EXTRA QUESTIONS (BMW)

EXAMPLE 1 CALCULATE $\int_0^\infty x^n e^{-x}\,dx$ IF $n \geq 0$ AND n INTEGER.

SOLUTION WE WILL USE IBP ONCE AND DERIVE A RECURSION RELATION.

DEFINE $I_n = \int_0^\infty x^n e^{-x}\,dx = \lim_{L\to\infty} \int_0^L x^n e^{-x}\,dx.$

LET $n > 0$ INTEGER. THEN USE IBP ONCE FOR $n \geq 1$ TO GET

$$u = x^n \longrightarrow du/dx = n x^{n-1}$$
$$\frac{dv}{dx} = e^{-x} \longrightarrow v = -e^{-x}.$$

SO $I_n = \lim_{L\to\infty} \left[-x^n e^{-x}\Big|_0^L + n\int_0^L x^{n-1} e^{-x}\,dx \right]$

NOW SINCE $\lim_{L\to\infty} L^n e^{-L} = 0$ (USING L'HOPITALS RULE) WE HAVE

$$I_n = n \lim_{L\to\infty} \int_0^L x^{n-1} e^{-x}\,dx \longrightarrow \boxed{I_n = n\, I_{n-1}} \text{ FOR } n \geq 1.$$

NOW THIS IS A 1-TERM RECURSION RELATION THAT NEEDS AN INITIAL VALUE. NOW $I_0 = \lim_{L\to\infty} \int_0^L e^{-x}\,dx = -\lim_{L\to\infty} e^{-x}\Big|_0^L = 1 - \lim_{L\to\infty} e^{-L} = 1.$

SINCE $I_0 = 1$ WE CAN USE $I_n = n\, I_{n-1}$ TO FIND I_n FOR ANY $n \geq 1$.

WE CALCULATE
$$I_1 = 1 \cdot I_0$$
$$I_2 = 2\, I_1 = 2 \cdot 1 \cdot I_0$$
$$I_3 = 3\, I_2 = 3 \cdot 2 \cdot 1 \cdot I_0$$
$$I_4 = 4 \cdot I_3 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot I_0$$

WE CONCLUDE BY INDUCTION THAT

$$I_n = n! = \int_0^\infty x^n e^{-x}\,dx \quad \text{FOR ANY INTEGER } n \geq 0.$$

REMARK KEY IDEA WAS TO USE IBP ONCE TO GET A RECURSION RELATION, THEN GET GENERAL PATTERN.

EXAMPLE 2 CALCULATE $\int_0^1 (\ln x)^n \, dx$ FOR $n \geq 0$ AND n INTEGER.

SOLUTION THIS IS AN IMPROPER INTEGRAL. WE DO ONE IBP AND THEN DERIVE A RECURSION RELATION.

$$I_n \equiv \int_0^1 (\ln x)^n \, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 (\ln x)^n \, dx.$$

NOW USE IBP: LET $n \geq 1$ INTEGER.

$$u = (\ln x)^n \qquad du/dx = n(\ln x)^{n-1}/x$$

$$\frac{dv}{dx} = 1 \qquad v = x$$

$$I_n = \lim_{\varepsilon \to 0^+} \left[x(\ln x)^n \Big|_\varepsilon^1 - n \int_\varepsilon^1 \frac{(\ln x)^{n-1} \cancel{x}}{\cancel{x}} \, dx \right]$$

$$I_n = \lim_{\varepsilon \to 0^+} \left[1(\ln 1)^n - \varepsilon(\ln \varepsilon)^n - n \int_\varepsilon^1 (\ln x)^{n-1} \, dx \right].$$

BUT $\ln 1 = 0$ AND $\lim_{\varepsilon \to 0^+} \varepsilon(\ln \varepsilon)^n = 0$ PROOF: let $\varepsilon = e^{-y}$

$$\lim_{\varepsilon \to 0^+} \varepsilon(\ln \varepsilon)^n = \lim_{y \to \infty} (-y)^n e^{-y} = 0.$$

THUS $I_n = -n I_{n-1}$ FOR $n \geq 1$.

NOW NOTICE THAT $I_0 = \int_0^1 1 \, dx = 1$.

SO THE RECURSION RELATION IS

$$I_n = -n I_{n-1} \quad \text{FOR } n = 1, 2, \ldots$$

WITH $I_0 = 1$.

