Part I: Real and Complex Analysis (Pure and Applied Exam)

1. Assume that $f : [0, 1] \to \mathbb{R}$ is a smooth function. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x)e^{inx^3} \, dx = 0.$$ 

2. a) Assume $f(x)$ is a strictly increasing continuous function with $f(0) = 0$ and with inverse $f^{-1}$. Show that

$$\int_0^a f(x) \, dx + \int_0^b f^{-1}(x) \, dx \geq ab,$$

for any two positive real numbers $a$ and $b$. For what $b$ does the equality hold?

b) Use this to prove Young's inequality, which states that if $p$ and $q$ are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$ 

3. a) Find a counter example to the following statement:

If $f_n(x)$ for $n > 0$ is a sequence of continuous real-valued functions on the unit interval $[0, 1]$ such that $\lim_{n \to \infty} f_n(x) = 0$ for all $x$. Then

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0.$$ 

b) Find a minimal extra condition that can not be simplified, which makes this statement true.
4. Use a contour integral to evaluate

\[ \int_0^\infty \frac{dx}{1 + x^{2n}}, \quad n \geq 1. \]

5. a) Show by contour integration that

\[ \int_0^{2\pi} \frac{d\theta}{x + \cos \theta} = \frac{2\pi}{\sqrt{x^2 - 1}}, \quad \text{if } x > 1. \]

b) Determine for which complex values of \( w \), the function \( f(w) \) defined as

\[ f(w) = \int_0^{2\pi} \frac{d\theta}{w + \cos \theta} \]

is analytic. Evaluate the integral for those \( w \). Simplify your answer as much as possible. Justify your reasoning with all details.

6. Consider the meromorphic function

\[ f(z) = \frac{1 - z^2}{2i(z^2 - (a + \frac{1}{a})z + 1)}, \quad |a| < 1. \]

Find the Laurent series expansion for \( f(z) \) valid in a neighborhood of the unit circle \(|z| = 1\).
1. a) Over the vector space $\mathcal{P}$ of all polynomials we consider the inner product

$$\langle P, Q \rangle = \int_0^1 P(x)Q(x) \, dx.$$ 

Find a polynomial of degree 2 that is orthogonal to $P_0(x) = 1$ and $P_1(x) = x$.

b) Let now $\mathcal{P}_n$ be the vector space of the polynomials of degree less or equal than $n$. Consider the linear mapping $\mathcal{F} : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$\mathcal{F}(P)(x) = (x-1)P'(x),$$ 

for $P \in \mathcal{P}_n$. Find the matrix $F$ that describes $\mathcal{F}$ with respect to the basis $\{1, x, x^2, \ldots, x^n\}$.

2. Let $A$ be an $n \times n$ matrix with real coefficients. Show the following:

a) If the sum of the elements in each of the columns of $A$ is 1, then $\lambda = 1$ is an eigenvalue of $A$.

b) If $A$ is invertible and $v$ is an eigenvector of $A$, then $v$ is also an eigenvector of both $A^2$ and $A^{-2}$. What are the corresponding eigenvalues?

c) If $AB = BA$ for all invertible matrices $B$, then $A = cI$ for some scalar $c$.

3. a) Let $A$ be an $n \times m$ matrix with real coefficients. Let $v_i$ denote the $i$-th row of $A$, and let $B$ be the matrix obtained from $A$ by the elementary row operation which replaces $v_j$ with $v_j - av_i$, for $a \in \mathbb{R}$ and $i \neq j$. Thus the rows $w_i$ of $B$ are given by $w_i = v_i$ if $i \neq j$, and $w_j = v_j - av_i$. Then show that there exists an invertible $n \times n$ matrix $E$ such that $B = EA$.

b) Use part a) to show that the rank of the row space of $A$ is equal to the rank of the column space of $A$. 

Please turn over
4. a) Let $K$ be a field and let $f(X), g(X)$ be monic irreducible polynomials with coefficients in $K$. Suppose there exists an extension $L/K$ and an element $\alpha \in L$ such that $f(\alpha) = g(\alpha) = 0$. Then show that $f = g$.

b) Let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$ be a monic polynomial with rational integer coefficients. Suppose there exists a rational number $\alpha$ with $f(\alpha) = 0$. Then show that $\alpha$ is a rational integer.

c) Show that the polynomial $X^4 + 1$ is irreducible in $\mathbb{Q}[X]$.

d) Find the Galois group over $\mathbb{Q}$ of the polynomial $X^3 - 2$.

5. a) Let $R$ be a commutative ring (with identity element) and let $I$ and $J$ be ideals of $R$. Show that the set $I + J = \{i + j | i \in I, j \in J\}$ is an ideal of $R$.

b) With the notations of part a), suppose that $I + J = R$. Then show that $R/IJ$ is isomorphic to $R/I \oplus R/J$.

6. a) Let $G$ be a finite group. If $x \in G$, let $G_x$ denote the set of elements in $G$ that are conjugate to $x$, namely, $z \in G_x \iff \exists y \in G$ with $yxy^{-1} = z$. Show that the cardinality of the set $G_x$ divides the order of $G$.

b) If the group $G$ has order $p^r$ where $p$ is a prime, then show that there exists some $x \neq 1 \in G$ such that $G_x = \{x\}$.