Fermi Liquids in Two Space Dimensions [1]

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§I Introduction

In this review, we consider a many-body system which is somewhat unusual in that the Fermi surface survives the turning on of all sufficiently weak short range interactions. The system lives in $d = 2$ space dimensions and consists of a gas of fermions with prescribed, strictly positive, density, together with a crystal lattice of magnetic ions. The fermions interact with each other through a two-body potential. The lattice provides periodic scalar and vector background potentials. As well, the ions oscillate, generating phonons and then the fermions interact with the phonons.

To start, turn off the fermion-fermion and fermion-phonon interactions. Then we have a gas of independent fermions, each with Hamiltonian

$$H_{\rm o} = \tfrac{1}{2m}\left(i\nabla + \mathbf{a}(\mathbf{x})\right)^2 + U(\mathbf{x})$$

The vector and scalar potentials $\mathbf{a}$, U are periodic with respect to some lattice Γ in $\mathbb{R}^2$. We use the convention that bold face characters are two component vectors. Because the Hamiltonian commutes with lattice translations it is possible to simultaneously diagonalize the Hamiltonian and the generators of lattice translations. Call the eigenvalues and eigenvectors $\varepsilon_\nu(\mathbf{k})$ and $\phi_{\nu,\mathbf{k}}(\mathbf{x})$ respectively. They obey

$$\begin{aligned} H_{\rm o}\phi_{\nu,\mathbf{k}}(\mathbf{x}) &= \varepsilon_\nu(\mathbf{k})\phi_{\nu,\mathbf{k}}(\mathbf{x}) \\ \phi_{\nu,\mathbf{k}}(\mathbf{x}+\gamma) &= e^{i<\mathbf{k},\gamma>}\phi_{\nu,\mathbf{k}}(\mathbf{x}) \qquad \text{for all } \gamma\in\Gamma \end{aligned} \tag{I.1}$$

The crystal momentum $\mathbf{k}$ runs over $\mathbb{R}^2/\Gamma^\#$ where

$$\Gamma^\# = \left\{\, b\in\mathbb{R}^2 \;\middle|\; <b,\gamma>\in 2\pi\mathbb{Z} \text{ for all } \gamma\in\Gamma \,\right\}$$

is the dual lattice to Γ. The band index $\nu\in\mathbb{N}$ just labels the eigenvalues for boundary condition $\mathbf{k}$ in increasing order.

In the grand canonical ensemble, the Hamiltonian H is replaced by $H-\mu N$ where N is the number operator and the chemical potential μ is used to control the density of the gas. At very low temperature, which is the physically interesting domain, only those pairs $\nu,\mathbf{k}$ for which $\varepsilon_\nu(\mathbf{k})\approx\mu$ are important. To keep things as simple as possible, we assume that $\varepsilon_\nu(\mathbf{k})\approx\mu$ only for one value $\nu_{\rm o}$ of ν and we put on a fixed ultraviolet cutoff so that we consider only those crystal momenta for which $|\varepsilon_{\nu_o}(\mathbf{k}) - \mu|$ is smaller than some fixed small constant.

Precisely, we denote $e(\mathbf{k}) = \varepsilon_{\nu_o}(\mathbf{k}) - \mu$ and make the following assumptions.

Hypothesis I: *The dispersion relation* $e(\mathbf{k})$ *is a real-valued, real analytic function on a compact subset* B *of* $\mathbb{R}^d$*. For all points* $\mathbf{p} \in \mathrm{B}$*,*

$$\nabla e(\mathbf{p}) \ \neq \ 0$$

Hypothesis II: *The Fermi curve*

$$\mathrm{F} \ = \ \left\{ \ \mathbf{p} \in \mathrm{B} \ \middle| \ e(\mathbf{p}) = 0 \ \right\}$$

for e *is a simple closed curve, whose curvature is bounded away from zero.*

Hypothesis III: *For all* $\mathbf{q} \in \mathbb{R}^d$*,*

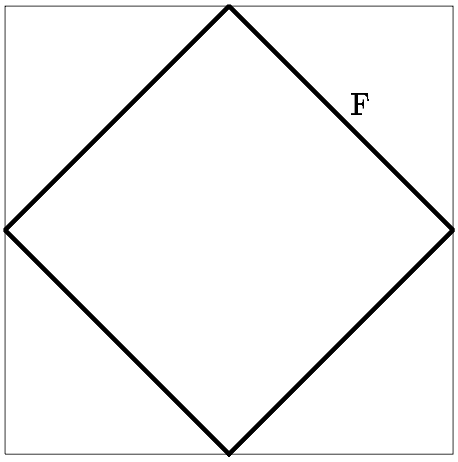
$$-\mathrm{F} + \mathbf{q} \neq \mathrm{F}$$

By definition,

$$-\mathrm{F} + \mathbf{q} \ = \ \left\{ \ \mathbf{p} \in \mathbb{R}^2 \ \middle| \ -\mathbf{p} + \mathbf{q} \in \mathrm{B} \text{ and } e(-\mathbf{p} + \mathbf{q}) = 0 \ \right\}$$

It is Hypothesis III that makes this class of models somewhat unusual and permits the system to remain a Fermi liquid when the interaction is turned on. If $\mathbf{a} = 0$ then, taking the complex conjugate of (I.1), we see that $\varepsilon_\nu(-\mathbf{k}) = \varepsilon_\nu(\mathbf{k})$ so that Hypothesis III is violated for $\mathbf{q} = 0$. Hence the presence of a nonzero vector potential is essential.

In order to have simple sounding hypotheses, we have made them much stronger than necessary. One model that violates these hypotheses, not only for technical reasons but because it exhibits different physics, is the Hubbard model at half filling. Its Fermi surface looks like



This Fermi curve is not smooth, violating Hypothesis I, has zero curvature almost everywhere, violating Hypothesis II and is reflection invariant so that F $=-$F, violating Hypothesis III with $\mathbf{q}=0$.

The interacting models are formally characterized by the Euclidean Green's functions

$$\left\langle \prod_{i=1}^{n} \psi_{p_i} \bar\psi_{q_i} \right\rangle = \frac{\int \left(\prod_{i=1}^{n} \psi_{p_i} \bar\psi_{q_i}\right) e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}} \tag{I.2a}$$

The action

$$\mathcal{A}(\psi,\bar\psi) \;=\; -\int đk\ \big(ik_0 - e(\mathbf{k})\big)\bar\psi_k\psi_k \;-\; \int đk\, \mathcal{E}(\lambda,\mathbf{k})\bar\psi_k\psi_k \;-\; \mathcal{V}(\psi,\bar\psi) \tag{I.2b}$$

We now take some time to explain this formula. The fermion fields are vectors

$$\psi_k = \begin{pmatrix} \psi_{k,\uparrow} \\ \psi_{k,\downarrow} \end{pmatrix} \qquad \bar\psi_k = (\,\bar\psi_{k,\uparrow} \quad \bar\psi_{k,\downarrow}\,)$$

whose components $\psi_{k,\sigma}, \bar\psi_{k,\sigma},\ k=(k_0,\mathbf{k}) \in \mathcal{B} = (-1,1)\times \mathrm{B},\ \sigma \in \{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other.

$$\overset{(-)}{\psi}_{k,\sigma}\overset{(-)}{\psi}_{p,\tau} = -\overset{(-)}{\psi}_{p,\tau}\overset{(-)}{\psi}_{k,\sigma}$$

We have deliberately chosen $\bar\psi$ to be a row vector and ψ to be a column vector so that

$$\bar\psi_k\psi_p = \bar\psi_{k,\uparrow}\psi_{p,\uparrow} + \bar\psi_{k,\downarrow}\psi_{p,\downarrow} \qquad \psi_k\bar\psi_p = \begin{pmatrix} \psi_{k,\uparrow}\bar\psi_{p,\uparrow} & \psi_{k,\uparrow}\bar\psi_{p,\downarrow} \\ \psi_{k,\downarrow}\bar\psi_{p,\uparrow} & \psi_{k,\downarrow}\bar\psi_{p,\downarrow} \end{pmatrix}$$

In the argument $k = (k_0, \mathbf{k})$, the last d components $\mathbf{k}$ are to be thought of as a crystal momentum and the first component k_0 as the dual variable to an imaginary time. Hence the $\sqrt{-1}$ in $ik_0 - e(\mathbf{k})$. For convenience only, we have put an ultraviolet cutoff on k_0 as well as on $\mathbf{k}$. In the full model k_0 runs over $\mathbb{R}$ and $\mathbf{k}$ is replaced by $(\nu, \mathbf{k})$ with ν summed over $\mathbb{N}$ and $\mathbf{k}$ integrated over $\mathbb{R}^d/\Gamma^\#$. The relationship between the position space field $\psi(\xi)$, with $\xi = (t, \mathbf{x})$ running over (imaginary)time$\times$space, and the momentum space field ψ_k is given, in our single band approximation, by

$$\begin{aligned} \psi_k &= \int d\xi \; e^{-ik_0 t} \phi_{\nu_o,\mathbf{k}}(\mathbf{x}) \psi(\xi) \\ \psi(\xi) &= \int đk \; e^{ik_0 t} \overline{\phi_{\nu_o,\mathbf{k}}(\mathbf{x})} \psi_k \end{aligned} \tag{I.3}$$

where

$$đk = \frac{dk_0}{2\pi} \; đ\mathbf{k} = \frac{d^{d+1}k}{(2\pi)^{d+1}}$$

The general spin independent form of the interaction is

$$\mathcal{V}(\psi, \bar\psi) = \tfrac{\lambda}{2} \int \prod_{i=1}^{4} đk_i \; (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \; \bar\psi_{k_1} \psi_{k_3} \left\langle k_1, k_2 \middle| \mathrm{V} \middle| k_3, k_4 \right\rangle \bar\psi_{k_2} \psi_{k_4} \tag{I.4}$$

