Problem 1: Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, and denote $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

1. Find the general solution of the system $\frac{d}{dt}\vec{x} = A\vec{x}$.

2. Solve the initial value problem

$$\frac{d}{dt}\vec{x} = A\vec{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Problem 2: Consider the following ODE for $x(t)$: $\frac{d^2 x}{dt^2} + \frac{dx}{dt} = x^3 - x$.

1. Rewrite this ODE as a system of two first-order equations.

2. Find all the critical points of the resulting system, and classify them as sinks, sources, saddles, or centres.

3. Show that if $x(t)$ is a solution, the quantity $(\frac{dx}{dt})^2 + x^2 - \frac{1}{2}x^4$ is non-increasing.

4. Prove that the constant solution $x(t) \equiv 0$ is stable in this sense: for any $\epsilon > 0$, every solution for which $(x(0))^2 + (\frac{dx}{dt}(0))^2$ is sufficiently small, satisfies $(x(t))^2 + (\frac{dx}{dt}(t))^2 < \epsilon$ for all $t > 0$. 
Problem 3: Consider the ODE eigenvalue problem:

\[(Ly)(x) = -(1 + x^2)y''(x) + A(x)y'(x) = \lambda y(x), \quad 0 < x < 1,\]

with either Dirichlet BCs: \(y(0) = y(1) = 0\); or Neumann BCs: \(y'(0) = y'(1) = 0\).

1. Find the function \(A(x)\) such that the ordinary differential operator \(L\) is self-adjoint (i.e., of Sturm-Liouville type) for either type of BC. For the rest of this question, take \(A(x)\) to be that function.

2. Listing the eigenvalues \(\lambda\) in order: \(\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots\), show that for Dirichlet BCs \(\lambda_0 > 0\), and for Neumann BCs \(\lambda_1 > 0\).

3. Find the solution \(u(x, t)\) of the following initial-boundary-value PDE problem in terms of the eigenvalues \(\lambda_0, \lambda_1, \lambda_2, \ldots\) and corresponding eigenfunctions (but you need not actually find these eigenvalues and eigenfunctions):

\[u_t = (1 + x^2)u_{xx} - A(x)u_x, \quad 0 < x < 1, \quad t > 0\]
\[u(0, t) = 1, \quad u(1, t) = 2, \quad u(x, 0) = 1 + x.\]
Problem 4: Find a matrix, $A \in \mathbb{R}^{2 \times 2}$, satisfying

$$A = A^T, \quad A_{1,1} + A_{2,2} = 5, \quad \sum_{i,j} A_{i,j} = 19, \quad -A_{1,1} + A_{2,1} + A_{1,2} = 11.$$ 

Answer: This is a set of 4 linear equations with 4 unknowns. It can be solved via Gaussian elimination to give

$$A = \begin{bmatrix} 3 & 7 \\ 7 & 2 \end{bmatrix}.$$
Problem 5: Let $P_2$ be the space of polynomials $a + bx + cx^2$ of degree at most 2 and with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x) dx.$$ 

1. Give an orthonormal basis for the orthogonal complement of $\text{span}(x)$.

2. Let $l$ be the functional defined by $l(p) := p(0)$ for each $p \in P_2$. Find $h \in P_2$ so that $l(p) = \langle h, p \rangle$ for each $p \in P_2$. 
Problem 6: Let $A, B \in \mathbb{R}^{3 \times 3}$. Let $I \in \mathbb{R}^{3 \times 3}$ be the 3 by 3 identity matrix. Suppose that $A$ has eigenvalues $\{-1, 4, 10\}$ and $B$ has eigenvalues $\{-2, 4, 7\}$. For each of the following matrices, if possible determine the eigenvalues. If not, state that there is insufficient information to determine the eigenvalues.

(a) $A^2$.
(b) $A \cdot B$.
(c) $A + B$.
(d) $A - 5 \cdot I$.
(e) $A + A^{-1}$. 