1. (7 points) Find all eigenvalues and a basis for each eigenspace of the operator \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by
\[
T(x, y, z) = (2x + y, y - z, 2y + 4z).
\]

2. (8 points) Let \( \vec{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \). Is \( A := \vec{u} \vec{u}^T \in \mathcal{M}_n(\mathbb{R}) \) diagonalizable?

3. (10 points) Let \( A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}) \).

a) Is \( A \) diagonalizable?
b) Give a basis for the \( \mathbb{R} \)-vector space \( \mathcal{C}(A) := \{ B \in \mathcal{M}_3(\mathbb{R}), AB = BA \} \).

4. (14 points) Consider the differential equation (DE)
\[
a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0,
\]
with constant coefficients \( a, b, c \). You are told that one solution of this DE is \( y_1(t) = e^{-2t} \), but the other solution is to be found. You also know the Wronskian of the solutions is
\[
W(y_1, y_2) = 5e^t.
\]

(a) Explain in 1-2 sentences what the Wronskian tells us about the solutions to the DE.
(b) Find the second solution, \( y_2(t) \).
(c) If \( a = 1 \), what are the values of the constants \( b, c \)?
(d) Determine the solution to the initial value problem given the initial conditions \( y(0) = 1, y'(0) = 8 \).
(e) Does the solution in (d) ever change sign for \( t > 0 \)? If so, at what time; if not, why not?

Recall that the Wronskian is defined as
\[
W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.
\]

5. (16 points) Consider the following initial value problem (IVP):
\[
\frac{\partial u}{\partial t} - 3x \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,
\]
with \( u(x, 0) = \begin{cases} x(4-x) & \text{if } 0 \leq x \leq 4, \\ 0 & \text{if } x < 0 \text{ or } x > 4. \end{cases} \)

You are asked to solve this IVP using the Method of Characteristics.

(a) Show that along the appropriate “characteristic curves” \( x = f(x_0, t) \), the PDE can be rewritten as
\[
\frac{du}{dt} = 0.
\]
(b) Find the equations of the characteristic curves and sketch them in the $x$ versus $t$ plane.

(c) Solve the IVP, that is, find $u(x,t)$.

(d) Interpret the behaviour of the solution, that is, say verbally what would be seen by an observer measuring $u$ at some location $x$ as time increases.

(e) Calculate the total mass as a function of time $t$,

$$U(t) = \int_{-\infty}^{\infty} u(x,t)dx.$$ 

6. (15 points) Consider the following initial boundary value problem on $0 \leq x \leq L$ and $t \geq 0$ for the concentration of a chemical $c(x,t)$ inside a pipe of length $L$:

$$c_t = Dc_{xx}, \quad c(x,0) = \phi(x), \quad c_x(0,t) = A, \quad c(L,t) = B,$$

where $A$ and $B$ are constants.

(a) Provide a physical interpretation of the above problem, by explaining what is happening inside the pipe and at its two ends.

(b) What will be the chemical profile in the pipe after a long time? Determine the form of the distribution $c_{ss}(x)$ as $t \to \infty$. Are there any conditions on $A, B, D, \phi$ that need to be satisfied for this solution to exist physically?

(c) Define $u(x,t) = c(x,t) - c_{ss}(x)$. What initial boundary value problem does $u$ satisfy?

(d) Use Separation of Variables to solve the problem in part (c). Leave your answer in terms of the constants in the original problem and the initial condition.