Spin independence is imposed purely for notational convenience. It plays no role. The delta function δ is that for $\mathbb{R}^d/\Gamma^\#$ and imposes the appropriate conservation of crystal momentum for the present setting. The function $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle$ implements the fermion-fermion and fermion-phonon interaction. Its precise value does not concern us. We just assume

Hypothesis IV *The interaction is short range. That is $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle \in C^\infty$.*

The net coefficient $e(\mathbf{k}) - \mathcal{E}(\lambda, \mathbf{k})$ of $\bar\psi_k \psi_k$ in $\mathcal{A}$ has been deliberately split into two parts, with $\mathcal{E}(\lambda, \mathbf{k})$ chosen to satisfy an explicit renormalization condition. This is called renormalization of the dispersion relation. It is done to ensure that $\left\langle \prod_{i=1}^n \psi_{p_i} \bar\psi_{q_i} \right\rangle$ is C^∞ in λ at $\lambda = 0$. Define the proper self energy $\Sigma(p)$ for the action $\mathcal{A}$ by the equation

$$\Big(ip_0 - e(\mathbf{p}) - \Sigma(p) \Big)^{-1} (2\pi)^{d+1} \delta(p - q) = \frac{\int \psi_p \bar\psi_q \; e^{\mathcal{A}(\psi)} \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}{\int \; e^{\mathcal{A}(\psi)} \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}$$

The counterterm $\mathcal{E}(\lambda, \mathbf{k})$ is chosen so that

$$\Sigma\big(0, \mathbf{p}\big)\Big|_{\mathbf{p} \in \mathbf{F}} = 0$$

To give a rigorous definition of (I.2) one must introduce cutoffs and then take the limit in which the cutoffs are removed. To impose an infrared cutoff in the spatial directions one may put the system in a finite periodic box $\mathbb{R}^d/L\Gamma$. To impose an infrared cutoff in the zero direction one may make the inverse temperature $\beta < \infty$. Then momenta $k = (k_0, \mathbf{k})$ are restricted to lie on the lattice

$$\begin{aligned} k_0 &\in \tfrac{\pi}{\beta}\big(2\mathbb{Z}+1\big) \\ \mathbf{k} &\in \tfrac{1}{L}\Gamma^{\#} \end{aligned}$$

The ultraviolet cutoffs further restrict $|k_0| \le 1,\ |e(\mathbf{k})| \le 1$. Then the Grassmann algebra becomes finite dimensional and (I.2b) with the integral symbol reinterpreted as

$$\int đk\ f(k) \;=\; \frac{1}{\beta} \sum_{\substack{k_0 \in \frac{\pi}{\beta}\left(2\mathbb{Z}+1\right) \\ |k_0| \le 1}} \frac{1}{L^2} \sum_{\substack{\mathbf{k} \in \frac{1}{L}\Gamma^{\#} \\ |e(\mathbf{k})| \le 1}} f(k)$$

is a well-defined element of that algebra.

Theorem. *Let $d = 2$ and Hypotheses I-IV be satisfied. There is an $r > 0$ and a dispersion relation counterterm $\mathcal{E}(\lambda, \mathbf{k})$, such that the limits*

$$\lim_{\beta, L \to \infty} \frac{\int \prod_{i=1}^n \psi_{p_i} \bar\psi_{q_i}\ e^{\mathcal{A}(\psi,\bar\psi)} \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}{\int\ e^{\mathcal{A}(\psi,\bar\psi)} \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}} \tag{I.5}$$

exist in the sense of distributions and are independent of the order in which the limits are taken. The counterterm and the limit are both analytic functions of the coupling constant λ for $|\lambda| < r$. Furthermore, there is a jump in the average occupation number $n_{\mathbf{k}}$ at the Fermi curve. Precisely, if

$$n_{\mathbf{k}} \;=\; \lim_{x_0 \searrow 0} \int đk_0\, e^{ik_0 x_0} \Big(ik_0 - e(\mathbf{k}) - \Sigma(k_0, \mathbf{k})\Big)^{-1}$$

then

$$\begin{aligned} \lim_{\varepsilon \searrow 0} n_{\mathbf{p}-\varepsilon\nu_{\mathbf{p}}} - n_{\mathbf{p}+\varepsilon\nu_{\mathbf{p}}} &= \Big(1 + i\tfrac{\partial}{\partial k_0}\Sigma(0,\mathbf{p})\Big)^{-1} \\ &\ge 1 - O(\lambda) \end{aligned}$$

for all $\mathbf{p}$ on the Fermi curve $\mathbf{F}$. Here, $\nu_{\mathbf{p}}$ is the outward pointing unit normal to $\mathbf{F}$ at $\mathbf{p}$. In other words, the infinite volume system is a Fermi liquid.

Our main goal here is to explain why this Theorem is true, though the complete proof [FKLT1] is too long to include. There are two main aspects to that proof: the control of four legged Feynman diagrams and the control of high orders of perturbation theory. The first aspect is discussed in §III while the second is discussed in §II.

§II Analyticity of Greens Functions

In this section we give an outline of the main ideas which are necessary for controlling large orders of perturbation theory. Roughly speaking, what we want to show is the following:

Theorem. *"The sum of all graphs that contribute to (I.5) and do not contain nontrivial four legged subgraphs is analytic."*

This theorem is true no matter whether you have $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$ where the Fermi curve is a circle or a dispersion relation $e(\mathbf{k})$ obeying the hypothesises of the first section, where the Fermi curve is not perfectly round. In particular, Theorem II.1 below is also the starting point for a rigorous construction of the theory of BCS-superconductivity. It means that the physicial behaviour of the model is completely determined by the four legged subgraphs. In the case of a dispersion relation of §I, four legged graphs are summable making the whole Greens functions analytic, whereas $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$ produces logarithmic divergences which drive the renormalization group flow to a superconducting fixed point.

To prove the above theorem, one has to do two things. First, one has to control the magnitude of each graph. When four legged subgraphs are removed, this can be done by power counting and renormalization of two legged subgraphs. Second, one has to control the number of graphs, since, if one expands the exponential in (I.5), there is a $\frac{1}{n!}$ from that expansion but one gets $(2n)! \approx \mathrm{const}^{\,n} n!^2$ Feynman graphs after the evaluation of the functional integral with $4n$ fields. Thus, one is left with $\sum_{n=0}^{\infty} \mathrm{const}^{\,n} n!\, \lambda^n$ which has radius of convergence zero. That is, we are not allowed to expand completely to Feynman graphs. Instead of this, one has to use the antisymmetry of the fermionic integral to exploit some cancellations of the $(2n)!$ graphs.

The strategy will be to decompose the propagator into scales and integrate out one scale at a time, removing four legged kernels by hand after each integration. Then, after renormalization of the two legged kernels, the result has summable power counting. To control the number of graphs, one Taylor expands the fields at each scale to sufficiently large order and uses the fact that the functional integral vanishes if two fields are equal.

Let us first give a precise mathematical formulation of what we have proven. Since we want to give an inductive proof which uses only a single scale expansion, instead of considering (I.5) we start with the generating functional for the connected amputated Greens functions given by

$$\mathcal{G}(\phi,\bar\phi) = \log \frac{1}{Z}\int e^{-\mathcal{V}(\psi+\phi,\bar\psi+\bar\phi)}\, d\mu_C(\psi,\bar\psi)\,,$$

This is convenient for an inductive proof. Here $d\mu_C$ is the Grassmann Gaussian measure with covariance

$$C(x,x') = C(x-x') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-ik_0(\tau-\tau')}\frac{1}{ik_0-e(\mathbf{k})}\,,$$

where $e(\mathbf{k})$ is either the dispersion relation of §I or, in this section, $\frac{\mathbf{k}^2}{2m}-\mu$. The coordinate and momentum space variables are

$$x=(x_0,\mathbf{x})=(x_0,x_1,x_2)\in\mathbb{R}^3\,,\quad k=(k_0,\mathbf{k})=(k_0,k_1,k_2)\in\mathbb{R}^3\,.$$

For simplicity, we do everything in infinite volume and at zero temperature. A more careful treatment, which starts at positive temperature and in finite volume will be given in [FKLT1].

First let us introduce scales $j=0,-1,-2,\cdots$ which select shells of thickness M^j around the Fermi curve $k_0=0,\ e(\mathbf{k})=0$ which is the singular locus of $C(k)$.

$$C(k)=\frac{1}{ik_0-e(\mathbf{k})}=\sum_{j=-\infty}^{0}\frac{f_j(k)}{ik_0-e(\mathbf{k})}+\frac{h(k)}{ik_0-e(\mathbf{k})}=\sum_{j=-\infty}^{0}C^j(k)+C^{>0}(k)\,,$$

where the f_j are smooth functions with support on $\{M^j\le|ik_0-e(\mathbf{k})|\le M^2M^j\}$ and the ultraviolet part h has support on $\{1\le|ik_0-e(\mathbf{k})|\}$. Here, $M\ge 2$ is a constant which eventually has to be chosen sufficiently large.

