AVERAGES, CENTER OF MASS, MOMENTS

CONSIDER DATA POINTS $f_1, \ldots, f_N$. THE ARITHMETIC AVERAGE $\bar{f}_{ave}$ IS

$$\bar{f}_{ave} = \frac{1}{N}\left[f_1 + \ldots + f_N \right] = \frac{1}{N}\sum_{i=1}^{N} f_i .$$

NOW SUPPOSE WE WANT TO DEFINE THE "AVERAGE" OF A FUNCTION $f(x)$ ON $a \le x \le b$. WE PARTITION X-AXIS INTO N EQUAL SEGMENTS $[x_{i-1}, x_i]$ WITH $x_0 = a$ AND $x_N = b$ AND $\Delta x = \frac{b-a}{N}$. LET x_i^* BE ANY POINT IN $[x_{i-1}, x_i]$.

THEN THE AVERAGE OF $f(x_1^*), \ldots, f(x_N^*)$ IS SIMPLY

$$\bar{f}_{ave} = \frac{1}{N}\left[f(x_1^*) + \ldots + f(x_N^*)\right] = \frac{1}{N}\sum_{i=1}^{N} f(x_i^*).$$

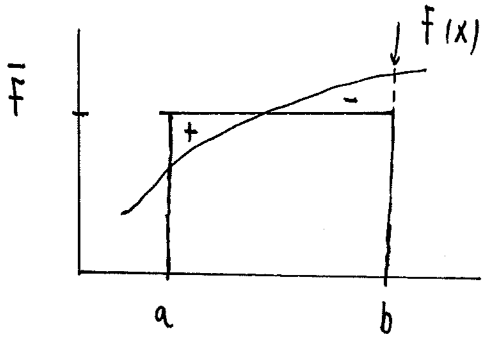
MULTIPLY BY $b-a$ TOP AND BOTTOM AND USE $\Delta x = \frac{b-a}{N}$.

SO
$$\bar{f}_{ave} = \frac{1}{b-a}\sum_{i=1}^{N} f(x_i^*)\,\Delta x .$$

NOW LET $N \to \infty$, $\Delta x \to 0$ AND OBSERVING THAT WE HAVE A RIEMANN SUM WE DEFINE

$$\bar{f} \equiv \frac{1}{b-a}\int_a^b f(x)\,dx . \qquad (1)$$

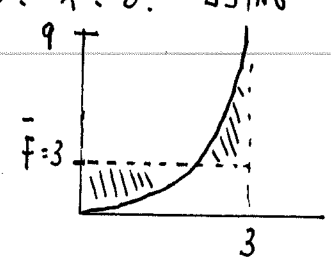
FOR A POSITIVE FUNCTION $f(x)$ THIS YIELDS THE GEOMETRIC INTERPRETATION:



AREA OF RECTANGLE $\bar{f}(b-a)$ IS AREA $\int_a^b f(x)\,dx$ SO $-$, $+$ LOBES CANCEL IN AREA.

PROBLEM 1 FIND THE AVERAGE OF $f(x) = x^2$ ON $0 \le x \le 3$. USING (1) WE GET

$$\bar{f} = \frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{9}x^3\Big|_0^3 = 3.$$



SHADED AREAS ARE THE SAME.

PROBLEM 2 LET $v(t)$ BE THE SPEED OF A PARTICLE ON A TIME INTERVAL $0 \le t \le T$. WHAT IS THE AVERAGE SPEED?

SOL'N
$$V_{ave} = \frac{1}{T-0}\int_0^T v(t)\,dt.$$

BUT $v(t) = dx/dt$ SO THAT $V_{ave} = \frac{1}{T}\int_0^T \frac{dx}{dt}\,dt = \frac{1}{T}\,x(t)\Big|_0^T.$

$$\longrightarrow \quad V_{ave} = \frac{1}{T}\left[x(T) - x(0)\right] = \frac{\text{TOTAL distance}}{\text{TOTAL TIME}}.$$

PROBLEM 3 IF A CUP OF COFFEE HAS TEMPERATURE $95^\circ C$ IN A ROOM WHERE THE TEMPERATURE IS $20^\circ C$, THEN FROM NEWTON'S LAW OF COOLING THE TEMPERATURE OF THE COFFEE AFTER t MINUTES IS
$$T(t) = 20 + 75e^{-t/50}.$$
WHAT IS THE AVERAGE TEMPERATURE OF THE COFFEE DURING THE FIRST HALF HOUR?

