ be the vector space of continuous, real-valued functions on the interval $[0, \pi]$, with the inner-product
\[ \langle f, g \rangle := \int_0^\pi f(x)g(x)dx. \]

(a) Find an orthonormal basis for the subspace $S := \text{span} \{1, \sin(x)\}$.
(b) Compute the distance of $\sin^2(x)$ from $S$. 

Part II

1. Consider the vector field\( \mathbf{F}(x, y, z) = (yz + x^4)\hat{i} + (x(1 + z) + e^y)\hat{j} + (xy + \sin(z))\hat{k} \). Let \( C \) be a circle of radius \( R \) lying in the plane \( 2x + y + 3z = 6 \). What are the possible values of the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \)?

2. A forced mass-spring system is governed by the following ODE for \( y(t) \):
\[
y'' + ky = f(t) \quad (*)
\]
where \( k > 0 \) is a constant, and \( f \) is a smooth, odd, \( T \)-periodic function.
(a) Find the general solution when \( f = 0 \).
(b) By expanding \( f \) in a Fourier series, find a formal (i.e. infinite series) particular solution of (*)
(c) Under what conditions on \( f \) and \( k \) will the system exhibit resonance?

3. Show that \( 2e^{-z} - z + 3 \) has exactly one root in the right half-plane \( \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \).

4. Let \( a, b, c, d \) be real numbers, not all zero. Find the eigenvalues of the following \( 4 \times 4 \) matrix and describe the eigenspace decomposition of \( \mathbb{R}^4 \):
\[
\begin{pmatrix}
aa & ab & ac & ad \\
ba & bb & bc & bd \\
ca & cb & cc & cd \\
da & db & dc & dd \\
\end{pmatrix}
\]

5. Define \( f(x) = x^2 \) for \( -\pi < x \leq \pi \), and extend it to be \( 2\pi \)-periodic.
(a) Find the Fourier series of \( f \).
(b) Use (a) to evaluate \( \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \) (state clearly any theorems you use).

6. Consider the following PDE for \( u(x, t) \):
\[
u_t - au_{xx} - bu + cu^3 = 0, \quad -\infty < x < \infty, \quad t > 0
\]
\((a, b, c > 0 \text{ are constants})\).
(a) Use scaling to reduce the problem to the form
\[
w_t - w_{xx} - w + w^3 = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (*)
\]
(b) Suppose \( w(x, t) \) is a smooth solution of (*) with \( w_x(x, t), w_t(x, t) \to 0 \) as \( x \to \pm\infty \). Show that the quantity
\[
\int_{-\infty}^{\infty} \left\{ \frac{1}{2} w_x^2(x, t) + \frac{1}{4} (w^2(x, t) - 1)^2 \right\} dx
\]
(if it is finite) is a non-increasing function of time.

(c) Suppose further that

\[
  w(x, t) \to \begin{cases} 
    -1 & x \to -\infty \\ 
    1 & x \to +\infty
  \end{cases}.
\]

Suppose the solution tends to a steady-state, \( v(x) = \lim_{t \to \infty} w(x, t) \). Find the form of \( v(x) \).