We consider only the infrared part of the model and introduce an infrared cuttoff at scale $r>-\infty$. Furthermore, to renormalize two legged kernels, we introduce a counterterm

$$\delta e_r(\mathbf{k},\lambda)=\sum_{j=r}^{-1}\delta e_r^j(\mathbf{k},\lambda)=\sum_{j=r}^{-1}\sum_{l=1}^{\infty}\delta e_{r,l}^j(\mathbf{k})\,\lambda^l\,,$$

$$\delta\mathcal{V}_r(\psi)=\int\frac{d^3k}{(2\pi)^3}\delta e_r(\mathbf{k},\lambda)\,\bar\psi(k)\psi(k)$$

which is an analytic function of λ and will be determined below. If the Fermi curve is a circle, δe_r is independent of $\mathbf{k}$, but in general, it will depend on $\mathbf{k}$. So we consider

$$\mathcal{G}^r(\phi) = \log \frac{1}{Z_{r+1}} \int e^{(-\mathcal{V}+\delta\mathcal{V}_r)(\sum_{j=r+1}^{0} \psi^j+\phi)} \prod_{j=r+1}^{0} d\mu_{C^j}(\psi^j)\,.$$

Then $\mathcal{G}^r$ can be computed inductively $(r+1 \le j \le 0)$

$$\mathcal{G}^0 = -\mathcal{V} + \delta\mathcal{V}_r\,, \qquad \mathcal{G}^{j-1}(\psi^{\le j-1} + \phi) = \log \frac{1}{\tilde{Z}_j} \int e^{\mathcal{G}^j(\psi^{\le j}+\phi)} d\mu_{C^j}(\psi^j)$$

where

$$\tilde{Z}_j = \int e^{\mathcal{G}^j(\psi^j)} d\mu_{C^j}(\psi^j)$$

The $2q$-point functions G_q^{j-1} at scale $j-1$ are defined by

$$\begin{aligned}\mathcal{G}^{j-1}(\phi) &= \sum_{q=1}^{\infty} \mathcal{G}_q^{j-1}(\phi)\\ &= \sum_{q=1}^{\infty} \int d\xi_1 \cdots d\xi_{2q}\, G_q^{j-1}(\xi_1, \cdots, \xi_{2q})\, \phi(\xi_1) \cdots \phi(\xi_{2q})\end{aligned}$$

where $G_q^{j-1}(\xi_1, \cdots, \xi_{2q})$ is some antisymmetric kernel and $\xi = (x, \sigma, b) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\} \times \{0, 1\}$,

$$\phi(\xi) = \begin{cases} \phi(x,\sigma) & \text{if } b = 0 \\ \bar{\phi}(x,\sigma) & \text{if } b = 1 \end{cases} \qquad \text{and} \qquad \int d\xi = \sum_{b\in\{0,1\}} \sum_{\sigma\in\{\uparrow,\downarrow\}} \int d^3x\,.$$

Define an operator Q_4 which projects out nontrivial four legged subgraphs, that is four legged subgraphs of order at least $O(\lambda^2)$, by

$$Q_4 \sum_{q=1}^{\infty} \mathcal{G}_q^{j-1}(\phi) = \sum_{\substack{q=1\\ q\neq 2}}^{\infty} \mathcal{G}_q^{j-1}(\phi) + \lambda \tfrac{d}{d\lambda} \mathcal{G}_2^{j-1}(\phi)\big|_{\lambda=0}$$

Theorem II.1 *Define the effective potential without four legged subgraphs $\mathcal{W}^j$ inductively by $\mathcal{W}^0 = -\mathcal{V} + \delta\mathcal{V}_r$ and for $r+1 \le j \le 0$*

$$\mathcal{W}^{j-1}(\psi^{\le j-1} + \phi) = Q_4 \log \frac{1}{Y_j} \int e^{\mathcal{W}^j(\psi^{\le j}+\phi)} d\mu_{C^j}(\psi^j)\,,$$

$$Y_j = \int e^{\mathcal{W}^j(\psi^j)} d\mu_{C^j}(\psi^j)\,.$$

Then there is an $\varepsilon > 0$ which is independent of the infrared cuttoff r and a function $\delta e_r(\mathbf{k}, \lambda)$ which is analytic in λ for $|\lambda| < \varepsilon$ such that $\mathcal{W}^r$ is analytic for $|\lambda| < \varepsilon$. Furthermore, for all test functions $f_1, \cdots, f_{2q}$,

$$\int \prod_{i=1}^{2q} d\xi_i \, |f_i(\xi_i)| \, |W_q^r(\xi_1, \cdots, \xi_{2q})| \, \leq \, |\lambda|^{\frac{q}{2}} \prod_{i=1}^{2q} \Big(\|f_i\|_{L^1} + \|f_i\|_{L^\infty} \Big) .$$

Remark. $\mathcal{W}^r$ is not really the sum of all graphs (with propagators $C^{\geq r}$) without four legged subgraphs. Rather, $\mathcal{W}^r$ may be expressed as a sum of labelled graphs, with each line of each graph labelled by a fixed scale. Then only those four legged subgraphs for which the maximal scale of the external legs is strictly less than the minimal scale of the internal lines are forbidden.

In the rest of this section, we sketch the proof of Theorem II.1, stating without proof the main Lemmata. Details will be given in [FKLT1]. Also see [FMRT1].

Let us first take a look at the power counting of the graphs. In coordinate space, we use the following norms. Fix test functions $f_k \in L^1 \cap L^\infty$, $1 \leq k \leq 2q$. Let $G_q = G_q(\xi_1, \cdots, \xi_{2q})$ be some kernel. Then

$$|\!|\!| G_q |\!|\!|_\emptyset = \sup_i \sup_{\xi_i} \int \prod_{\substack{k=1 \\ k \neq i}}^{2q} d\xi_k \, |G_q(\xi_1, \cdots, \xi_{2q})|$$

and for $S \subset \{1, \cdots, 2q\}$, $S \neq \emptyset$

$$|\!|\!| G_q |\!|\!|_S = \int \prod_{i=1}^{2q} d\xi_i \prod_{k \in S} |f_k(\xi_k)| \, |G_q(\xi_1, \cdots, \xi_{2q})| \, .$$

Lemma II.2 (Power Counting) *Let G_q be a connected amputated graph with $2q$ external legs built up from generalized, $2q_v$ legged vertices or subgraphs I_{q_v} with $|\!|\!| I_{q_v} |\!|\!|_\emptyset < \infty$. Suppose each line of the graph has a covariance C^j with*

$$\|C^j\|_{L^\infty} \leq c\, M^j \, , \qquad \qquad \|C^j\|_{L^1} \leq c\, M^{-j} \, .$$

Then there are the following bounds

$$\|G_q\|_\emptyset \le c^{\sum_v q_v - q} \prod_{v\in V} \left(\|I_{q_v}\|_\emptyset M^{(q_v-2)j}\right) M^{-(q-2)j},$$

$$\begin{aligned}\|G_q\|_S \le c^{\sum_v q_v - q} &\prod_{v\in V_{int}} \left(\|I_{q_v}\|_\emptyset M^{(q_v-2)j}\right)\times \\ &\prod_{v\in V_{ext}} \left(\|I_{q_v}\|_{S_v} M^{\frac{1}{2}(2q_v-|S_v|)j}\right) M^{-\frac{1}{2}(2q-|S|)j}.\end{aligned}$$

Here a vertex is called external, if at least one of its legs is integrated against a test function.

There is an analogous bound in momentum space. Then the L^∞ and L^1 norms reverse roles. In momentum space, one easily verifies that $C^j(k) = \frac{f^{(j)}(k)}{ik_0 - e(\mathbf{k})}$ obeys $|C^j(k)| \le c\,M^{-j}$ and $\int d^3k\,|C^j(k)| \le c\,M^j$, so that one obtains the above bound. In coordinate space, one has to work harder (see [FT1], lemma V.2,3).

The reason we have formulated the above lemma in coordinate space is, that the whole expansion for the generating functional is done in coordinate space since the fields have to be Taylor expanded in coordinate space. Then, to get small factors from the Taylor expansion, the covariance has to be decomposed further into pieces $C^{j,\ell}$ and the power counting lemma will be applied to graphs, or more precisely, to kernels which are sums of graphs where each line has covariance $C^{j,\ell}$. The estimates are done in coordinate space and one effectively obtains the above bound.

If one iteratively applies Lemma II.1 for all scales one gets a similar bound with $M^{(q_v-2)j}$ replaced by $M^{(q_v-2)(j-i_v)}$ where $i_v > j$ is the scale of the generalized vertex v. This scale must be summed over, which yields

$$\sum_{i_v=j+1}^{0} M^{(q_v-2)(j-i_v)} \sim \begin{cases} M^{-(q_v-2)} \le M^{-\frac{1}{3}q_v} \le 1 & \text{if } q_v \ge 3 \\ |j| & \text{if } q_v = 2 \\ M^{|j|} & \text{if } q_v = 1 \end{cases}$$

Vertices with at least six external legs are summable. In fact, they produce a small factor $M^{-\frac{1}{3}q_v}$ which can be used to control the q_v-sums coming with each vertex. Four legged vertices give a $|j| = |\log M^j| = |\log e(\mathbf{k})|$. In the next section it is shown that this logarithm is really there if the Fermi curve is a circle, but it is absent if the dispersion relation $e(\mathbf{k})$

satisfies hypotheses I-III of section I. Finally, a two legged vertex gives the exploding factor $M^{|j|}$, but this factor can be eliminated by renormalization of the two legged kernels.

As we mentioned earlier, we will give an inductive proof which uses only a single scale expansion. We will not write down the scale sums explicitly. They are replaced by a suitable induction hypothesis on V_q^{j-1}, see (II.16,17) below. We hope that this makes the proof clearer.

When renormalization is performed, the scale j can no longer be used as the induction index, because the definition of the counterterm at scale j involves the sum of all scales below j. In that case all scales are treated simultaneously and the induction is on "iteration step", corresponding to the depth of the tree in the Gallavotti Nicolo tree expansion. This was also the method used in [FT1,2]. The corresponding formalism is presented in Lemma II.5 below.

We start with an easier case, in which we remove both two and nontrivial four legged subgraphs. That is, we replace the Q_4 in Theorem II.1 with $Q_{2,4}$ defined by

$$Q_{2,4}\sum_{q=1}^{\infty}\mathcal{G}_q^{j-1}(\phi)=\lambda\tfrac{d}{d\lambda}\mathcal{G}_2^{j-1}(\phi)\big|_{\lambda=0}+\sum_{q=3}^{\infty}\mathcal{G}_q^{j-1}(\phi)\,.$$

Then one does not have to renormalize and the induction is on scales. We want to show that $\mathcal{W}^{j-1}$ defined inductively by $\mathcal{W}^0=-\lambda\mathcal{V}$ and

$$\mathcal{W}^{j-1}(\psi^{\le j-1}+\phi)=Q_{2,4}\log\frac{1}{Y_j}\int e^{\mathcal{W}^j(\psi^{\le j}+\phi)}d\mu_{C^j}(\psi^j)\,,\qquad Y_j=\int e^{\mathcal{W}^j(\psi^j)}d\mu_{C^j}(\psi^j)$$

is analytic for all sufficiently small λ, independent of $j>-\infty$.

