Part I: Real and Complex Analysis (Pure and Applied Exam)

1. (a) Find all polynomials that are uniformly continuous on $\mathbb{R}$.
   (b) Let $A$ be a nonempty subset of $\mathbb{R}$ and let $f$ be a real-valued function defined on $A$. Further let $\{f_n\}$ be a sequence of bounded functions on $A$ which converge uniformly to $f$. Prove that
   $$\frac{f_1(x) + \cdots + f_n(x)}{n} \to f(x)$$
   uniformly on $A$ as $n \to \infty$.

2. (a) Prove the Logarithmic Test
   \[\textbf{Theorem 1.} \quad \text{Suppose that } a_k \neq 0 \text{ for large } k \text{ and that} \]
   $$p = \lim_{k \to \infty} \frac{\log(1/|a_k|)}{\log k} \text{ exists.}$$
   \begin{itemize}
   \item If $p > 1$ then $\sum_{k=1}^{\infty} a_k$ converges absolutely, and
   \item If $p < 1$ then $\sum_{k=1}^{\infty} |a_k|$ diverges.
   \end{itemize}
   (b) Let $\{a_k\}$ be a sequence of non-zero real numbers and suppose that
   $$p = \lim_{k \to \infty} k \left(1 - \left|\frac{a_{k+1}}{a_k}\right|\right) \text{ exists}$$
   Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely when $p > 1$.

3. Evaluate the integral
   $$\iiint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$
   where $S$ is the region of the plane $y = z$ lying inside the unit ball centred at the origin, and $\mathbf{F} = (xy, xz, -yz)$, and $\mathbf{n}$ is the upward-pointing normal.
   Note that it might be helpful to remember that
   $$\int 2 \sin^2 t \, dt = t - \sin t \cos t.$$
4. In the following, justify your answer.
   (a) (6 points) Prove or disprove:
   There exists a holomorphic function \( f \) on \( \mathbb{C} \) (thus an entire function) such that
   \( f(D) = Q \) where \( D \) is the unit disk \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and \( Q \) is the square
   \( Q = \{ z \in \mathbb{C} \mid -1 < \text{Re} \, z, \text{Im} \, z < 1 \} \).
   (b) (7 points) Find all holomorphic functions \( f(z) \) on \( \mathbb{C} \setminus \{0\} \) such that
   \[ f(1) = 1, \quad |f(z)| \leq \frac{1}{|z|^3} \]
   (c) (7 points) Find a holomorphic function \( f(z) \) on \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \), which maps
   \( D \) onto the infinite sector
   \[ S = \{ z = re^{i\theta} \in \mathbb{C} \mid 0 < \theta < \pi/4 \} \].

5. (a) (6 points) Prove or disprove:
   There exists a nonconstant holomorphic function \( f(z) \) from \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \)
   into \( \mathbb{C} \) such that the area of its image, \( \text{area} \, f(D) = 0 \).
   (b) (7 points) Show that there is no holomorphic function \( f(z) \) on \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \)
   such that \( |f(z)| = |z|^{1/2} \) for all \( z \in D \).
   (c) (7 points) Find all harmonic functions \( u(x, y) \) on \( \mathbb{R}^2 \) such that \( e^{u(x,y)} \leq 10 + (x^2 + y^2) \)
   and \( u(1, 1) = 0 \).

6. (20 points) Evaluate the following integral, using contour integration, carefully justifying each step:

   \[
   \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx
   \]
Linear Algebra

1. Determine the eigenvalues and a basis of the corresponding eigenspaces for the linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $A$ with respect to the standard basis, where:

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

*Note:* all eigenvalues are rational numbers.

2. Let $\mathcal{N}_n \subset M_n(\mathbb{R})$ be the set of nilpotent matrices, that is the set of $n \times n$ matrices $A$ such that $A^k = 0$ for some $k$. Show that $\mathcal{N}_n$ is a closed subset of $M_n(\mathbb{R})$ (identify the latter with $\mathbb{R}^{n^2}$).

3. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

   (a) Show that there is a unique integer $0 \leq k \leq \min \{n, m\}$ for which there are bases $\{\mathbf{u}_i\}_{i=1}^n \subset \mathbb{R}^n$ $\{\mathbf{v}_i\}_{i=1}^m \subset \mathbb{R}^m$ such that the matrix of $T$ with respect to these bases is $D^{(k)}$, where

$$D^{(k)} = \begin{cases} 1 & 1 \leq i = j \leq k \\ 0 & \text{otherwise} \end{cases},$$

   that is $D^{(k)}$ has zeroes everywhere except that the first $k$ entries on the main diagonal are 1.

   (b) Show that the row rank and column rank of any matrix $A \in M_{m,n}(\mathbb{R})$ are equal.
Differential Equations

1. Consider the differential equation
   \[ 4x^2 \frac{d^2 y}{dx^2} + y = 0. \]
   (a) For \( x > 0 \) find all solutions \( y(x) \).
   (Hint: look for solutions of the form \( y(x) = \sqrt{x} f(x) \).)
   (b) Determine \( y(x) \) in the limit \( x \to +0 \).

2. The following system of differential equations:
   \[
   \begin{align*}
   \frac{dx_1}{dt} &= 2x_1 - x_2 + t \\
   \frac{dx_2}{dt} &= 3x_1 - 2x_2
   \end{align*}
   \]
   has a linear solution. Determine the set of all solutions \((x_1(t), x_2(t))\).

3. Consider the initial value problem
   \[
   \begin{align*}
   u_{tt} - u_{xx} &= f(x) \cos t \\
   u(x, 0) &= 0, & u_t(x, 0) &= 0, & -\infty < x < \infty, 0 \leq t < \infty
   \end{align*}
   \]
   for a continuous function \( f(x) \) on \( \mathbb{R} \), which vanishes for \(|x| > R\).
   (a) Solve the initial value problem.
   \textit{Note:} The solution is of the form \( u(x, t) = u_p(x, t) + u_h(x, t) \). Use separation of variables to find a particular solution \( u_p(x, t) \) of \( u_{tt} - u_{xx} = f(x) \cos t \) (ignoring the initial values). Then, \( u_h(x, t) \) is a solution to the homogenous PDE with appropriately adjusted initial conditions.
   (b) The particular solution \( u_p(x, t) \) is not unique. Because of that it is not obvious whether the solution \( u(x, t) \) is unique. Prove that it is.