SOLUTION CONVERTING TO MINUTES WE WANT T_{ave} OVER $0 \le t \le 30$.

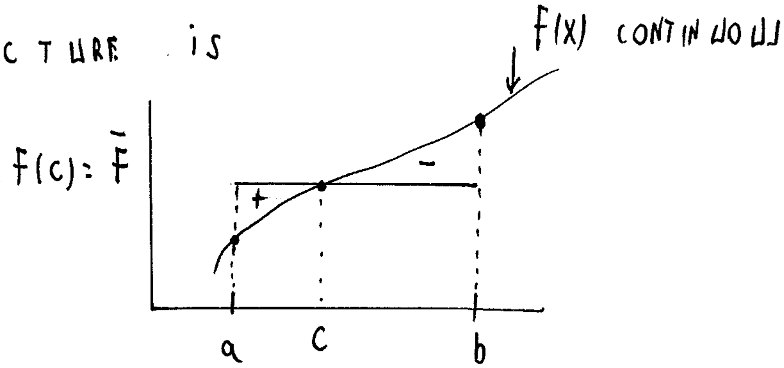
SO
$$T_{ave} = \frac{1}{30}\int_0^{30} T(t)\,dt = \frac{1}{30}\int_0^{30}\left[20 + 75e^{-t/50}\right]dt$$
$$= \frac{1}{30}\left[20(30) - \frac{75(50)}{30}e^{-t/50}\Big|_0^{30}\right] = 20 - \frac{75(50)}{30}\left(e^{-30/50} - 1\right)$$
$$= 20 + \frac{(75)(5)}{3}\left(1 - e^{-3/5}\right) = 20 + 125\left(1 - e^{-3/5}\right) \cong 76.4^\circ C.$$

OPTIONAL (MEAN-VALUE THEOREM FOR INTEGRALS)

THEOREM IF $f(x)$ IS CONTINUOUS ON $[a,b]$ THEN THERE IS A NUMBER c IN $[a,b]$ SUCH THAT $\int_a^b f(x)\,dx = f(c)(b-a)$.

INTERPRETATION $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$, i.e. IF $f(x)$ IS A CONTINUOUS FUNCTION THEN SOMEWHERE IN $[a,b]$ THE FUNCTION WILL TAKE ON ITS AVERAGE VALUE.

THE PICTURE IS



EXAMPLE (OPTIONAL) FIND THE NUMBER C SO THAT THE MVT FOR INTEGRALS IS SATISFIED FOR $f(x) = x^2 + 3x + 2$ ON $1 \leq x \leq 4$.

SOLUTION $\bar{F} = \frac{1}{4-1}\int_1^4 f(x)\,dx$. WANT $\bar{F} = f(c)$ FOR SOME C.

SO $$c^2 + 3c + 2 = \frac{1}{3}\int_1^4 (x^2 + 3x + 2)\,dx = \frac{1}{3}\left(\frac{x^3}{3} + \frac{3}{2}x^2 + 2x\right)\Big|_1^4 .$$

THUS $$3c^2 + 9c + 6 = \frac{1}{3}(4^3 - 1) + \frac{3}{2}(16 - 1) + 6 = \frac{1}{3}(63) + \frac{3(15)}{2} + 6 .$$

SO $$3c^2 + 9c + 6 = 21 + \frac{45}{2} + 6 = \frac{42 + 45 + 12}{2} = \frac{99}{2}$$

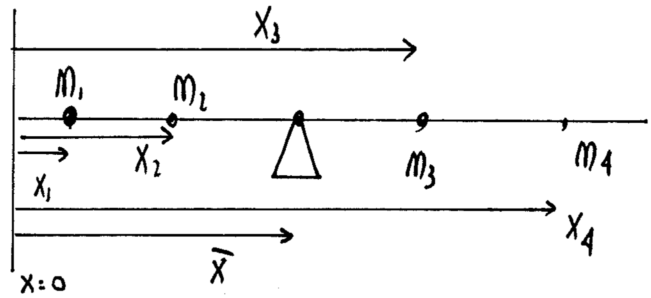
SO $$3c^2 + 9c - \frac{87}{2} = 0 \rightarrow c_{\pm} = \frac{-3 \pm \sqrt{67}}{2}$$

ONLY c_+ IS IN $[1, 4]$ SO $c_+ = \frac{-3 + \sqrt{67}}{2}$.