First write

$$\mathcal{W}^{j-1}=\sum_{i=j-1}^{0}\mathcal{V}^i$$

where

$$\mathcal{V}^0(\psi^{\le j-1}+\phi)=-\mathcal{V}(\psi^{\le j-1}+\phi)$$

and for $j\le 0$

$$\begin{aligned}\mathcal{V}^{j-1}(\psi^{\le j-1}+\phi)&=Q_{2,4}\log\frac{1}{Y_j}\int e^{\mathcal{W}^j(\psi^{\le j}+\phi)-\mathcal{W}^j(\psi^{\le j-1}+\phi)}d\mu_{C^j}(\psi^j)\\&=Q_{2,4}\left\{\log\frac{1}{Y_j}\int e^{\mathcal{W}^j(\psi^{\le j}+\phi)}d\mu_{C^j}(\psi^j)-\mathcal{W}^j(\psi^{\le j-1}+\phi)\right\}\end{aligned}$$

Since $\mathcal{W}^j(\psi^{\le j-1}+\phi)$ is subtracted in the exponential, $\mathcal{V}^{j-1}$ must contain at least one contraction at scale $j-1$. This is not the case for $\mathcal{W}^{j-1}$. From a technical point of view, the $\mathcal{V}^{j-1}$'s are the basic objects. In particular, the induction hypothesis is stated in terms of them.

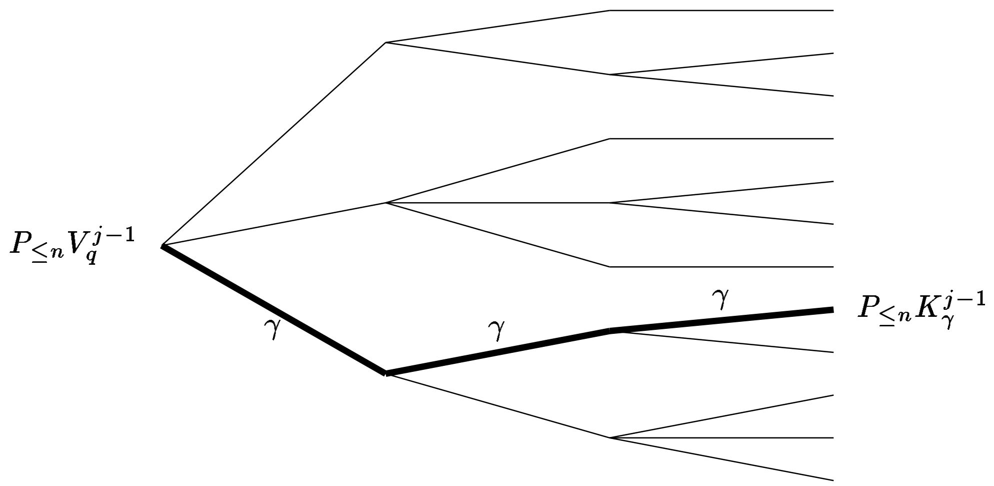
Let $P_{\le n}$ be the operator which projects onto contributions up to $n^{\rm th}$ order in λ. That is $P_{\le n}\sum_{k=0}^{\infty} a_k\lambda^k = \sum_{k=0}^{n} a_k\lambda^k$ and let V_q^{j-1}, $j\le 0$, be given by

$$\mathcal{V}^{j-1}(\phi) = \sum_{q=3}^{\infty} \mathcal{V}_q^{j-1}(\phi) = \sum_{q=3}^{\infty}\int d\xi_1\cdots d\xi_{2q}\, V_q^{j-1}(\xi_1,\cdots,\xi_{2q})\,\phi(\xi_1)\cdots\phi(\xi_{2q})\,.$$

There will be an expansion

$$P_{\le n}V_q^{j-1}(\eta_1,\cdots,\eta_{2q}) = \sum_{\gamma\in\mathcal{P}_{n,q}} P_{\le n}K_\gamma^{j-1}(W_{q_v}^j)(\eta_1,\cdots,\eta_{2q})$$

where the sum is over all paths γ joining the root to some other extremal vertex of a tree $\mathcal{P}_{n,q}$, called the expansion or partial integration tree. This tree is **not** the tree of the Gallavotti-Nicolo Tree Expansion. The expansion tree contains a description of the entire expansion. Each fork corresponds to a substep of the expansion process and each branch leaving the fork corresponds to a possible outcome for that substep.



To give a bound on $P_{\le n}V_q^{j-1}$ one has to bound the kernels $K_\gamma^{j-1}(W_{q_v}^j)$ and to control the sum $\sum_{\gamma\in\mathcal{P}_{n,q}}$. The sum is controlled by

Lemma II.3 (Combinatorial Tree Lemma) *Let $\mathcal{T}$ be a tree, $w_\ell > 0$ a weight factor* *assigned to the line ℓ and K_γ a real number assigned to the end of the path $\gamma \in \mathcal{T}$. Then*

$$\sum_{\gamma\in\mathcal{T}} |K_\gamma| \le \sup_{\gamma\in\mathcal{T}} \left\{ \prod_{f\in\gamma} b(f) \prod_{\ell\in\gamma} w_\ell \, |K_\gamma| \right\}$$

where $b(f) = \sum\limits_{\ell\in f} \frac{1}{w_\ell}$ is a generalized branching number for the fork f.

In the case of $\mathcal{P}_{n,q}$ we can choose the w_ℓ's so that

$$\prod_{f\in\gamma} b(f) \prod_{\ell\in\gamma} w_\ell \le \mathrm{const}^{\sum_{v\in V(\gamma)} q_v}$$

where $\sum_{v\in V(\gamma)} q_v$ is half the number of legs of the vertices of which the kernel K_γ is made of. Since power counting gives a factor $M^{-\frac{1}{3}q_v}$ coming with each vertex, the above factor can be controlled by choosing M sufficiently large. This is a general rule for the expansion: you can allow all kinds of sums as long as the branching numbers or generalized branching numbers times the weight factors is bounded by $\mathrm{const}^{\sum_v q_v}$. Factorials like $\prod_v q_v!$ are not allowed. Weight factors are necessary when there are infinite sums. In our model, this will only be the case for the q_v-sums. One may take 2^{q_v} as a weight factor.

Before we write down what the kernels are, let us first explain how the expansion is generated. We start with the fundamental theorem of calculus $f(1) = f(0) + \int_0^1 d\varepsilon \frac{d}{d\varepsilon} f(\varepsilon)$:

$$\mathcal{V}^{j-1}(\psi^{\le j-1}) = Q_{2,4} \log \frac{1}{Y_j} \int e^{\mathcal{W}^j(\psi^j+\varepsilon\psi^{\le j-1}) - \mathcal{W}^j(\varepsilon\psi^{\le j-1})} d\mu_{C^j}(\psi^j) \Big|_{\varepsilon=0}^{\varepsilon=1}$$

$$= Q_{2,4} \sum_{q=3}^{\infty} \sum_{\substack{J\subset\{1,\cdots,2q\} \\ J\ne\emptyset}} \int \prod_{i=1}^{2q} d\xi_i \, W_q^j(\underline{\xi}) \prod_{k\in J^c} \psi^{\le j-1}(\xi_k) \times$$

$$\int_0^1 d\varepsilon \, \frac{d}{d\varepsilon} \left\{ \varepsilon^{|J^c|} \right\} \frac{\int \prod_{i\in J} \psi^j(\xi_i) \, e^{\mathcal{W}^j(\psi^j+\varepsilon\psi^{\le j-1}) - \mathcal{W}^j(\varepsilon\psi^{\le j-1})} d\mu_{C^j}(\psi^j)}{\int e^{\mathcal{W}^j(\psi^j+\varepsilon\psi^{\le j-1}) - \mathcal{W}^j(\varepsilon\psi^{\le j-1})} d\mu_{C^j}(\psi^j)} . \qquad \text{(II.1)}$$

The J-sum comes from multplying out the $2q$ brackets $\left(\psi^j(\xi_i) + \varepsilon\psi^{\le j-1}(\xi_i)\right)$. The condition $J \ne \emptyset$ ensures that there is at least one contraction. Together with the q-sum, they yield the first branching of the expansion tree.

Now we do integration by parts. That is, we eliminate all the fields $\prod_{i\in J}\psi^j(\xi_i)$ in the numerator of (II.1) by repeatedly applying the formula

$$\int \psi(\xi)\, F(\psi)\, d\mu_C = \int d\eta\, C(\xi,\eta) \int \frac{\delta F(\psi)}{\delta\psi(\eta)}\, d\mu_C \tag{II.2}$$

where, if $\xi=(x,\sigma,b)$ and $\eta=(y,\tau,c)$,

$$C(\xi,\eta)=\delta_{b,1-c}\delta_{\sigma,\tau}\begin{cases} C(x,y) & \text{if } b=0 \\ -C(y,x) & \text{if } b=1 \end{cases} \quad \text{and} \quad \int d\eta = \sum_{c\in\{0,1\}}\sum_{\tau\in\{\uparrow,\downarrow\}}\int d^3y\,.$$

The result is formalized in the following general

Lemma II.4 (Integration by Parts) *Let C be some covariance and*

$$\mathcal{W}(\psi)=\sum_{q=1}^{\infty}\int d\eta_1\cdots d\eta_{2q}\, W_q(\eta_1,\cdots,\eta_{2q})\,\psi(\eta_1)\cdots\psi(\eta_{2q})$$

