Solve all six problems, and start each problem on a new page.

1. Prove or disprove: The series
\[ \sum_{n=1}^{\infty} \frac{\sin x}{1 + n^2 x^2} \]
converges uniformly on \([-\pi, \pi]\).

2. Let \(Q = \{0 < x < 1, 0 < y < 1\}\). For what values of \(a, b\) is the function
\[ x^a y^b \int_0^{\infty} \frac{1}{(x + t)(y^2 + t^2)} \, dt \]
bounded on \(Q\)?

3. (a) Does \(p_N = \prod_{n=2}^{N} (1 + \frac{(-1)^n}{n})\) converge to a nonzero limit as \(N \to \infty\)? Explain your answer!
(b) Prove that \(\int_0^{\infty} \cos(t^2) \, dt\) converges.

4. Let \(f(z) = \int_0^{\infty} e^{-zt^2} \, dt\).
(a) Show that \(f(z)\) is analytic in the domain \(\text{Re}(z) > 0\).
(b) Assume that \(f(1) = \frac{1}{2} \sqrt{\pi}\). Find the analytic continuation of \(f(z)\) into the domain \(\mathbb{C} \sim (-\infty, 0]\). This is the domain that is the whole complex plane except the negative real axis and zero. [Hint: compute \(f(x)\) for \(x > 0\].
(c) Let \(F(z)\) denote the analytic continuation of \(f(z)\) from part (b). Evaluate \(F(i)\).
(d) Evaluate \(\int_0^{\infty} \cos(t^2) \, dt\).

5. Calculate the following integrals:
(a) \(\int_C (z)^2 \, dz\)
where \(C\) is the circle \(|z + 1| = 4\), oriented counterclockwise.
(b) \(\int_C z \sin(z^{-1}) \, dz\)
where \(C\) is the circle \(|z| = 100\), oriented counterclockwise.
(c) \(\int_C \frac{\sin 3z}{(z - 1)^4} \, dz\)
where \(C\) is the circle \(|z| = 2\), oriented counterclockwise.

6. \(J = \int_0^{\infty} \frac{(\ln x)^2}{x^2 + 9} \, dx\).
Evaluate \(J\), explaining all steps and calculations carefully.
UBC Math Qualifying Exam January 9th 2010: Afternoon Exam, 1pm-4pm
APPLIED exam

Solve all seven problems, and start each problem on a new page.

1. Let $x$ be a unit vector in $\mathbb{R}^n$ and let $A = I - \beta xx^T$.
   
   (a) Show that $A$ is symmetric.
   
   (b) Find all values of $\beta$ for which $A$ is orthogonal.
   
   (c) Find all values of $\beta$ for which $A$ is invertible.

2. Let $U$ and $W$ be subspaces of a finite-dimensional vector space $V$.
   Show that $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$.

3. If $A$ and $B$ are real symmetric $n$-by-$n$ matrices with all eigenvalues positive, show that $A + B$
   has the same property.

4. Show that any two commuting matrices with complex entries share a common eigenvector.

5. Consider the ODE given by $x^2(1 - x^2)y'' + 2x(1 - x)y' - y = 0$.
   
   (a) Find and classify the finite singular points of the ODE.
   
   (b) Find the form of a general series solution of the ODE about $x = \frac{1}{2}$. (Do not evaluate
   the coefficients but do indicate which are arbitrary.)
   
   (c) For which values of $x$ would the series solution of (b) converge absolutely?
   
   (d) Find the form of a general series solution of the ODE valid near $x = 0$. (Do not evaluate
   the coefficients but do indicate which are arbitrary.)
   
   (e) For which values of $x$ would the series solution of (d) converge absolutely?
6. Consider the Frenet-Serret formulas for a space curve \( r(s) \) given by the system of differential equations

\[
\frac{dT}{ds} = \kappa(s)N \\
\frac{dN}{ds} = -\kappa(s)T + \tau(s)B \\
\frac{dB}{ds} = -\tau(s)N
\]

\( T = dr/ds \) is the tangent to the curve and \( N \) and \( B \) are unit vectors. Show that \( r(s) \) lies on a circular path if and only if \( \kappa(s) = \text{constant} = K, \tau(s) = 0 \). Find the radius of the circular path.

7. Suppose \( K(x, \xi, t, \tau) \) satisfies

\[
K_t - K_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 < x, \xi < 1, \quad t > 0 \\
K(x, \xi, 0, \tau) = 0 \\
K_x(0, \xi, t, \tau) = K_x(1, \xi, t, \tau) = 0.
\]

(a) Find \( K(x, \xi, t, \tau) \).

(b) In terms of \( K(x, \xi, t, \tau) \) and given data \( \{F(x, t), f(x), h(t), k(t)\} \), find the solution \( u(x, t) \) of the boundary value problem given by

\[
u_t - u_{xx} = F(x, t), \quad 0 < x < 1, \quad t > 0 \\
u(x, 0) = f(x), \quad 0 \leq x \leq 1 \\
u_x(0, t) = h(t), \quad u_x(1, t) = k(t), \quad t > 0
\]

(c) In terms of \( K(x, \xi, t, \tau) \) found in (a), is your solution \( u(x, t) \) useful for large or small times \( t \)? Explain your answer.