CENTER OF MASS

IF YOU SUPPORT A BODY AT ITS CENTER OF MASS (IN UNIFORM GRAVITY) IT BALANCES PERFECTLY. CONSIDER A 1-D PROBLEM OF POINT MASSES m_i AT POSITIONS x_i. THE CENTER OF MASS $\underline{X}$ IS DEFINED AS

$$\bar{X} = \frac{\sum_{i=1}^{N} x_i m_i}{\sum_{i=1}^{N} m_i}$$



(N = 4 MASSES).

A WAY TO THINK OF THIS IS TO IDENTIFY FROM BALANCING FORCES THAT

$$\sum_{i=1}^{N} m_i \left(\bar{X} - x_i \right) = 0 \quad \text{IS CONDITION FOR CENTER OF MASS.}$$

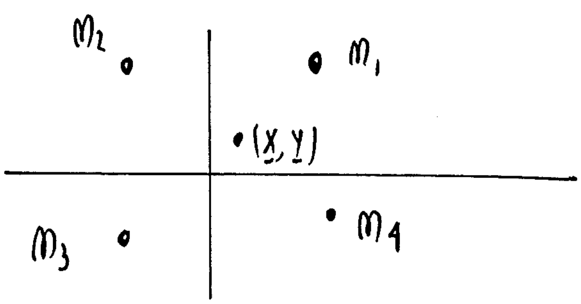
THIS GIVES

$$\bar{X} \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} m_i x_i \longrightarrow \bar{X} = \sum_{i=1}^{N} m_i x_i \Big/ \sum_{i=1}^{N} m_i = \frac{\text{"MOMENT" ABOUT ORIGIN}}{\text{TOTAL MASS}}$$

THUS THE SYSTEM BEHAVES LIKE A POINT MASS OF "SIZE" $\sum_{i=1}^{N} m_i$ CENTERED AT $\bar{X}$.

NOW IN A 2-D SCENARIO WITH POINT MASSES $m_1, \ldots, m_N$ CENTERED AT $(x_1, y_1), (x_2, y_2), \ldots (x_N, y_N)$ THE X AND Y COORDINATES OF THE CENTER OF MASS ARE GIVEN BY

$$\bar{X} = \sum_{i=1}^{N} x_i m_i \Big/ \sum_{i=1}^{N} m_i \qquad \bar{Y} = \sum_{i=1}^{N} y_i m_i \Big/ \sum_{i=1}^{N} m_i$$

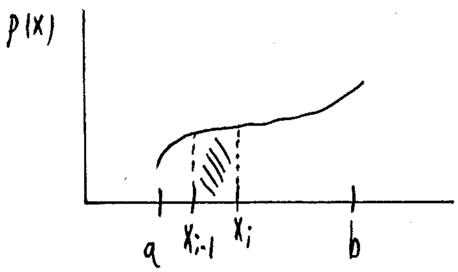


NOW RETURN TO THE 1-D CASE

IF THE BODY CONSISTS OF MASS DISTRIBUTED ALONG A STRAIGHT LINE WITH DENSITY $\rho(x)$ (kg/m) WITH $a \le x \le b$ THE CENTER OF MASS $\bar{x}$ IS

$$\bar{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx} = \frac{\text{MOMENT ABOUT ORIGIN}}{\text{TOTAL MASS}}$$

REMARK FROM THE DISCRETE MASS CASE WE CAN WRITE



THE MASS IN CHUNK $x_{i-1} < x < x_i$ IS $m_i = \rho(x_i^*)\,\Delta x$. SO IN DISCRETE CASE WE HAVE WITH $\Delta x = (b-a)/N$

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i^*\,\rho(x_i^*)\,\Delta x}{\sum_{i=1}^{N} \rho(x_i^*)\,\Delta x} \rightarrow \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

(OPTIONAL)

EXAMPLE A METAL ROD IS 50 cm LONG. ITS LINEAR DENSITY AT THE POINT x cm FROM LEFT END IS $\rho(x) = \frac{1}{100-x}$ $(\mathrm{gm}/\mathrm{cm})$. FIND THE MASS AND THE CENTER OF MASS FOR THE ROD.