Then one has

$$\int \psi(\xi_1)\psi(\xi_2)\cdots\psi(\xi_p)\, e^{\mathcal{W}(\psi+\phi)} d\mu_C(\psi) =$$

$$\sum_{A\subset\{1,\cdots,p\}}\ \sum_{\substack{q_i=1\\ \text{if } i\in A}}^{\infty}\ \sum_{\substack{J_i\subset\{1,\cdots,2q_i\}\\ \text{if } i\in A}}\ \sum_{\substack{l_i\in J_i\\ \text{if } i\in A}}\ \sum_{U\subset W(A,\underline{J},\underline{l})} \operatorname{sign}(\underline{J},\underline{l},U)\int\prod_{i\in A}\left\{\prod_{k=1}^{2q_i} d\eta_k^i\, W_{q_i}(\eta_1^i,\cdots,\eta_{2q_i}^i)\times\right.$$

$$\left. C(\xi_i,\eta_{l_i}^i)\prod_{k\in J_i^c}\phi(\eta_k^i)\right\}\int\prod_{i=1}^{p}\chi_{_{A,U,i}}(\psi)\, d\mu'_C(\psi',\psi)\int\prod_{i\in A}\prod_{(i,k)\in U^c}\psi(\eta_k^i)\, e^{\mathcal{W}(\psi+\phi)}d\mu_C(\psi)$$

where

$$\chi_{_{A,U,i}}(\psi)=\begin{cases}\psi'(\xi_i) & \text{if } i\notin A\\ \prod_{(i,k)\in U}\psi(\eta_k^i) & \text{if } i\in A\,,\end{cases}$$

and

$$\int\prod_{i=1}^{p}\chi_{_{A,U,i}}(\psi)\, d\mu'_C(\psi',\psi)=\det\left[\begin{array}{c}\text{same matrix as for the}\\ \text{functional integral without primes,}\\ \text{but with } C(\eta_l^i,\eta_m^l) \text{ replaced by } 0\\ \text{and } C(\xi_i,\eta_k^j) \text{ replaced by } 0 \text{ if } i<j\end{array}\right].$$

Furthermore, if the tree $\mathcal{P}$ is defined by

$$\sum_{\gamma\in\mathcal{P}}=\sum_{A\subset\{1,\cdots,p\}}\ \sum_{\substack{q_i=1\\ if\ i\in A}}^{\infty}\ \sum_{\substack{J_i\subset\{1,\cdots,2q_i\}\\ if\ i\in A}}\ \sum_{\substack{l_i\in J_i\\ if\ i\in A}}\ \sum_{U\subset W(A,\underline{J},\underline{l})}$$

with weight factors $w_{q_i} = 2^{q_i}$ *for the* q_i*-sums and all other weights being one, then*

$$\prod_{f\in\gamma} b(f) \prod_{\ell\in\gamma} w_\ell \le \text{const}^{\,p+\sum_{v\in V(\gamma)} q_v}$$

for all $\gamma \in \mathcal{P}$. *Finally,* $W(A,\underline{J},\underline{l}) = \cup_{i\in A} \cup_{k\in J_i\setminus\{l_i\}} \{(i,k)\}$, $J_i^c = \{1,\cdots,2q_i\} \setminus J_i$, $U^c = W(A,\underline{J},\underline{l}) \setminus U$ *and* $\text{sign}(\underline{J},\underline{l},U) \in \{1,-1\}$.

Let us briefly explain how the five sums in the lemma arise. The first sum tells you which fields of the p 'downstairs' fields $\psi(\xi_1),\cdots,\psi(\xi_p)$ differentiate the exponential when you apply (II.2). Each time one hits the exponential, say with the field $\psi(\xi_i)$, the sum

$$\sum_{q_i=1}^{\infty} \int d\eta_1^i \cdots d\eta_{2q_i}^i \, W_{q_i}(\eta_1^i,\cdots,\eta_{2q_i}^i)\,(\phi+\psi)(\eta_1^i)\cdots(\phi+\psi)(\eta_{2q_i}^i) \tag{II.3}$$

is brought down from the exponential. This explains the second sums in the lemma. The third sums come from multiplying out the $(\phi+\psi)(\eta_k^i)$ brackets and the fourth sums are there because of the derivative in $\frac{\delta}{\delta\psi} e^{\mathcal{W}} = \frac{\delta\mathcal{W}}{\delta\psi}\, e^{\mathcal{W}}$. Finally, the last sum tells you which of the new downstairs fields in (II.3) are differentiated by some later $\psi(\xi)$ fields which do not hit the exponential. That is, $U \subset W(A,\underline{J},\underline{l})$, is the set of η-fields (the new downstairs fields) which are contracted to some ξ-fields. The sum of all possible contractions between them is given by the "primed integral"

$$\int \prod_{i=1}^{p} \chi_{{}_{A,U,i}}(\psi)\, d\mu'_C(\psi',\psi) = \det \begin{bmatrix} \text{same matrix as for the} \\ \text{functional integral without primes,} \\ \text{but with } C(\eta_l^i,\eta_m^l) \text{ replaced by 0} \\ \text{and } C(\xi_i,\eta_k^j) \text{ replaced by 0 if } i<j \end{bmatrix}. \tag{II.4}$$

We used a prime on $d\mu'_C$ to indicate that this is not the sum of all contractions given by the usual functional integral. It is restricted to contractions initiated by ξ-fields so that (η,η)-contractions are forbidden. Furthermore, an η-field which is generated by, say, $\psi(\xi_p)$ hitting the exponential, cannot contract to $\psi(\xi_1)$, $\cdots$, $\psi(\xi_{p-1})$, since, again by construction, we begin the integration by parts procedure with $\psi(\xi_1)$ and at that time the η-field has not yet been produced. Therefore the primed ξ-fields in (II.4) can only contract to η-fields sitting on the left of the ξ-field, or to an arbitrary other primed ξ-field. A rigorous proof of the Integration by Parts Lemma will be given in [FKLT1].

Let us return to (II.1). Eliminate the fields $\prod_{i\in J}\psi^j(\xi_i)$ in the numerator of (II.1),

$$\int\prod_{i\in J}\psi^j(\xi_i)\, e^{\mathcal{W}^j(\psi^j+\varepsilon\psi^{\le j-1})-\mathcal{W}^j(\varepsilon\psi^{\le j-1})}d\mu_{C^j}(\psi^j)\,, \tag{II.5}$$

by doing one round of partial integration. That is, apply the above lemma one time. This produces a big sum of terms. There are terms where U^c, which labels the fields in the new functional integral, is empty. Then the remaining functional integral is just

$$\int e^{\mathcal{W}^j(\psi^j+\varepsilon\psi^{\le j-1})-\mathcal{W}^j(\varepsilon\psi^{\le j-1})}d\mu_{C^j}(\psi^j)$$

and cancels against the denominator in (II.1). On the other hand, when U^c is not the empty set, apply the integration by parts lemma again. Repeat as necessary.

Consider a term where, after n steps of partial integration, the functional integral has not cancelled. Then, in each step there must have been a field which hit the exponential. Since each $\mathcal{W}^j$ comes at least with one λ and (II.1) already has one $\mathcal{W}^j_q$ downstairs, the term must be of order at least λ^{n+1}. Thus, to isolate all contributions up to $n^{\rm th}$ order, it suffices to do n rounds of partial integration discarding all terms for which the functional integral has not cancelled. So we may write

$$P_{\le n}V_q^{j-1}(\eta_1,\cdots,\eta_{2q})=\sum_{\gamma\in\mathcal{P}_{n,q}}P_{\le n}K_\gamma^{j-1}(W^j_{q_v})(\eta_1,\cdots,\eta_{2q}) \tag{II.6}$$

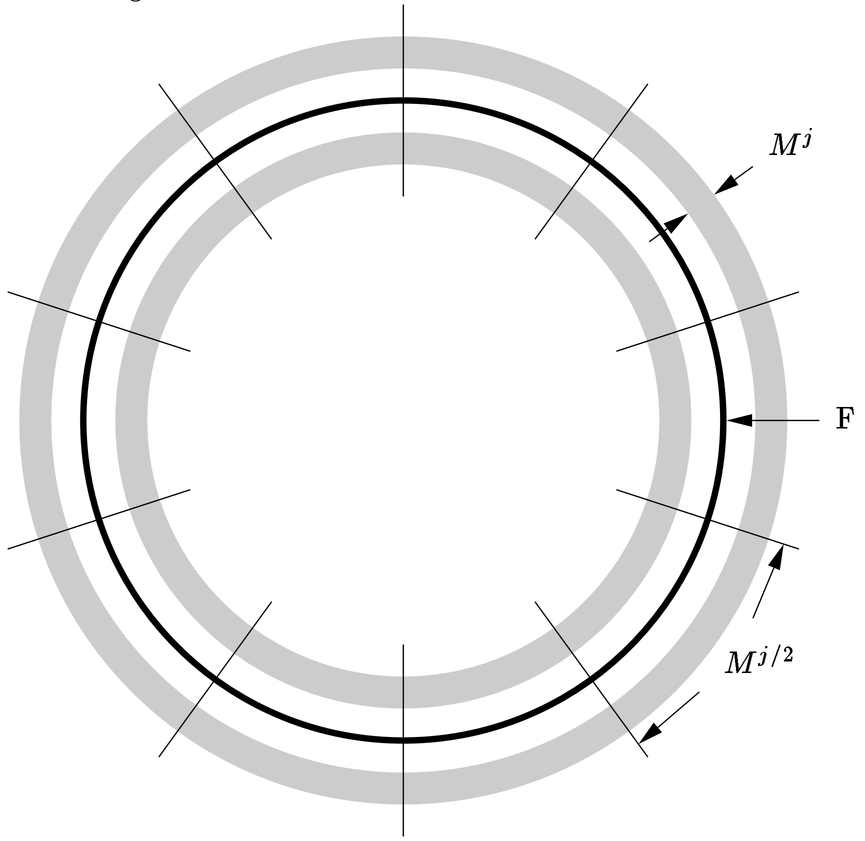
where the expansion tree $\mathcal{P}_{n,q}$ is obtained by iterating the partial integration tree $\mathcal{P}$ of Lemma II.2 n times and removing all paths which lead to contributions in which functional integral has not cancelled or the number of external fields is not equal to $2q$. The kernels are given by

$$\begin{aligned}K_\gamma^{j-1}(W^j_{q_v})(\eta_1,\cdots,\eta_{2q})&\equiv K_\gamma^{j-1}(W^j_{q_v})(\underline{\eta}^{ext})\\&=\int d\underline{\eta}^{int}\prod_{v\in V_\gamma}W^j_{q_v}(\underline{\eta}^v)\prod_{l\in T_\gamma}C^j(\eta_{i_1(l)},\eta_{i_2(l)})\prod_{r=1}^{n}I'_r(\underline{\eta})\end{aligned} \tag{II.7}$$

where V_γ is the set of all vertices of K_γ^{j-1} and T_γ is a spanning tree for K_γ^{j-1}. The primed integral produced at the r'th step of partial integration is abbreviated by $I'_r(\underline{\eta})$ in (II.7).

