PARTIAL FRACTIONS

WE LET $f(x) = \dfrac{P(x)}{Q(x)}$ AND SUPPOSE $\deg P \geq \deg Q$.

WE NEED TO DO LONG DIVISION FIRST. WE GIVE TWO EXAMPLES OF THIS.

EXAMPLE CALCULATE $I = \displaystyle\int \frac{5x^2-3x-1}{x^2-1}\,dx$.

step 1

$$\begin{array}{r l} & 5 \\ x^2-1 \,\big) & 5x^2-3x-1 \\ & -(5x^2 \quad -5) \\ \hline & -3x+4 \end{array}$$

REMARK WE CAN ALSO ADD AND SUBTRACT $\dfrac{5(x^2-1)-3x+5-1}{x^2-1} = 5 + \dfrac{(-3x+4)}{(x^2-1)}$

SO $\dfrac{5x^2-3x-1}{x^2-1} = 5 + \dfrac{(-3x+4)}{x^2-1}$ $(*)$ NOW DO PFD ON THIS TERM

NOW $\dfrac{-3x+4}{x^2-1} = \dfrac{-3x+4}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} = \dfrac{A(x+1)+B(x-1)}{(x-1)(x+1)}$

SO $-3x+4 = A(x+1) + B(x-1)$ $\quad x=1 \rightarrow A = 1/2$

$x=-1 \quad B = -7/2$.

SO IN $(*)$ UPON SUBSTITUTING $\dfrac{5x^2-3x-1}{x^2-1} = 5 + \dfrac{1}{2(x-1)} - \dfrac{7}{2(x+1)}$

SO $I = \displaystyle\int \frac{5x^2-3x-1}{x^2-1}\,dx = 5x + \frac{1}{2}\ln|x-1| - \frac{7}{2}\ln|x+1| + C.$

EXAMPLE $I = \displaystyle\int \frac{9x^3+1}{x(x^2+1)}\,dx.$

step 1 $x(x^2+1) = x^3+x \rightarrow$

$$\begin{array}{r l} & 9 \\ x^3+x \,\big) & 9x^3+1 \\ & -(9x^3+9x) \\ \hline & 1-9x \end{array}$$

SO $\dfrac{9x^3+1}{x(x^2+1)} = 9 + \dfrac{(1-9x)}{x(x^2+1)}$ $(*)$

$$\frac{1-9x}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ IS THE PFD SINCE x^2+1 IS AN IRREDUCIBLE QUADRATIC FACTOR

SO $A(x^2+1)+(Bx+C)x = 1-9x.$

UPON CROSS-MULTIPLYING.

set $x=0: \longrightarrow A=1$

COMPARE $x^2: \rightarrow A+B=0 \rightarrow B=-1$

COMPARE $x: \longrightarrow C=-9$

SO $$\frac{1-9x}{x(x^2+1)} = \frac{1}{x} + \frac{(-x-9)}{x^2+1}$$

sub INTO (*)

CAN integrate WITH $u=x^2+1$ SUBSTITUTION

$$\frac{9x^3+1}{x(x^2+1)} = 9 + \frac{1}{x} - \frac{x}{x^2+1} - \frac{9}{x^2+1}$$

SO $$I = \int \frac{9x^3+1}{x(x^2+1)}\,dx = 9x + \ln|x| - \frac{1}{2}\ln|x^2+1| - 9\arctan x + C.$$