SOLUTION MASS $= \int_0^{50} \rho(x)\,dx = \int_0^{50} \frac{1}{100-x}\,dx = -\log(100-x)\Big|_0^{50} = \log(100) - \log(50)$

SO MASS $= \log(2)$.

NOW THE MOMENT ABOUT $x=0$ IS $\int_0^{50} x\,\rho(x)\,dx = \int_0^{50} \frac{x}{100-x}\,dx$

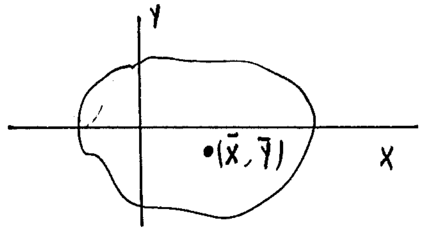
$$\text{MOMENT} = \int_0^{50} \left(\frac{x-100}{100-x} + \frac{100}{100-x} \right) dx = -50 - 100\log(100-x)\Big|_0^{50}$$

$$= -50 - 100\left(\log(50) - \log(100)\right) = -50 + 100\log(2)$$

THUS $\bar{x} = \frac{100\log 2 - 50}{\log 2} = 100 - \frac{50}{\log 2} = 28.06$ cm.

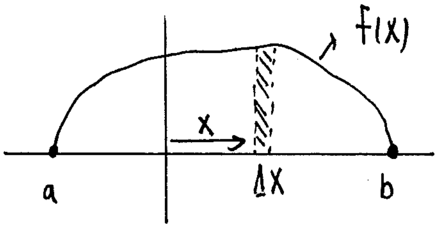
CENTROID OF A LAMINA

LAMINA IS A THIN "PLATE", WHICH OCCUPIES SOME AREA IN THE X,Y PLANE.



WE WILL ASSUME THAT THE DENSITY ρ IS CONSTANT IN THE LAMINA AND WANT TO CALCULATE THE X,Y COORDINATES, LABELLED BY $\bar{X}$ AND $\bar{Y}$ OF THE CENTER OF MASS, I.e. THE CENTROID.

CASE I CONSIDER A LAMINA WITH CONSTANT DENSITY ρ WHOSE LOWER BOUNDARY IS THE X-AXIS AND UPPER BOUNDARY IS $y = f(x)$ AS SHOWN



WE MUST FIRST FIND MOMENT ABOUT Y-AXIS. WE TAKE A CHUNK Δx AT A SIGNED DISTANCE X FROM VERTICAL AXIS. THE AREA OF THE STRIP IS

$$\text{AREA} = (f(x)\,\Delta x)$$

SO MASS OF STRIP IS $\quad \text{MASS} = (f(x)\,\Delta x)\,\rho$

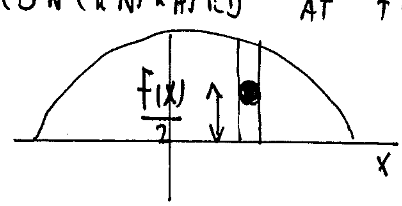
THE MOMENT OF THE STRIP IS, ABOUT Y-AXIS, LABELLED BY ΔM_y IS $\Delta M_y = x\,[\,f(x)\,\Delta x\,\rho\,]$. INTEGRATING OVER ALL SUCH STRIPS

THEN $$\bar{X} = \frac{M_y}{M} = \frac{\int_a^b x\,f(x)\,\rho\,dx}{\int_a^b f(x)\,\rho\,dx}$$ WHERE $M = \int_a^b f(x)\,\rho\,dx$ IS TOTAL MASS OF LAMINA.