As mentioned above, the sum over the expansion tree is estimated with the combinatorial tree lemma and causes no problem. The only place where factorials may arise are the primed integrals. In fact, full expansion of the determinant in (II.4) for all primed integrals in (II.7) generates all Feynman graphs. Our expansion is designed to produce all contributions up to n^{th} order in such a way that potential factorials are isolated and can be controlled.

To eliminate the factorials produced by the primed integrals, one has to do two things. First, one introduces sectors, that is, one decomposes the shells around the Fermi curve into smaller pieces as shown in the following figure. The decomposition for a nonspherical Fermi curve is analogous.



Second, one Taylor expands the fields (in coordinate space) at the beginning of each step of partial integration.

We now explain why. One starts with a functional integral like (II.5). Suppose that two fields in (II.5) are equal. Then the functional integral vanishes. If you were to integrate by parts with these fields, you would produce terms whose sum is zero but which are assigned to different paths in the expansion tree. If one applies the combinatorial tree lemma, one does not see the cancelations between these terms.

The reason for sectors is the following: Each covariance C^j comes with a decay factor $(1+M^j|x|)^{-N}$ where N can be chosen arbitrary large. That is, one has summable decay between boxes of length M^{-j}. Using this decay, one can bound, as in the classical cluster expansion, the primed integral by a power counting factor times a product of "local" factorials,

$$\prod_{\substack{\Delta\in\mathcal{D}_j\\ b\in\{0,1\}\\ \sigma\in\{\uparrow,\downarrow\}}} \pi(\underline{\xi},\Delta,b,\sigma)!$$

where $\underline{\xi}$ is the set of coordinates of the downstairs fields in (II.5) and $\pi(\underline{\xi},\Delta,b,\sigma)$ is the number of those fields which have $x_i\in\Delta$, $b_i=b$ and $\sigma_i=\sigma$. Here $\mathcal{D}_j$ is a set of space time boxes of length M^{-j} which cover $\mathbb{R}^3$.

For a given set of coordinates $\underline{\xi}$ and a given field $\psi(\xi_i)$, divide $\Delta\ni x_i$ into $\pi(\underline{\xi},\Delta,b_i,\sigma_i)$ subboxes d of length

$$\frac{M^{-j}}{\pi(\underline{\xi},\Delta,b_i,\sigma_i)^{\frac{1}{3}}}\,.$$

Let c_i be the center of the subbox $d\subset\Delta$ to which ξ_i belongs, that is, for which $x_i\in d$. The Taylor expansion produces powers of $((x_i-c_i)\cdot\nabla)$. Now, by construction,

$$|x_i-c_i|\le\sqrt{3}\,\frac{M^{-j}}{\pi(\underline{\xi},\Delta,b_i,\sigma_i)^{\frac{1}{3}}}\,. \tag{II.8}$$

Thus, the Taylor expansion will generate a net small factor if each derivative in coordinate space gives an M^j. This would be the case if the covariance had a singularity at a single point in momentum space, for example if $C^j(x)=\int d^{d+1}k\,e^{ikx}\frac{f_j(k)}{|k|^{\frac{d+1}{2}}}$. However, in our model the singularity is on the Fermi curve $k_0=0$, $e(\mathbf{k})=0$ and, in C^j, $\mathbf{k}$ is only localized in a shell of thickness M^j around the Fermi curve. Each coordinate space derivative brings down a factor of k from e^{ikx}, which is of order one rather than M^j. Alternatively, one may say that the phase space volume $M^{-3j}\times M^{2j}=M^{-j}$ is too big. Therefore the spatial momentum has to be localized further. This is done by introducing sectors ℓ. Then, if the order t of the the Taylor expansion is choosen sufficiently large, the operators $(x_i-c_i)\cdot\nabla$ give small factors (up to constants)

$$\prod_{d\in\Delta}\frac{1}{\pi(\underline{\xi};\Delta,b,\sigma,\ell)^{\frac{t}{3}(\pi(\underline{\xi};d,b,\sigma,\ell)-3^t)}}\le\frac{\mathrm{const}^{\,\pi(\underline{\xi};\Delta,b,\sigma,\ell)}}{\pi(\underline{\xi};\Delta,b,\sigma,\ell)^{2\pi(\underline{\xi};\Delta,b,\sigma,\ell)}}$$

which kill the local factorials produced by the primed integrals. The number $\pi(\underline{\xi}; d, b, \sigma, \ell) - 3^t$ arises as follows. Expand each field

$$\psi^{j,\ell}(\xi_k) = \psi^{j,\ell}(c) + (x_k - c) \cdot \nabla \psi^{j,\ell}(c) + \cdots + \frac{1}{(t-1)!} \big((x_k - c) \cdot \nabla \big)^{t-1} \psi^{j,\ell}(c)$$

$$+ \frac{1}{(t-1)!} \int_0^1 dw\, (1-w)^{t-1} \big((x_k - c) \cdot \nabla \big)^t \psi^{j,\ell}(c + w(x_k - c))$$

$$\equiv \sum_{s=0}^{t} T^s_{\underline{\xi}} \psi^{j,\ell}(\xi_k)\,, \tag{II.9}$$

Since the functional integral (II.5) vanishes if two fields are equal, at most $1 + 3 + 3^2 + \cdots + 3^{t-1} \le 3^t$ fields can fail to be Taylor remainders. So at least $\pi(\underline{\xi}; d, b, \sigma, \ell) - 3^t$ fields must be Taylor remainders.

The most natural thing would be to break up the Fermi curve into sectors of length M^j. However, there is a further subtlety which forces one to use sectors of length $M^{\frac{j}{2}}$ instead of M^j. Of course, one has to control the sector sums associated with the decomposition of the covariance and fields. Since the primed integrals are to be estimated without full evaluation, one no longer has Feynman graphs. There no longer are momentum loops and one can use conservation of momentum only at each generalized vertex to control the sector sums. To be more precise on this point: if one computes the primed integral, one obtains for each contraction of, say, ψ^{j,ℓ_1} and ψ^{j,ℓ_2} a Kronecker delta δ_{ℓ_1,ℓ_2}, that mimics conservation of momentum in a line. However, to have access to all these delta's one must fully expand the primed integrals and then the crucial bound (II.14) below fails.

If $\zeta_\ell(\mathbf{k})$ has support on a sector $\ell \in \Sigma_j$ of length $M^{\frac{j}{2}}$ and

$$C^j(k) = \frac{f_j(k)}{ik_0 - e(\mathbf{k})} = \sum_{\ell \in \Sigma_j} \frac{f_j(k)\zeta_\ell(\mathbf{k})}{ik_0 - e(\mathbf{k})} = \sum_{\ell \in \Sigma_j} C^{j,\ell}(k) \tag{II.10}$$

is the corresponding decomposition of the covariance, then one can prove that

$$\left| e^{i\mathbf{q}_\ell(\mathbf{x}-\mathbf{x}')} D_0^{n_0} D_{\|}^{n_1} D_{\perp}^{n_2} \left(e^{-i\mathbf{q}_\ell(\mathbf{x}-\mathbf{x}')} C^{j,\ell}(x,x') \right) \right| \le \text{const } M^{(\frac{3}{2}+n_0+n_1+\frac{n_2}{2})j} \rho^{j,\ell}(x,x')^{-N}\,, \tag{II.11}$$

where the decay factor is given by

$$\rho^{j,\ell}(x,x') = 1 + M^j |x_0 - x_0'| + M^j |\mathbf{x}_{\|} - \mathbf{x}'_{\|}| + M^{\frac{j}{2}} |\mathbf{x}_\perp - \mathbf{x}'_\perp|\,,$$

and

$$D_0 = \frac{\partial}{\partial x_0}, \quad D_{\|} = \frac{\mathbf{q}_\ell}{|\mathbf{q}_\ell|} \cdot \nabla_{\mathbf{x}_1}, \quad D_\perp = \hat{\pi}_\ell \cdot \nabla_{\mathbf{x}_1}$$

where $\mathbf{q}_\ell$ denotes center of the sector ℓ, $\hat{\pi}_\ell$ is some unit vector perpendicular to $\mathbf{q}_\ell$ and $\mathbf{x} = (\mathbf{x}, \mathbf{q}_\ell)\mathbf{q}_\ell + (\mathbf{x}, \pi_\ell)\pi_\ell = \mathbf{x}_{\|} + \mathbf{x}_\perp$. A derivative in the π_ℓ direction only gives an $M^{\frac{j}{2}}$ but the decay rate in that direction is also $M^{-\frac{j}{2}}$ instead of M^{-j}. Thus the phase space volume is still one. $M^{\frac{j}{2}}$-sectors are the largest ones for which this can be achieved.

So we introduce the decomposition (II.10) and write the fields $\psi^j = \sum_{\ell \in \Sigma_j} \psi^{j,\ell}$ as a sum of independent variables. At the beginning of each step of partial integration, we Taylor expand the downstairs fields as in (II.9), where the expansion point $c = c(\underline{\xi}, \xi_k)$ is determined by the number of fields which are localized in the box $\Delta \ni x_k$. Observe that, because of the anisotropic decay of the covariance (II.11), one has to use boxes of size $M^{-j} \times M^{-j} \times M^{-\frac{j}{2}}$ in the x_0-, $\mathbf{q}_\ell$- and π_ℓ-direction.

