Part I

1. Let $b$ and $c$ be real numbers, $c > 0$. Use contour integration to evaluate the integral

$$
\int_{-\infty}^{\infty} \frac{\cos(x-b)}{x^2 + c^2} \, dx.
$$

2. Let $A$ be an $n \times n$ real symmetric matrix with smallest eigenvalue $\lambda_1$ and largest eigenvalue $\lambda_n$. Show that for any vector $v \neq 0$ in $\mathbb{R}^n$,

$$
\lambda_1 \leq \frac{\langle v, Av \rangle}{\langle v, v \rangle} \leq \lambda_n
$$

(here $\langle v, w \rangle = v^T w$ is the standard inner-product on $\mathbb{R}^n$).

3. (a) Find the eigenvalues $\lambda$ and the eigenfunctions for the eigenvalue problem

$$
u_{xx} + u_{yy} = \lambda u, \quad (x, y) \in (0, 1) \times (0, 1)
$$

$$u(x, 0) = u_x(0, y) = u_y(x, 1) = u(1, y) = 0. \quad (*)$$

(b) Now suppose $w(x, y, t)$ solves the heat equation

$$w_t = w_{xx} + w_{yy}, \quad (x, y) \in (0, 1) \times (0, 1), \quad t > 0$$

with the same boundary conditions as $(*)$, and with initial condition $w(x, y, 0) \equiv 1$. Find the leading-order behaviour of $w$ as $t \to \infty$.

4. Consider the vector field

$$
\mathbf{F}(x, y, z) = \frac{x \hat{i} + y \hat{j} + z \hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.
$$

(a) Verify that $\nabla \cdot \mathbf{F} = 0$ on $\mathbb{R}^3 \setminus \{0\}$.

(b) Let $S$ be a sphere centred at the origin, with “outward” orientation. Show that

$$
\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi. \quad (*)
$$

(c) Now let $E \subset \mathbb{R}^3$ be an open region with smooth boundary $S$ (given “outward” orientation), and suppose $0 \in E$. Show that $(*)$ still holds.
5. In the complex plane, let $C_1$ be the circle passing through $-2, -i,$ and $2$, and let $C_2$ be the circle passing through $-2, (2/3)i,$ and $2$. Let $\Omega$ be the intersection of the open disks whose boundaries are $C_1$ and $C_2$ (so $\Omega$ is the region bounded by $C_1$ and $C_2$).

(a) Find a transformation of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}$$

(i.e. a fractional linear transformation) which maps $-2$ to $0$, $-i$ to $1$, and $2$ to $\infty$.

(b) Find the image of $\Omega$ under this mapping.

(c) Find the angle between the circles $C_1$ and $C_2$ at the point $-2$.

6. Let $V$ be the vector space of all polynomials $p(x)$ with real coefficients. Let $A$ and $B$ denote the linear transformations on $V$ of (respectively) multiplication by $x$, and differentiation. That is, $A : p(x) \mapsto xp(x)$, and $B : p(x) \mapsto p'(x)$.

(a) Show that $A$ has no eigenvalues, and that $0$ is the only eigenvalue of $B$.

(b) Compute the transformation $BA - AB$.

(c) Show that no two linear transformations $A, B$ on a finite dimensional real vector space can satisfy $BA - AB = I$ (here $I$ denotes the identity transformation).

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Part II

1. (a) Show that a continuous function on $\mathbb{R}$ cannot take every real value exactly twice.

(b) Find a continuous function on $\mathbb{R}$ that takes each real value exactly 3 times.

2. A nonlinear oscillator is described by the following ODE for $y(t)$:

$$y'' + \gamma(y)y' + g(y) = 0 \quad (*)$$

where $\gamma(y)$ and $g(y)$ are smooth functions, with $g(0) = 0$.

(a) Verify that the constant function $y_0(t) \equiv 0$ is a solution of $(*)$.

(b) Re-write $(*)$ as a first-order system.

(c) What conditions on $\gamma$ and $g$ ensure that the constant solution $y_0$ is stable? Unstable?

3. Show that an entire function $f(z)$ satisfying $\lim_{|z| \to \infty} |f(z)| = c$ (for some $c \in (0, \infty)$) is constant.
4. For a vector \( v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n \), define \( \|v\|_1 := \sum_{j=1}^{n} |v_j| \), and for an \( n \times n \) matrix \( A \), define
\[
\|A\|_1 := \sup_{v \in \mathbb{R}^n ; v \neq 0} \frac{\|Av\|_1}{\|v\|_1}.
\]
Show that if \( A = (a_{ij}) \), then
\[
\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|.
\]

5. Let \( f(x) \) be a periodic function with period 1 whose Fourier series is \( \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} \).
(a) Find an integral formula for the coefficients \( a_n \) (in terms of \( f \)).
(b) Show that if \( f \) has \( m \) continuous derivatives, then \( |a_n| \leq C/|n|^m \) (where \( C \) is a constant depending on \( f \)).

6. Consider the following PDE for \( u(x,t) \):
\[
u_t + auu_x + bu_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0
\]
(\( a, b > 0 \) are constants).
(a) Use scaling to reduce the problem to the form
\[
w_t + ww_x + w_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0. \quad (*)
\]
(b) Suppose \( w(x,t) \) is a smooth solution of \( (*) \) for which \( w \) and its derivatives decay rapidly to 0 as \( x \to \pm \infty \). Show that the quantity
\[
\int_{-\infty}^{\infty} w^2(x,t)dx.
\]
is constant in time.
(c) Equation \( (*) \) has solutions of the form \( w(x,t) = \phi(x-ct) \) (\( c \) a constant) with \( \phi > 0 \), and \( \phi(y) \) (and its derivatives) tending to 0 as \( y \to \pm \infty \). Find the equation satisfied by the function \( \phi \), and solve it. (This last part is somewhat involved!).