SINCE DENSITY IS CONSTANT, IT CANCELS AND

$$\bar{X} = \frac{M_y}{M} = \frac{\int_a^b x\,f(x)\,dx}{\int_a^b f(x)\,dx}. \quad (1)$$

NOW TO FIND THE MOMENT ABOUT THE X-AXIS. THE CENTER OF MASS OF THE STRIP OF WIDTH Δx IS AT $\frac{y}{2} = f(x)$ AND ITS MASS CAN BE THOUGHT OF AS CONCENTRATED AT THE POINT $(x, y/2)$. THIS MASS IS $(f(x)\,\Delta x\,\rho)$



● COORDINATES $(x, y/2) = (x, \frac{f(x)}{2})$

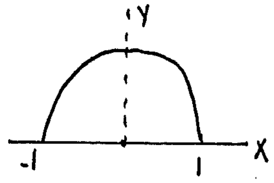
THUS $$M_X = \int_a^b \left(\frac{f(x)}{2}\right) \underbrace{(f(x)\,\rho\,dx)}_{\text{MASS IN SLICE}}$$

THE CENTER OF MASS IN LAMINA IS $\bar{y} = \dfrac{M_X}{M}$ OR

$$\bar{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\,dx}{\int_a^b f(x)\,dx} \qquad (2).$$

EXAMPLE FIND THE CENTER OF MASS OF A PARABOLIC PLATE $y = 1 - x^2$ ABOVE $y = 0$ AND $-1 \le x \le 1$. ASSUME CONSTANT DENSITY.

SOLUTION



- BY SYMMETRY WE SHOULD HAVE $\bar{x} = 0$.

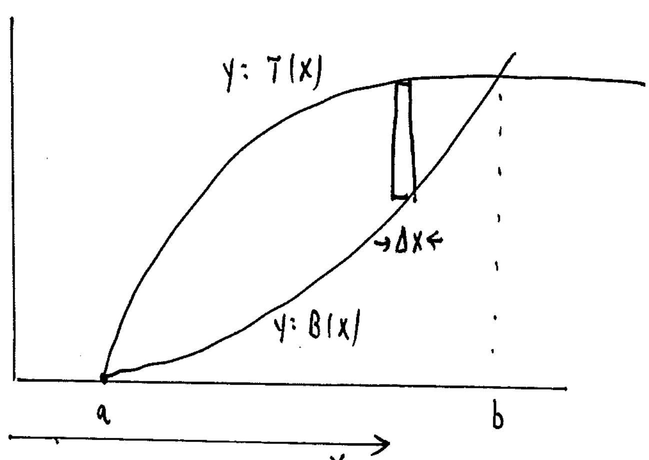
  CHECK $f(x) = 1 - x^2$ SO $\bar{x} = \dfrac{\int_{-1}^1 \underbrace{x}_{\text{ODD}}\underbrace{(1-x^2)}_{\text{EVEN}}\,dx}{\int_{-1}^1 (1-x^2)\,dx} = 0.$

- NOW $$\bar{y} = \frac{1}{2}\frac{\int_{-1}^1 (1-x^2)^2\,dx}{\int_{-1}^1 (1-x^2)\,dx} = \frac{1}{2}\frac{\int_0^1 (1-x^2)^2\,dx}{\int_0^1 (1-x^2)\,dx} = \frac{1}{2}\frac{\int_0^1 (1-2x^2+x^4)\,dx}{(1-1/3)}$$

$$\bar{y} = \frac{3}{4}\int_0^1 (1-2x^2+x^4)\,dx = \frac{3}{4}\left(1 - \frac{1}{3} + \frac{1}{5}\right) = \frac{3}{4}\left(\frac{15-10+3}{15}\right) = \frac{3}{4}\left(\frac{8}{15}\right)$$

THUS $\bar{y} = \dfrac{2}{5}$.

NEXT, WE DEVELOP A FORMULA FOR THE CENTROID OF THE LAMINA DEFINED BY $a \le x \le b$, $B(x) \le y \le T(x)$ AS SHOWN.



ASSUME CONSTANT DENSITY ρ.

THE TOTAL MASS M IS

$$M = \rho\int_a^b [T(x) - B(x)]\,dx.$$

NOW THE MASS IN SLICE ΔX IS $\Delta \text{MASS} = (T(x) - B(x))\Delta x\, \rho$.

