1. (10 points) For each value of the real constant \( a > 0 \), discuss the convergence of the series
\[
\sum_{n=1}^{\infty} \frac{a^n}{(n!)^\frac{1}{n}}.
\]

**Solution:** By using the obvious inequality \( n! \leq n^n \), we get
\[
\frac{a^n}{(n!)^\frac{1}{n}} \geq \frac{a^n}{n}.
\]
Thus if \( a \geq 1 \), then the series diverges.
On the other hand, if \( 0 < a < 1 \), then
\[
\frac{a^n}{(n!)^\frac{1}{n}} \leq a^n
\]
and the series converges by using comparison test.

2. Let \( \vec{i}, \vec{j}, \vec{k} \) be the usual unit vectors in \( \mathbb{R}^3 \). Let \( \vec{F} \) be the vector field
\[
(x^2 + y)i + (xy)j + (xz + z^2)k.
\]

a) (3 points) Compute \( \nabla \times \vec{F} \).
b) (7 points) Compute the integral of \( \nabla \times \vec{F} \) over the surface \( x^2 + y^2 + z^2 = 4, \ z \geq 0 \).

**Solution:**
\[
\nabla \times \vec{F} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2 + y & xy & xz + z^2
\end{vmatrix} = (-z)\vec{j} + (y - 1)\vec{k}.
\]
Let \( \Omega = \{(x, y, z) \in \mathbb{R}^3|x^2 + y^2 + z^2 = 4, \ z \geq 0\} \), \( D = \{(x, y, 0) \in \mathbb{R}^3|x^2 + y^2 \leq 4\} \). Note that \( \Omega \) and \( D \) have the same boundary. By using Stokes’ Theorem, we get
\[
\int_{\Omega} \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial \Omega} \vec{F} \cdot d\vec{l} = \int_{\partial D} \vec{F} \cdot d\vec{l} = \int_{D} (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{D} ((-z)\vec{j} + (y - 1)\vec{k}) \cdot \vec{k} \, dx \, dy = -4\pi.
\]
3. (10 points) Let \(f : \mathbb{R} \to \mathbb{R}\) be a twice differentiable function such that \(f \geq 0\) and \(f'' \leq 0\) everywhere. Prove that \(f\) must be a constant.

**Solution:** Let \(x_0 \in \mathbb{R}\). Enough to show \(f'(x_0) = 0\). Now observe that for any \(t\), we have
\[
0 \leq f(x_0 + t) = f(x_0) + f'(x_0)t + \frac{f''(\xi)}{2}t^2 \leq f(x_0) + f'(x_0)t.
\]
Since \(t\) is arbitrary, the result follows.

4. (10 points) Three sets of entire functions are described below. For each set, do two things:

(i) Explain why there is a parametric representation of the form
\[
f(z) = c_0 + c_1 z + \ldots + c_N z^N, \quad (c_0, c_1, \ldots, c_N) \in S,
\]
where \(N \geq 0\) is an integer and \(S\) is a subset of \(\mathbb{C}^{1+N}\).

(ii) Describe the value of \(N\) and the conditions defining \(S\) as completely as possible.

Here are the sets:

(a) All entire functions \(f\) such that \(\text{Im} \{f(z)\} \leq 0\) for all \(z \in \mathbb{C}\).

(b) All entire functions \(f\) such that \(|f(z)| \leq 2015 + |z|^10\) for all \(z \in \mathbb{C}\).

(c) All entire functions \(f\) such that \(|f''(z)| \leq |z|\) for all \(z \in \mathbb{C}\).

**Solution:**

(a) Given any such \(f\), let \(g(z) = \exp(-if(z))\). Then \(g\) is entire, with
\[
|g(z)| = e^{|\text{Im}(f(z))|} \leq 1, \quad z \in \mathbb{C}.
\]
By Liouville’s Theorem, \(g\) must be constant. Since \(f\) is continuous, it follows that \(f\) must also be constant. To match the requested pattern, take \(N = 0\) and let \(S\) denote the set of \(c \in \mathbb{C}\) where \(\text{Im} \{c\} \leq 0\).

(b) A direct application of the Extended Liouville Theorem implies that any \(f\) satisfying the given condition is a polynomial of degree at most 10. So \(N = 10\) will work in the desired representation. A detailed description of \(S\) is not possible.

(c) Any \(f\) of the given family will make \(g(z) = f''(z)/z\) analytic at all points \(z \neq 0\), and bounded in a neighbourhood of \(z = 0\). Therefore \(g\) has a removable singularity at 0 and we can treat \(g\) as if it were entire. With this interpretation,
\[
|g(z)| \leq 1, \quad z \in \mathbb{C},
\]
so Liouville’s Theorem implies that \(g(z) = k\) for some complex \(k\) with \(|k| \leq 1\). Consequently \(f''(z) = kz\), which leads to
\[
f'(z) = \frac{k}{2} z^2 + c_1, \quad f(z) = \frac{k}{6} z^3 + c_1 z + c_0.
\]
Thus \(N = 3\) fits the desired pattern, with
\[
S = \left\{ (c_0, c_1, c_2, c_3) : \ c_2 = 0, \ |c_3| \leq \frac{1}{6} \right\}.
\]
5. (a) (10 points) For each real constant $a$ in the interval $-1 < a < 1$, present simple closed-form expressions for the integrals below:

$$I(a) = \int_{0}^{2\pi} \frac{d\theta}{1 + a \sin \theta}, \quad J(a) = \int_{0}^{2\pi} \frac{d\theta}{1 + a \cos \theta}.$$

(b) Evaluate $I(4i/3)$, where $I$ denotes the integral defined in part (a).

