Every problem is worth 10 points.

**Problem 1**: Let $D$ be the open unit disc, $\partial D$ its boundary and $\bar{D}$ its closure. Let $f_n$ be a sequence of functions holomorphic in $D$, continuous on $\bar{D}$, and converging uniformly on $\partial D$. Show that $f_n$ converges uniformly on $\bar{D}$ (you may use that a sequence $a_n$ converges uniformly iff $\{a_n\}$ is uniformly Cauchy).
Problem 2: Use residues to calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^3} \, dx$
**Problem 3:** Consider the upper half plane $U = \{ z : \text{Im} z \geq 0 \}$. Let $f$ be continuous on the closure $\bar{U}$ such that $f(x)$ is real for $x$ real, and $f$ is holomorphic on $U$. Let $F(z)$ be the extension of $f$ to lower half plane defined by $F(z) = \overline{f(\overline{z})}$ where $\text{Im} z < 0$. Show that $F(z)$ is entire. (you may assume $F(z)$ is continuous. You may also assume that in the statement of Moreras theorem, the closed loops are all rectangles).
Problem 4:
For each of the following vector fields \( \mathbf{F} \), determine if \( \mathbf{F} \) is conservative on \( \mathbb{R}^3 \). For each that is conservative (i.e., a gradient vector field), find all potentials \( f \) for \( \mathbf{F} \), i.e., all \( C^1 \) functions \( f \) such that \( \nabla(f) = \mathbf{F} \).

1. \( \mathbf{F} = (xz, xy, yz) \)
2. \( \mathbf{F} = (2y \sin(yz), 2x \sin(yz) + 3z + 2xyz \cos(yz), 2xy^2 \cos(yz) + 3y) \)
Problem 5: Let \( a_{m,n} \geq 0 \), and assume that each \( a_{m+1,n} \leq a_{m,n} \) and \( a_{m,n+1} \leq a_{m,n} \). Show that \( \lim_{m} \lim_{n} a_{m,n} = \lim_{n} \lim_{m} a_{m,n} = a \), for some \( a \geq 0 \).
Problem 6: Let $M$ be a compact metric space and $f : M \to M$ be continuous.

1. Let $M \times M$ be given metric $ho((x, y), (x', y')) = d(x, x') + d(y, y')$. Show that the function $d(x, y)$ from $M \times M$ to $\mathbb{R}$ is continuous.

2. Let $r := \inf_{x \in M} d(x, f(x))$. Show that $r = d(x, f(x))$ for some $x \in M$.

3. Suppose that

$$\text{for all } x, y \in M \text{ s.t. } x \neq y, \ d(f(x), f(y)) < d(x, y).$$

Show that $f$ has a unique fixed point.