Qualifying Exam Problems: Differential Equations

1. (10 points) Let $A \in M_{n,n}(\mathbb{R})$ be a matrix of rank $n-1$. Let $L_A : M_{n,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ be the function given by $L_A(B) = A \cdot B$.
   (a) Show that $L_A$ is a linear map.
   (b) Find the dimension of the image of $L_A$.
   (c) Find a basis for the image of $L_A$.

2. (10 points) Let $k \in \mathbb{N}$, let $A_1, \ldots, A_k \in M_{n,n}(\mathbb{R})$ and let
   \[ B = \sum_{i=1}^{k} A_i \cdot A_i^t, \]
   where for each matrix $C$, we denote by $C^t$ its transpose.
   (a) Prove that $B$ is a symmetric matrix.
   (b) Prove that $B$ is a positive definite matrix, i.e. for each vector $v \in M_{n,1}(\mathbb{R})$, the dot product $\langle Bv, v \rangle$ is nonnegative.
   (c) Prove that $\det(B) \geq 0$.

3. (10 points) Solve the following system of linear equations:
   \[
   \begin{align*}
   x_1 + x_2 + x_3 + x_4 &= 0 \\
   2x_1 + 4x_2 + 8x_3 + 10x_4 &= 2 \\
   -2x_1 - x_2 + x_3 + 2x_4 &= 1 \\
   -10x_1 - 8x_2 - 4x_3 - 2x_4 &= 2 \\
   \end{align*}
   \]
   Explain your answer.

4. (10 points) Find the solution $y(t)$ of $y'' + y' - 2y = 4te^{2t}$ satisfying $y(0) = y'(0) = 0$.

5. (10 points) Consider the autonomous system
   \[
   \begin{align*}
   \dot{x} &= 2y - xf(x, y) \\
   \dot{y} &= -x - yf(x, y) \\
   \end{align*}
   \]
   where $f(x, y)$ is a smooth real valued function on the plane.
   (a) Find the critical point(s) of the system.
   (b) Find a function $f(x, y)$ so that all solutions $x(t), y(t)$ are bounded for all $t$, but do not all converge to a critical point as $t \to \infty$.
   (c) Determine the long time behaviour of solutions if $f(x, y) > 0$.
   (d) Find an example of a function $f(x, y)$ and an initial condition so that the solution $x(t), y(t)$ blows up in finite time.

6. (10 points) Consider the Sturm-Liouville problem
   \[ y''(x) + \lambda y(x) = 0 \]
   on the interval $0 \leq x \leq 1$, with boundary conditions
   \[
   \begin{align*}
   y(0) &= 0 \\
   y(1) + y'(1) &= 0. \\
   \end{align*}
   \]
   Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ be the eigenvalues listed in increasing order.
(a) Show that $\lambda_1 > 0$.

(b) Write down the equation that determines the eigenvalues and give a qualitative description of the large $n$ behaviour of $\lambda_n$.

(c) Determine the eigenfunctions $\varphi_n(x)$, normalized so that $\int_0^1 \varphi_n^2(x)dx = 1$, in terms of the eigenvalues you found in the previous part.

(d) Use an eigenfunction expansion to solve
\[ u_t(x, t) = u_{xx}(x, t) + t \]
for $0 \leq x \leq 1$ and $t \geq 0$ where
\[
\begin{align*}
u(0, t) &= 0 \\
u(1, t) + u_x(1, t) &= 0 \\
u(x, 0) &= 0
\end{align*}
\]