*Note:* Since the input $a = 4i/3$ does not obey the assumptions in part (a), a complete solution must *interpret* and *explain* the term “evaluate” as well as producing a numerical value.

**Solution:**

(a) One has $I(a) = J(a)$ for all $a$, thanks to the change of variable $\phi = \theta - \pi/2$. So focus on $I(a)$, recognizing $\sin \theta = (e^{i\theta} - e^{-i\theta})/(2i)$. The parametrization $z = e^{i\theta}$ makes $dz = ie^{i\theta} d\theta$, so $d\theta = dz/(iz)$ and

$$I(a) = \int_{|z|=1} \frac{dz/(iz)}{(1 + a(z - 1/z)(2i))} = \int_{|z|=1} \frac{2dz}{az^2 + 2iz - a} = \int_{|z|=1} f(z) dz,$$

where

$$f(z) := \frac{2}{az^2 + 2iz - a} = \frac{2/a}{(z - z_0)(z - z_1)}.$$

The poles of $f$ can be determined using the quadratic formula:

$$z = -2i \pm \sqrt{-4 + 4a^2} = \frac{i}{a} \left[ -1 \pm \sqrt{1 - a^2} \right].$$

Both are purely imaginary; we name them $z_0 = \frac{i}{a} \left[ -1 + \sqrt{1 - a^2} \right], z_1 = \frac{i}{a} \left[ -1 - \sqrt{1 - a^2} \right].$

Now

$$|z_1| = \frac{1 + \sqrt{1 - a^2}}{|a|} \geq \frac{1}{|a|} > 1,$$

so $z_1$ lies outside the disk of interest, and (from the factorization above)

$$|z_0 z_1| = |1| = 1 \implies |z_0| = \frac{1}{|z_1|} < 1.$$

It follows that $I(a) = 2\pi i \text{Res} (f; z_0)$. To find this residue, suppose $A$ and $B$ make

$$\frac{2/a}{(z - z_0)(z - z_1)} = f(z) = \frac{A}{z - z_0} + \frac{B}{z - z_1}.$$

Then $2/a = A(z - z_1) + B(z - z_0)$, and sending $z \to z_0$ gives

$$\text{Res} (f; z_0) = A = \frac{2/a}{z_0 - z_1} = \frac{1}{i\sqrt{1 - a^2}}.$$

Finally, recalling $I(a) = 2\pi i \text{Res} (f; z_0)$,

$$J(a) = I(a) = \frac{2\pi}{\sqrt{1 - a^2}}.$$

(b) Analytic extension of $I(z)$ from the real interval $-1 < z < 1$ to a superset having nonempty interior in $\mathbb{C}$ requires some kind of branch cut linking the points $z = \pm 1$. Go the long way, discarding all points $z = x + i0$ for which $|x| \geq 1$. (Sketch.) Then

$$I(4i/3) = \frac{2\pi}{\sqrt{1 + 16/9}} = \frac{6\pi}{5}.$$
6. (10 points) Prove that this equation has precisely four solutions in the annulus $\frac{3}{2} < |z| < 2$:

$$z^5 + 15z + 1 = 0.$$ 

Include a statement of the main theorem (or theorems) on which your analysis is based.

**Solution:** This is a double application of Rouché’s Theorem. A simple form of this result says, “Let $\gamma$ be a simple closed curve. Suppose $f$ and $g$ are analytic at all points on and inside $\gamma$, and

$$|f(z) - g(z)| < |g(z)|, \quad z \in \gamma.$$ 

Then $f$ and $g$ have the same number of zeros of $f$ inside $\gamma$, counted according to multiplicity.”

(There is a more elaborate form, which allows a finite number of poles for $f$ and $g$ inside $\gamma$.)

We use $f(z) = z^5 + 15z + 1$ in both cases.

First, take $g(z) = 15z + 1$ and let $\gamma$ be the circle where $|z| = \frac{3}{2}$. Clearly $g$ has exactly one zero inside $\gamma$, at $z = -1/15$. And on $\gamma$, the triangle inequality gives both

$$|g(z)| = |15z + 1| \geq 15|z| - 1 = 15 \left(\frac{3}{2}\right) - 1 = \frac{43}{2} \geq \frac{42}{2} = 21$$

and

$$|f(z) - g(z)| = |z^5| = |z|^5 = \frac{243}{32} \leq \frac{256}{32} = 8.$$ 

Thus the conditions for Rouché’s Theorem are in force, and we deduce that $f$ has exactly one zero in the set where $|z| < \frac{3}{2}$.

Second, take $g(z) = z^5 + 15$ and let $\gamma$ be the circle where $|z| = 2$. This time each $z$ on $\gamma$ obeys

$$|g(z)| = |z^5 + 15| \geq |z| \left(|z|^4 - 15\right) = 2 (16 - 15) = 2$$

and

$$|f(z) - g(z)| = 1.$$ 

Thus the conditions for Rouché’s Theorem are in force, and we deduce that $f$ has the same number of zeros as $g$ has inside $\gamma$. Clearly $g(z) = z(z^4 + 15)$ has one zero at the origin and another four on the circle $|z| = 15^{1/4} < 2$, so $f$ has 5 zeros with $|z| < 2$.

Combining the results above, we find that all 5 roots of $f$ obey $|z| < 2$, and exactly one satisfies $|z| < \frac{3}{2}$. So there are exactly 4 zeros obeying $3/2 \leq |z| < 2$. To get the chain of strict inequalities requested in the setup, it would suffice to re-run the first application of Rouché’s Theorem on any circle of radius slightly larger than $3/2$. The gap between 21 and 8 noted above is positive, so there exists some $\epsilon > 0$ for which the desired inequality remains valid on $|z| = \frac{3}{2} + \epsilon$, and this completes the proof.