THE MOMENT ABOUT Y-AXIS IS $\Delta M_y = x\,(T(x) - B(x))\Delta x\, \rho$.

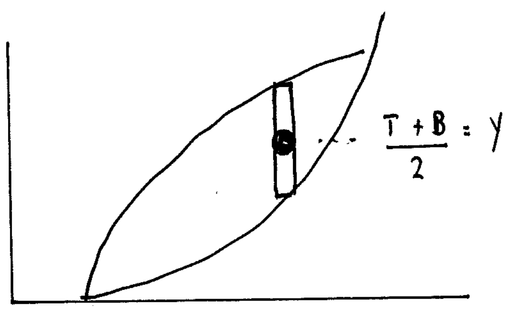
SO THE TOTAL MOMENT ABOUT Y-AXIS IS

$$M_y = \int_a^b x\,[T(x) - B(x)]\,\rho\, dx$$

THIS YIELDS

$$\bar{X} = \frac{M_y}{M} = \frac{\int_a^b x\,[T(x) - B(x)]\,dx}{\underbrace{\int_a^b [T(x) - B(x)]\,dx}_{\leftarrow \text{AREA}}} \qquad (3).$$

SINCE ρ CANCELS OUT. NOW TO FIND MOMENT ABOUT Y-AXIS WE OBSERVE THAT CENTER OF MASS OF SLICE IS $\frac{T(x)+B(x)}{2}$ AND WE CAN PUT ALL MASS IN SLICE CENTERED AT THIS POINT.



SO MOMENT IS ABOUT X-AXIS:

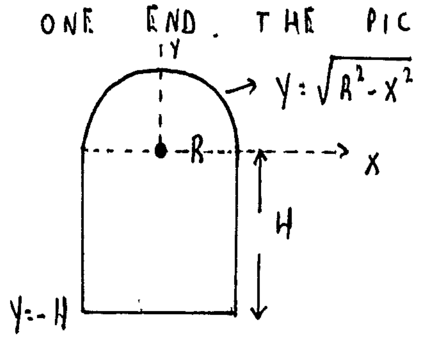
$$\Delta M_X = \left(\frac{T(x)+B(x)}{2}\right)\left[T(x) - B(x)\right]\rho\,\Delta x$$

SUMMING OVER SLICES AND CANCELLING ρ WE GET

$$\bar{y} = \frac{M_x}{M} = \frac{1}{2}\,\frac{\int_a^b \left[(T(x))^2 - (B(x))^2\right] dx}{\int_a^b (T(x) - B(x))\,dx} \qquad (4).$$

FORMULAS (3) AND (4) ARE OUR KEY RESULTS.

<u>EXAMPLE 1</u> FIND THE CENTROID OF A REGION CONSISTING OF A RECTANGLE OF WIDTH $2R$ AND HEIGHT H WHICH HAS A SEMICIRCLE OF RADIUS R ON ONE END. THE PICTURE IS



NOW BY SYMMETRY $\bar{X} = 0$, CENTROID MUST LIE ON Y-AXIS.

$$\text{AREA} = \underbrace{\tfrac{1}{2}\pi R^2}_{\text{semi-circle}} + \underbrace{2RH}_{\text{Rectangle}}$$

NOW $T(x) = \sqrt{R^2 - x^2}$ AND $B(x) = -H$.

WE HAVE $\bar{y} = \frac{1}{2\,\text{AREA}} \int_{-R}^{R} (T^2 - B^2)\,dx = \frac{1}{2\,\text{AREA}} \int_{-R}^{R} (R^2 - x^2 - H^2)\,dx = \frac{1}{\text{AREA}} \int_{0}^{R} (R^2 - H^2 - x^2)\,dx.$

SO $\bar{y} = \frac{1}{\text{AREA}} \left[(R^2 - H^2)R - \frac{R^3}{3} \right] = \frac{1}{\text{AREA}} \left[\frac{2R^3}{3} - HR^2 \right] = \frac{1}{\frac{1}{2}\pi R^2 + 2RH} \left(\frac{2R^3}{3} - HR^2 \right).$

