Part I

1. Let $b$ and $c$ be real numbers, $c > 0$. Use contour integration to evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{\cos(x-b)}{x^2 + c^2} \, dx.
\]

2. Let $A$ be an $n \times n$ real symmetric matrix, and define the matrix $e^A$ by the convergent series
\[
e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots.
\]
Show that $e^A$ is non-singular.

3. Let $p$ be a prime number and $G$ a group of order $p^3$. Show that for any $g, h \in G$
\[
g^p h = h g^p.
\]

4. Consider the vector field
\[
\mathbf{F}(x, y, z) = \frac{x \hat{i} + y \hat{j} + z \hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.
\]
(a) Verify that $\nabla \cdot \mathbf{F} = 0$ on $\mathbb{R}^3 \setminus \{0\}$.
(b) Let $S$ be a sphere centred at the origin, with “outward” orientation. Show that
\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi. \quad (*)
\]
(c) Now let $E \subset \mathbb{R}^3$ be an open region with smooth boundary $S$ (given “outward” orientation), and suppose $0 \in E$. Show that $(*)$ still holds.

5. (a) Let $f(z)$ be an analytic function on an open, bounded, connected region $\Omega \subset \mathbb{C}$, and suppose $f$ is continuous on the boundary $\partial \Omega$. Suppose also that $|f(z)|$ is constant on $\partial \Omega$. Show that either $f$ has a zero in $\Omega$, or $f$ is constant.
(b) Find all functions which are analytic in the open unit disk $\{z \in \mathbb{C} \ | \ |z| < 1\}$, continuous on the closed unit disk, and which satisfy $|z| \leq |f(z)| \leq 1$ for $|z| \leq 1$.

6. Let $V$ be the vector space of all polynomials $p(x)$ with real coefficients. Let $A$ and $B$ denote the linear transformations on $V$ of (respectively) multiplication by $x$, and differentiation. That is, $A : p(x) \mapsto xp(x)$, and $B : p(x) \mapsto p'(x)$.
(a) Show that $A$ has no eigenvalues, and that $0$ is the only eigenvalue of $B$.
(b) Compute the transformation $BA - AB$.
(c) Show that no two linear transformations $A, B$ on a finite dimensional real vector space can satisfy $BA - AB = I$ (here $I$ denotes the identity transformation).
Pure Math Qualifying Exam: Jan. 8, 2005

Part II

1. (a) Show that a continuous function on \( \mathbb{R} \) cannot take every real value exactly twice.
   (b) Find a continuous function on \( \mathbb{R} \) that takes each real value exactly 3 times.

2. Let \( R \) be the ring \( R = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z} + \mathbb{Z}\sqrt{-3} \).
   (a) Show that \( 2R \subset R \) is not a prime ideal.
   (b) Show that 2 is an irreducible element of \( R \); i.e., if \( 2 = uv \) then either \( u \) or \( v \) is a unit.
   (c) Is \( R \) a principal ideal domain (PID)?

3. Show that an entire function \( f(z) \) satisfying \( \lim_{|z| \to \infty} |f(z)| = c \) (for some \( c \in (0, \infty) \)) is constant.

4. For a vector \( v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n \), define \( \|v\|_1 := \sum_{j=1}^n |v_j| \), and for an \( n \times n \) matrix \( A \), define
   \[
   \|A\|_1 := \sup_{v \in \mathbb{R}^n; v \neq 0} \frac{\|Av\|_1}{\|v\|_1}.
   \]
   Show that if \( A = (a_{ij}) \), then
   \[
   \|A\|_1 = \max_{1 \leq i \leq n} \sum_{i=1}^n |a_{ij}|.
   \]

5. Let \( \{f_n\}_{n=1}^\infty \) be a sequence of continuous functions on \( [0, 1] \) satisfying
   \[
   |f_n(x) - f_n(y)| \leq L|x - y|
   \]
   for all \( x, y \in [0, 1] \), and for all \( n \) (here \( L \) is a fixed constant), and suppose \( f_n \) converges pointwise to a function \( f \). Show that \( f_n \) converges to \( f \) uniformly, and that
   \[
   |f(x) - f(y)| \leq L|x - y|
   \]
   for all \( x, y \).

6. Let \( \alpha \in \mathbb{C} \) be an algebraic number and \( p \) a prime. Show that there exist field extensions of finite degree
   \[
   \mathbb{Q} \subset F \subset K,
   \]
   such that \( \alpha \in K \), the degree \( |K : F| \) is a power of \( p \) and \( |F : \mathbb{Q}| \) is prime to \( p \).