NOW THIS GIVES

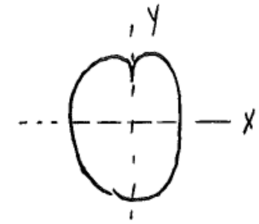
$$I_1 = -I_0$$
$$I_2 = -2 I_1 = -2(-1) I_0 = 2 I_0$$
$$I_3 = -3 I_2 = -3(2) I_0 = -3! \, I_0$$
$$I_4 = -4 I_3 = 4 \cdot 3! = 4! \, I_0.$$

CONTINUING ON AND USING $I_0 = 1$ WE GET

$$I_n = (-1)^n n! = \int_0^1 (\ln x)^n \, dx.$$

NOTICE: $(-1)^n = 1$ IF $n = 0, 2, 4, \ldots$ AND $(-1)^n = -1$ IF $n = 1, 3, 5, 7, \ldots$

EXAMPLE 3 SINCE WE ALL LOVE CALCULUS SO MUCH CONSIDER THE HEART SHAPED REGION SHOWN WHICH IS THE REGION INSIDE THE CURVE $x^2 + \frac{9}{4}\left(y - \sqrt{|x|}\right)^2 = 1$.



(i) FIND THE AREA OF THE HEART-SHAPED CURVE.

(ii) FIND THE y COORDINATE OF THE CENTROID IN THE FORM $\bar{y} = b\int_0^1 \sqrt{x-x^3}\,dx$ FOR SOME b THAT YOU ARE TO FIND. (DO NOT TRY TO EVALUATE THE INTEGRAL.)

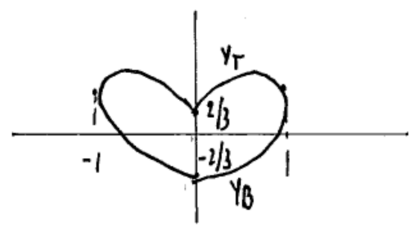
SOLUTION (i) TO GET AN IDEA HOW TO PROCEED, IF WE HAD A CIRCLE $x^2 + (y-1)^2 = 1$ WE WOULD WRITE $(y-1)^2 = 1 - x^2$ AND $y - 1 = \pm\sqrt{1-x^2}$; SO THAT $y_T = 1 + \sqrt{1-x^2}$ AND $y_B = 1 - \sqrt{1-x^2}$.

LET'S PROCEED THE SAME WAY: $\frac{9}{4}\left(y - \sqrt{|x|}\right)^2 = 1 - x^2 \rightarrow \left(y - \sqrt{|x|}\right)^2 = \frac{4}{9}(1-x^2)$.

THUS $y - \sqrt{|x|} = \pm \frac{2}{3}\sqrt{1-x^2}$.

THUS $y_T(x) = \sqrt{|x|} + \frac{2}{3}\sqrt{1-x^2}$

$y_B(x) = \sqrt{|x|} - \frac{2}{3}\sqrt{1-x^2}$



NOTICE: IF $x = 0$ $9y^2/4 = 1$ SO $y = \pm 2/3$

NOTICE $y_T = y_B$ AT $x = \pm 1$. THUS WE HAVE REGION AS SHOWN.

THEN $A = \int_{-1}^{1}(y_T - y_B)\,dx = \underset{\text{EVEN}}{2}\int_0^1 (y_T - y_B)\,dx = 2\cdot\left(\frac{2}{3}\cdot 2\right)\int_0^1 \sqrt{1-x^2}$.

$A = \frac{8}{3}\int_0^1 \sqrt{1-x^2}\,dx = \frac{8}{3}\left(\pi/4\right) = \frac{2\pi}{3}$ SO $A = 2\pi/3 \Rightarrow$ AREA $= \frac{2\pi}{3}$

(ii) BY SYMMETRY $\bar{x} = 0$. NOW FROM OUR FORMULA $\bar{y} = \frac{1}{2\,\text{AREA}}\int_{-1}^{1}\left[y_T^2 - y_B^2\right]dx$.

BY USING EVENNESS $\bar{y} = \frac{1}{\text{AREA}}\int_0^1\left[\left(\sqrt{|x|} + \frac{2}{3}\sqrt{1-x^2}\right)^2 - \left(\sqrt{|x|} - \frac{2}{3}\sqrt{1-x^2}\right)^2\right]dx$

BUT ON $x > 0$ AND FROM CANCELLATION, $\bar{y} = \frac{1}{\text{AREA}}\int_0^1 2\cdot\frac{4}{3}\sqrt{x}\,\sqrt{1-x^2}\,dx$.

$\bar{y} = \frac{8}{3\,\text{AREA}}\int_0^1 \sqrt{x-x^3}\,dx$. NOW THIS MEANS

$\bar{y} = b\int_0^1 \sqrt{x-x^3}\,dx$ WITH $b = \frac{8}{3\cdot\text{AREA}} = \frac{4}{\pi}$.