The sums coming with the Taylor expansion add new branchs to the expansion tree, which is still denoted by $\mathcal{P}_{n,q}$, while the sector sums are left explicit rather than bounded by Lemma II.3. The result of the integration by parts expansion with sectors and Taylor expansion is

$$P_{\leq n} \mathcal{V}^{j-1}_{q,\leq n}(\psi^{\leq j-1}) = \sum_{\gamma \in \mathcal{P}_{n,q}} P_{\leq n} \mathcal{K}^{j-1}_\gamma(\psi^{\leq j-1}) \tag{II.12}$$

where

$$\begin{aligned} \mathcal{K}^{j-1}_\gamma(\psi^{\leq j-1}) = & \prod_{v \in V} \Bigg(\sum_{j_v = j}^{-1} \sum_{\ell^v_k \in \Sigma_{j_v+1}} \sum_{m^v_k \in \Sigma_j(\ell^v_k)} \int \prod_{k=1}^{2q_v} d\eta^v_k \, V^{j_v}_{q_v}(\underline{\eta}^v; \underline{\ell}^v) \, \delta_{q_v}(\underline{m}_v) \Bigg) \times \\ & \prod_{l \in T} \delta_{m_{i_1(l)}, m_{i_2(l)}} T^{s_l}_{\underline{\eta}} C^{j, m_{i_1(l)}}(\eta_{i_1(l)}, \eta_{i_2(l)}) \prod_{r=1}^{n} I'_r(\underline{\eta}; \underline{m}) \prod_{\substack{\text{external} \\ \text{legs}}} \psi^{\leq j-1, m_k}(\eta_k) \\ = & \sum_{m_1, \cdots, m_{2q} \in \Sigma_j} \int \prod_{k=1}^{2q} d\eta_k \, K^{j-1}_\gamma(\underline{\eta}; \underline{m}) \, \psi^{\leq j-1, m_1}(\eta_1) \cdots \psi^{\leq j-1, m_{2q}}(\eta_{2q}) . \end{aligned} \tag{II.13}$$

Here the first sector sums $\sum_{\ell^v_k}$ are over the set Σ_{j_v+1} of sectors of scale $j_v + 1$ where the vertex $V^{j_v}_{q_v}$ was created. The second sector sum decomposed each sector ℓ^v_k into $M^{-\frac{j-(j_v+1)}{2}}$ sectors m^v_k of length $M^{\frac{j}{2}}$. Furthermore, the sector delta function

$$\delta_q(m_1, \cdots, m_{2q}) = \begin{cases} 1, & \text{if } \int \prod_{i=1}^{2q} \Big(d^{d+1}k_i \, f_j(k_i) \zeta_{m_i}(\mathbf{k}_i) \Big) \delta(k_1 + \cdots + k_{2q}) > 0 \\ 0, & \text{if } \int \prod_{i=1}^{2q} \Big(d^{d+1}k_i \, f_j(k_i) \zeta_{m_i}(\mathbf{k}_i) \Big) \delta(k_1 + \cdots + k_{2q}) = 0 \end{cases}$$

ensures conservation of momentum at each vertex.

The effect of the Taylor expansion is that

$$\sup_{\underline{\eta},\underline{m}} \left\{ \prod_{r=1}^{n} |I'_r(\underline{\eta};\underline{m})| \right\} \le \Big(\underbrace{\text{const}_M \, M^{\frac{3}{2}j}}_{=\|C^{j,m}\|_{L^\infty}}\Big)^{\frac{1}{2}\left(\substack{\text{number of fields in} \\ \text{the primed integrals}}\right)} . \tag{II.14}$$

One gets the same bound if one estimates a single diagram.

Because there are sector sums on the external legs, we introduce new norms which are suitable for an inductive treatment.

$$|\!|\!| K_\gamma |\!|\!|_{\emptyset,\Sigma_j} = \sup_i \sup_{\eta_i,\, m_i} \sum_{\substack{m_k \in \Sigma_j \\ k \ne i}} \int \prod_{\substack{k=1 \\ k\ne i}}^{2q} d\eta_k \, |K_\gamma(\underline{\eta};\underline{m})| \, ,$$

$$|\!|\!| K_\gamma |\!|\!|_{S,\Sigma_j} = \sum_{\substack{m_k \in \Sigma_j \\ k \in S^c}} \int \prod_{k=1}^{2q} d\eta_k \prod_{i\in S} |f_i(\eta_i)| \, |K_\gamma(\underline{\eta};\underline{m})| \, .$$

In particular

$$|\!|\!| K_\gamma |\!|\!|_{\{1,\cdots,2q\},\Sigma_j} = |\!|\!| K_\gamma |\!|\!|_{\{1,\cdots,2q\}} = \int \prod_{k=1}^{2q} d\eta_k \, |f_k(\eta_k)| \, |K_\gamma(\underline{\eta})| \, .$$

Furthermore, if $K_\gamma = \sum_{k=1}^\infty \lambda^k K_{\gamma,k}$, we define

$$|\!|\!| K_\gamma |\!|\!|_{\emptyset,\Sigma_j,\le n} = \sum_{k=1}^{n} |\lambda|^k \, |\!|\!| K_{\gamma,k} |\!|\!|_{\emptyset,\Sigma_j} \, .$$

Then one obtains the following bound

$$|\!|\!| K_\gamma^{j-1} |\!|\!|_{\emptyset,\Sigma_j,\le n} \le \prod_{v\in V} \left(M^{\frac{j}{2}(q_v-2)} \sum_{j_v=j}^{-1} M^{\frac{j_v+1}{2}(2q_v-3)} |\!|\!| V^{j_v}_{q_v,k_v} |\!|\!|_{\emptyset,\Sigma_{j_v+1},\le n} \right) \times$$

$$c_1^{\sum_v q_v - q} M^{-\frac{j}{2}(q-2)} M^{-\frac{j}{2}(2q-3)} \, . \tag{II.15}$$

Let us briefly explain how the different powers of M^j arise. The primed integrals give the factors

$$\left(M^{\frac{3}{2}j}\right)^{\frac{1}{2}\left(\substack{\text{number of fields in} \\ \text{the primed integrals}}\right)} = \left(M^{\frac{3}{2}j}\right)^{\frac{1}{2}\left(\substack{\text{number of fields} \\ \text{not on the tree}}\right)} = M^{\frac{3}{2}j\left(\sum_v q_v - q - \sum_v 1 + 1\right)} \, .$$

On the tree T in (II.13), the L^1-norm of $C^{j,m}$ has to be taken which give the factors

$$\left(M^{\frac{3}{2}j}M^{-\frac{5}{2}j}\right)^{\sum_v 1-1} = M^{-j(\sum_v 1-1)}.$$

The sector sums $\sum_{\ell_k^v\in\Sigma_{j_v+1}}$ are contained in the norms $|||V^{j_v}_{q_v,k_v}|||_{\emptyset,\Sigma_{j_v+1}}$. The sector sums $\sum_{m_k^v\in\Sigma_j(\ell_k^v)}$ are estimated by Lemma II.3' of [FMRT1]. This lemma covers only the spherical case, but there is an analogous version for the $e(\mathbf{k})$'s specified in section I. It says that, if one fixes the sector of one leg of the vertex v, the number of independent sector sums for the remaining legs is at most $2q_v-3$. By conservation of momentum, one can get rid of two sector sums at each generalized vertex. This gives the factor

$$\prod_v M^{-\frac{j-(j_v+1)}{2}(2q_v-3)}$$

since for all but one vertex one sector is fixed by the Kroenecker delta's $\delta_{m_{i_1(l)},m_{i_2(l)}}$ on the tree T and for the last vertex one sector is fixed by the definition of the norm $|||\cdot|||_{\emptyset,\Sigma_j}$.

Altogether, one obtains

$$M^{\frac{3}{2}j(\sum_v q_v-q-\sum_v 1+1)}M^{-j(\sum_v 1-1)}M^{-\sum_v(2q_v-3)\frac{j-(j_v+1)}{2}} =$$

$$\prod_v\left(M^{\frac{j}{2}(q_v-2)}M^{\frac{j_v+1}{2}(2q_v-3)}\right)M^{-\frac{j}{2}(q-2)}M^{-\frac{j}{2}(2q-3)}$$

which coincides with the power counting of (II.15).

Now we apply the combinatorial tree lemma to obtain

$$\begin{aligned}
|||V_q^{j-1}|||_{\emptyset,\Sigma_j,\le n} &\le \sum_{\gamma\in\mathcal{P}_{n,q}} |||K_\gamma^{j-1}|||_{\emptyset,\Sigma_j,\le n}\\
&\le \sup_{\gamma\in\mathcal{P}_{n,q}}\left\{\prod_{f\in\gamma} b(f)\prod_{\ell\in\gamma} w_\ell\ |||K_\gamma^{j-1}|||_{\emptyset,\Sigma_j,\le n}\right\}\\
&\le \sup_{\gamma\in\mathcal{P}_{n,q}}\left\{c_2^{\sum_v q_v}c_1^{\sum_v q_v-q}\prod_{v\in V}\left(M^{\frac{j}{2}(q_v-2)}\sum_{j_v=j}^{-1}M^{\frac{j_v+1}{2}(2q_v-3)}|||V^{j_v}_{q_v}|||_{\emptyset,\Sigma_{j_v+1},\le n}\right)\times\right.\\
&\qquad\left. M^{-\frac{j}{2}(q-2)}M^{-\frac{j}{2}(2q-3)}\right\}. \qquad\qquad (\mathrm{II}.16)
\end{aligned}$$