NOW MULTIPLYING TOP AND BOTTOM BY 6, WE GET

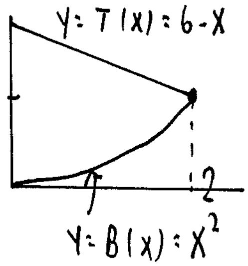
$$\bar{y} = \frac{4R^2 - 6H^2}{3\pi R + 12H}.$$

NOTICE: (i) IF $R \ll H$, $\longrightarrow$ $\bar{y} \approx -H/2$ AS EXPECTED

(ii) IF $H \ll R$, $\longrightarrow$ $\bar{y} \approx \frac{4R}{3\pi}$ (SEMI-CIRCLE).

EXAMPLE 2 FIND CENTROID OF THE REGION IN THE FIRST QUADRANT BOUNDED BY $y = x^2$ AND $y = 6 - x$.

SOLUTION



INTERSECTION POINT $6 - x = x^2$ SO $x^2 + x - 6 = 0$

OR $(x+3)(x-2) = 0$ SO $x = 2$, $x = -3$.

IF $x = 2$, $y = 4$.

NOW $\text{AREA} = \int_0^2 (6 - x - x^2)\,dx = 12 - \left(\frac{x^2}{2} + \frac{x^3}{3} \right)\Big|_0^2 = 12 - \left(2 + \frac{8}{3} \right) = 10 - \frac{8}{3} = \frac{22}{3}.$

$\text{AREA} = 22/3$.

NOW $\bar{x} = \frac{1}{\text{AREA}} \int_0^2 x(T(x) - B(x))\,dx = \frac{3}{22} \int_0^2 x(6 - x - x^2)\,dx = \frac{3}{22} \int_0^2 (6x - x^2 - x^3)\,dx$

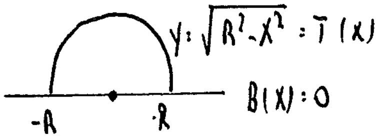
$\bar{x} = \frac{3}{22} \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]\Big|_0^2 = \frac{3}{22} \left[12 - \frac{8}{3} - 4 \right] = \frac{3}{22} \left(\frac{16}{3} \right) = \frac{8}{11} \Rightarrow \bar{x} = \frac{8}{11}.$

NOW $\bar{y} = \frac{1}{2\,\text{AREA}} \int_0^2 (T^2 - B^2)\,dx = \frac{3}{44} \int_0^2 \left((6-x)^2 - x^4 \right) dx = \frac{3}{44} \int_0^2 (36 - 12x + x^2 - x^4)\,dx$

$\bar{y} = \frac{3}{44} \left[72 - 6(4) + \frac{2^3}{3} - \frac{2^5}{5} \right] = \frac{3}{44} \left[48 + \frac{8}{3} - \frac{32}{5} \right] = \frac{3}{44} \left(\frac{664}{15} \right) = \frac{166}{55}$

SO $\bar{y} = \frac{166}{55}$ AND $\bar{x} = \frac{8}{11}$.

EXAMPLE 3 FIND CENTROID OF A SEMI-CIRCLE OF RADIUS R AS SHOWN.



$$\text{AREA} = \frac{1}{2}\pi R^2.$$

$\bar{x} = 0$ BY SYMMETRY

$$\bar{y} = \frac{1}{2\,\text{AREA}} \int_{-R}^{R} \left(T^2(x)\right) dx = \frac{2}{\pi R^2} \int_0^R \left(R^2 - x^2\right) dx = \frac{2}{\pi R^2}\left(R^3 - \frac{R^3}{3}\right)$$

SO $$\bar{y} = \frac{2}{\pi R^2}\left(\frac{2}{3}R^3\right) = \frac{4R}{3\pi}.$$

SEPARABLE DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS MODEL MANY PHYSICAL PROCESSES AND YOU WILL LEARN ABOUT THESE IN MATH 215. SEPARABLE ODE'S ARE ONES THAT HAVE THE FORM

$$(*) \quad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

(i.e. Right side is A FUNCTION OF x TIMES A FUNCTION OF y).

