Part I

1. Prove that the product of two uniformly continuous real-valued functions on $(0, 1)$ is also uniformly continuous on $(0, 1)$.

2. For what values of $r$ and $n$ is there an $n \times n$-matrix of rank $r$, with real entries, such that $A^2 = 0$? Here $0$ denotes the $n \times n$ zero matrix.

3. Determine all entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ that satisfy $|f(z)| \leq e^{\text{Re}(z)}$ for all complex $z$. (An entire function is one that is analytic for all complex $z$.)

4. Let $G$ be a group, $H$ be a subgroup of finite index $n$ and $g \in G$.
   (a) Show that $g^k \in H$ for some $0 < k \leq n$.
   (b) Show by example that $g^n$ may not lie in $H$.

5. Let $\phi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be bounded and continuous. For each $n \in \mathbb{N}$ let $F_n : [0, 1] \rightarrow \mathbb{R}$ satisfy
   
   \[ F_n(0) = \frac{1}{n}, \quad F_n'(t) = \phi(t, F_n(t)) \quad \text{for} \quad t \in [0, 1]. \]

   Here $F_n'(t)$ denotes the right derivative if $t = 0$ and the left derivative if $t = 1$.
   (a) Prove that there is a subsequence such that $\{F_{n_k}\}$ converges uniformly to a limit function $F$.
   (b) Prove that $F$ solves
   
   \[ F(0) = 0, \quad F'(t) = \phi(t, F(t)) \quad \text{for} \quad t \in [0, 1]. \]

6. Show that there is no real $n \times n$ matrix $A$ such that

   \[ A^2 = \begin{pmatrix} -a_1 & 0 & \ldots & 0 \\ 0 & -a_2 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & -a_n \end{pmatrix}, \]

   where $a_1, \ldots, a_n$ are distinct positive real numbers.
Part II

7. Use contour integration to evaluate the integral \[ \int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 + x + 1} \, dx. \]

8. Let \( \mathbb{Z} \) be the ring of integers, \( p \) a prime, and \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) the field with \( p \) elements. Let \( x \) be an indeterminate, and set \( R_1 = \mathbb{F}_p[x]/(x^2 - 2) \), \( R_2 = \mathbb{F}_p[x]/(x^2 - 3) \). Determine whether the rings \( R_1 \) and \( R_2 \) are isomorphic in each of the following cases:
   (a) \( p = 2 \),
   (b) \( p = 5 \),
   (c) \( p = 11 \).

9. Let \( C \) be a simple closed \( C^1 \)-curve in \( \mathbb{R}^2 \) with the positive orientation enclosing a region \( D \). Assume \( D \) has area 2 and centroid \((3, 4)\). Let \( \mathbf{F}(x, y) = (y^2, x^2 + 3x) \). Find the line integral \[ \int_C \mathbf{F} \cdot d\mathbf{s}. \]

10. Let \( A \) be a nilpotent \( n \times n \)-matrix, i.e., \( A^m = 0 \) for some \( m \geq 1 \), where 0 is the \( n \times n \)-zero matrix. Prove or disprove the following assertions:
   (a) \( A^n = 0 \),
   (b) \( \det(A + I) = 1 \). Here \( I \) denotes the \( n \times n \) identity matrix.
   (c) \( \det(D + A) = \det(D) \) for every diagonal \( n \times n \)-matrix \( D \)?

11. (a) Show that all the zeros of the polynomial \( f(z) = z^8 - 3z + 1 \) lie in the disk \( |z| < 5/4 \).
    (b) How many zeros does \( f \) have in the unit circle?

12. A complex number is called algebraic if it is a root of a non-zero polynomial with integer coefficients. Show that \( a = \sin(r^\circ) \) is an algebraic number for every rational number \( r \). Here \( r^\circ \) denotes the angle of \( r \) degrees or, equivalently, of \( \frac{\pi r}{180} \) radians.