1. (a) Find $A^{10}$ for the matrix $A$ given below:

$$A = \begin{bmatrix} -7 & -3 \\ 18 & 8 \end{bmatrix}.$$ 

(b) Let $P_2[x]$ denote the set of all polynomials in $x$ having real coefficients and degree at most 2. Determine a basis for $P_2[x]$ which contains both $1 + x + x^2$ and $2 + x + x^2$.

(c) Find all possible values of $\det(A + A^{-1})$, allowing arbitrary $3 \times 3$ matrices $A$ with eigenvalues $-1, 1, 2$.

2. Let $M_{3 \times 3}$ be the vector space consisting of all $3 \times 3$ matrices (with elementwise addition and the usual scalar multiplication). Let 0 denote the zero matrix in $M_{3 \times 3}$.

(a) Show that for each fixed $A \in M_{3 \times 3}$, the set $Z(A) = \{B \in M_{3 \times 3} : BA = 0\}$ is a subspace of $M_{3 \times 3}$.

(b) Find the dimension of $Z(0)$, where 0 denotes the zero matrix in $M_{3 \times 3}$.

(c) Suppose $A \in M_{3 \times 3}$ and $\text{rank}(A) = 2$. Find $\dim(Z(A))$.

3. Let $k$ vectors $u_1, u_2, \ldots, u_k$ in $\mathbb{R}^d$ be given. Assume that, for some constant $\alpha \in (0, 1)$,

$$u_i^T u_j = \begin{cases} 1, & \text{if } i = j, \\ \alpha, & \text{if } i \neq j. \end{cases}$$

(Such a collection of unit vectors is called equiangular. Notice that the statement $u_i^T u_j = 1$ implicitly specifies that each $u_j$ is a column vector of unit length.)

Consider the set of $d \times d$ matrices $S = \{u_i u_i^T : i = 1, 2, \ldots, k\}$. Prove the following.

(a) If the matrices in $S$ are linearly independent, then $k \leq \frac{d(d + 1)}{2}$.

(b) The matrices in $S$ are, in fact, linearly independent.

Hint: One approach starts by postulating the matrix equation $\sum_{i=1}^k a_i u_i u_i^T = 0$, then multiplying from the left by $u_j^T$ and from the right by $u_j$.

4. Suppose $a, b \in \mathbb{C}$. Show that the ideal $(x - a, y - b)$ in $\mathbb{C}[x, y]$ generated by $x - a$ and $y - b$ is a maximal ideal.

5. Suppose $p$ is a prime number. In parts (b)–(d) below, assume $d \geq 2$ is an integer. Recall that for any field $\mathbb{F}$, $GL_d(\mathbb{F})$ is the group of invertible $d \times d$ matrices with entries drawn from $\mathbb{F}$.

(a) Show that $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ is a field.

(b) Show that $GL_d(\mathbb{Z}/p\mathbb{Z})$ is a finite group and compute its order in terms of $d$.

(c) Let $U$ be the subgroup of $GL_d(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with all diagonal entries equal to 1. Find the order of $U$.

(d) Show that $d!$ divides the order of $GL_d(\mathbb{Z}/p\mathbb{Z})$. Hint: realize the symmetric group as a subgroup.
6. (10 points) Part (b) below has many correct answers. Choose one that will help with part (c).

(a) Suppose $m, n \geq 1$ are integers. Compute the group of morphisms

$$\text{Hom}_\mathbb{Z}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}).$$

(b) Find an example of a short exact sequence of abelian groups

$$0 \to G' \to G \to G'' \to 0.$$ 

(c) For the groups $G, G', G''$ in part (b), find an abelian group $K$ for which the following sequence is NOT exact:

$$0 \to \text{Hom}_\mathbb{Z}(K, G') \to \text{Hom}_\mathbb{Z}(K, G) \to \text{Hom}_\mathbb{Z}(K, G'') \to 0.$$