FOR INSTANCE, EACH OF THE FOLLOWING ARE SEPARABLE EQUATIONS:

$$\frac{dy}{dx} = xy, \qquad \frac{dy}{dx} = e^{x^2+y^2}, \qquad \frac{dy}{dx} = \frac{x^2+4}{y^2+1}$$

$\rightarrow f = x,\ g = 1/y \qquad g = e^{-y^2},\ f = e^{x^2} \qquad f = x^2+4,\ g = y^2+1$

FOR SEPARABLE ODE'S LIKE $(*)$ WE WANT TO FIND THE SOLUTION $y(x)$ THAT SATISFIES SOME CONSTRAINT $y(x_0) = y_0$, WHERE x_0, y_0 ARE GIVEN.

WE MULTIPLY BY $g(y)$ TO GET

$$g(y)\frac{dy}{dx} = f(x). \qquad (1)$$

NOW DEFINE $G(y) = \int_{y_0}^{y} g(\eta)\, d\eta$. THEN BY FTC I AND CHAIN RULE,

$$\frac{d}{dx} G[y(x)] = G'(y)\frac{dy}{dx} = g(y)\frac{dy}{dx} = f(x) \quad \text{BY } (1)$$

INTEGRATING WRT X WE GET USING $Y(X_0)=Y_0$ THAT

$$G(y) = \int_{X_0}^{X} f(\eta)\, d\eta$$

THIS GIVES $$\int_{Y_0}^{Y} g(\eta)\, d\eta = \int_{X_0}^{X} f(\eta)\, d\eta$$ AS AN IMPLICIT SOLUTION FOR $y(x)$.

MEMORY AID: WE WRITE (*) AS

$$g(y)\, dy = f(x)\, dx$$

AFTER "SEPARATING" X AND Y'S. INTEGRATING BOTH SIDES GIVES

$$\int g(y)\, dy = \int f(x)\, dx \qquad (+)$$

WHICH WILL THEN HAVE AN ARBITRARY CONSTANT "C" SO AS TO SATISFY THE CONSTRAINT $y(X_0) = Y_0$.

EXAMPLE FIND THE SOLUTION TO $\dfrac{dy}{dx} = \dfrac{y}{x^2+1}$ WITH $y(1)=1$.

SOLUTION SEPARATE VARIABLES AND USE (+) AS MEMORY AID:

$$\frac{dy}{y} = \frac{dx}{x^2+1}.$$ SINCE $y(1)=1$ AND $dy/dx > 0 \rightarrow y > 0$

INTEGRATE BOTH SIDES $\ln y = \arctan x + C$. BUT $y(1)=1$ AND USING $\ln 1 = 0$ AND $\arctan(1) = \pi/4$ GIVES $\ln y = \arctan x - \pi/4$. SO $y = e^{-\pi/4 + \arctan x}$.

EXAMPLE 2 FIND THE SOLUTION TO $\dfrac{dy}{dx} = -x(y-1)^2$ WITH $y(1)=2$.

SOLUTION SEPARATE VARIABLES TO GET

$$\frac{dy}{(y-1)^2} = -x\, dx \longrightarrow -\frac{1}{(y-1)} = -\frac{x^2}{2} + C.$$ NOW $y(1)=2$ GIVES $-1 = -\frac{1}{2} + C$

SO $C = -\frac{1}{2}$. THUS $\dfrac{-1}{(y-1)} = -\frac{1}{2}(x^2+1)$ OR $\dfrac{1}{y-1} = \dfrac{x^2+1}{2} \longrightarrow y-1 = \dfrac{2}{x^2+1}$. THUS, $y = 1 + \dfrac{2}{x^2+1}$.

EXAMPLE 3 SOLVE $\dfrac{dy}{dx} = -x e^{y}$ WITH $y(0)=0$. WE SEPARATE VARIABLES $\dfrac{dy}{e^y} = -x\, dx$.

THUS, $e^{-y}\, dy = -x\, dx$. INTEGRATE BOTH SIDES TO GET $-e^{-y} = -x^2/2 + C$.

NOW $y(0)=0 \rightarrow -1 = C$ SO $-e^{-y} = -x^2/2 - 1 \longrightarrow e^{-y} = 1 + x^2/2 \longrightarrow y = -\log(1 + x^2/2)$.