The constant c_1 comes from covariance estimates and depends on M but c_2 is a pure combinatorial constant. If some external legs in $S\subset\{1,\cdots,2q\}$ are integrated against test functions, one gets the following bound

$$\|V_q^{j-1}\|_{S,\Sigma_j,\leq n} \leq \tag{II.17}$$
$$\sup_{\gamma\in\mathcal{P}_{n,q}} \Bigg\{ c_2^{\sum_v q_v-|S|} c_1^{\sum_v q_v - q} \prod_{v\in V_{int}} \Bigg(M^{\frac{j}{2}(q_v-2)} \sum_{j_v=j}^{-1} M^{\frac{j_v+1}{2}(2q_v-3)} \|V_{q_v}^{j_v}\|_{\emptyset,\Sigma_{j_v+1},\leq n}\Bigg) \times$$
$$\prod_{v\in V_{ext}} \Bigg(M^{\frac{j}{2}(q_v-\frac{1}{2}|S_v|)} \sum_{j_v=j}^{-1} M^{\frac{j_v+1}{2}(2q_v-|S_v|)} \|V_{q_v}^{j_v}\|_{S_v,\Sigma_{j_v+1},\leq n}\Bigg) M^{-\frac{j}{2}(q-\frac{1}{2}|S|)} M^{-\frac{j}{2}(2q-|S|)} \Bigg\}.$$

By (II.16,17), $\|V_q^{j-1}\|_{{\emptyset \atop S},\Sigma_j,\leq n}$ is estimated in terms of $\|V_{q_v}^{j_v}\|_{{\emptyset \atop S},\Sigma_{j_v+1},\leq n}$ of higher scale $j_v > j-1$. Therefore we can proceed by induction on the scale with the following inductive hypothesis

$$\|V_q^{j-1}\|_{\emptyset,\Sigma_j,\leq n} \leq |\lambda|^{\frac{q}{2}} M^{-\frac{j}{2}(3q-5)}, \tag{II.18}$$
$$\|V_q^{j-1}\|_{S,\Sigma_j,\leq n} \leq \binom{2q}{|S|}^{-1} p(j)^{|S|} |\lambda|^{\frac{q}{2}} M^{-\frac{j}{2}(3q-\frac{3}{2}|S|-\frac{1}{4})} \prod_{k\in S} \Big(\|f_k\|_{L^1} + \|f_k\|_{L^\infty}\Big) \tag{II.19}$$

where $p(j) = \prod_{i=j+1}^0 (1+M^{\frac{1}{4}i}) \leq \prod_{i=-\infty}^0 (1+M^{\frac{1}{4}i}) = c_3 < \infty$. Using (II.16,17), (II.18,19) is verified for λ sufficiently small†. In particular, we obtain

$$\int \prod_{i=1}^{2q} d\xi_i\, |f_i(\xi_i)|\, |W_q^r(\xi_1,\cdots,\xi_{2q})| = \|W_q^r\|_{\{1,\cdots,2q\}} \leq \sum_{j=r}^{0} \|V_q^j\|_{\{1,\cdots,2q\}}$$
$$= \sum_{j=r}^{0} \|V_q^j\|_{\{1,\cdots,2q\},\Sigma_{j+1}} \leq 2\, c_3^{2q} |\lambda|^{\frac{q}{2}} \prod_{k=1}^{2q} \Big(\|f_k\|_{L^1} + \|f_k\|_{L^\infty}\Big). \tag{II.20}$$

This completes our outline of the proof of Theorem II.1 in the case without two and nontrivial four legged subgraphs. Now let us turn to the case where two legged subgraphs are allowed.

Renormalization of Two Legged Subgraphs

So now, for $r+1 \leq j \leq 0$, let $\mathcal{W}^{j-1}$ be the effective potential without four legged subgraphs defined in Theorem II.1. Then

$$\mathcal{W}^{j-1}(\psi^{\leq j-1}) = \sum_{i=j-1}^{0} \mathcal{V}^i(\psi^{\leq j-1}) + \delta\mathcal{V}_r(\psi^{\leq j-1})$$

† First choose M big enough such that $c_2^{\sum_v q_v} M^{-\frac{1}{6}\sum_v q_v} \leq 1$ and then make λ small enough such that $c_1^{\sum_v q_v - q} |\lambda|^{\sum_v \frac{q_v}{2} - \frac{q}{2}} \leq 1$.

where

$$\mathcal{V}^0(\psi^{\leq j-1}) = -\mathcal{V}(\psi^{\leq j-1})$$

and, for $i < 0$,

$$\begin{aligned}\mathcal{V}^i(\psi^{\leq i}) &= \mathcal{W}^i(\psi^{\leq i}) - \mathcal{W}^{i+1}(\psi^{\leq i}) \\ &= Q_4 \log \frac{1}{Y_{i+1}} \int e^{\mathcal{W}^{i+1}(\psi^{\leq i+1}) - \mathcal{W}^{i+1}(\psi^{\leq i})} d\mu_C^{i+1}(\psi^{i+1})\end{aligned} \tag{II.21}$$

and

$$Y_{i+1} = \int e^{\mathcal{W}^{i+1}(\psi^{i+1})} d\mu_C^{i+1}(\psi^{i+1}) .$$

The counterterm is given by

$$\delta\mathcal{V}_r(\psi) = \int \frac{d^3k}{(2\pi)^3} \delta e_r(\mathbf{k}, \lambda)\, \bar{\psi}(k)\psi(k)$$

where we write

$$\delta e_r(\mathbf{k}, \lambda) = \sum_{j=r}^{-1} \delta e_r^j(\mathbf{k}, \lambda) = \sum_{j=r}^{-1} \sum_{l=1}^{\infty} \delta e_{r,l}^j(\mathbf{k})\, \lambda^l .$$

Define a localization operator $\mathbf{L}$ which annihilates all but quadratic monomials by

$$\mathbf{L} \int \frac{d^3k}{(2\pi)^3} G(k)\, \psi(k)\bar{\psi}(k) = \int \frac{d^3k}{(2\pi)^3} LG(k)\, \psi(k)\bar{\psi}(k) , \tag{II.22}$$

$$\mathbf{L}\mathcal{G}_q = 0 \quad \forall q \neq 1 ,$$

$$LG(k_0, \mathbf{k}) = G(0, \pi_F \mathbf{k}) .$$

Here we assume that on a tubular neighbourhood $\mathcal{N}(F)$ of the Fermi curve $F = \{\mathbf{k} \in \mathbb{R}^2 \mid e(\mathbf{k}) = 0\}$ we can define a projection $\pi_F : \mathcal{N}(F) \to F$ such that

$$|e(\mathbf{k})| \leq M^j \ \Rightarrow \ |\pi_F \mathbf{k} - \mathbf{k}| \leq c\, M^j$$

where c is independent of j and $\mathbf{k}$. In the event that F is a circle of radius k_F, we simply let $\pi_F \mathbf{k} = k_F \frac{\mathbf{k}}{||\mathbf{k}||}$ and then $LG(k) = G(0, k_F \frac{\mathbf{k}}{||\mathbf{k}||}) = G(0, k_F)$ is independent of $\mathbf{k}$ by the rotation invariance of G.

The interaction for $\mathcal{V}^{j-1}$ is $\mathcal{W}^j$ which may be decomposed into a renormalized and a local part. Let $\mathbf{R} = 1 - \mathbf{L}$, then

$$\begin{aligned}
\mathcal{W}^j(\psi^{\le j}) &= \mathbf{R}\mathcal{W}^j(\psi^{\le j}) + \mathbf{L}\mathcal{W}^j(\psi^{\le j}) \\
&= \sum_{i=j}^{0} \mathbf{R}\mathcal{V}^i(\psi^{\le j}) + \sum_{i=j}^{-1} \mathbf{L}\mathcal{V}^i(\psi^{\le j}) + \delta\mathcal{V}_r(\psi^{\le j}) \\
&= \sum_{i=j}^{0} \mathbf{R}\mathcal{V}^i(\psi^{\le j}) + \int \frac{d^3k}{(2\pi)^3} \Big(\sum_{i=j}^{-1} LV_1^i(\mathbf{k},\lambda) + \delta e_r(\mathbf{k},\lambda) \Big)\, \psi^{\le j}(k)\bar\psi^{\le j}(k)\,.
\end{aligned}$$

provided we define $\delta e_r(\mathbf{k},\lambda)$ to be invariant under $\mathbf{L}$.

If the quadratic part appears as a two legged vertex in the computation of $\mathcal{V}^{j-1}$, the Power Counting Lemma II.2 gives a factor (neglecting sectors)

$$\Big|\!\Big|\!\Big| \sum_{i=j}^{-1} LV_1^i + \delta e_r \Big|\!\Big|\!\Big|_{\emptyset} \times M^{-j}$$

which is big. On the other hand, the $|\!|\!|\ \cdot\ |\!|\!|_{\emptyset}$ norm of a two legged graph without two and four legged subgraphs is bounded by M^j which is a small number. Therefore one should choose δe_r such that $\sum_{i=j}^{-1} LV_1^i + \delta e_r$ can be treated as a kernel of a two legged graph of scale $\le j-1$. That is, choose $\delta e_r = -\sum_{i=r}^{-1} LV_1^i$ or, if $LV_1^i(\mathbf{k},\lambda) = \sum_{l=1}^{\infty} LV_{1,l}^i(\mathbf{k})\lambda^l$,

$$\delta e_{r,l}^i(\mathbf{k}) = -LV_{1,l}^i(\mathbf{k}) \qquad \forall r \le i \le -1,\ l \ge 1. \tag{II.23}$$

Note that $\mathcal{V}_1^i$ is defined through a functional integral whose integrand contains $\mathcal{W}^{i+1}$ and hence

$$\sum_{m=i+1}^{-1} LV_1^m + \delta e_r = -\sum_{m=r}^{i} LV_1^m = \sum_{m=r}^{i} \sum_{l=0}^{\infty} \delta e_{r,l}^m(\mathbf{k})$$

Thus (II.23) is of the form

$$\delta e_{r,l}^i = f_{r,l}^i\Big((\delta e_{r,l'}^m)_{\substack{r\le m\le -1 \\ 1\le l'\le l-1}} \Big) \tag{II.23a}$$

In particular, for $l=1$, (II.23) is

$$\int \frac{d^3k}{(2\pi)^3} \delta e_{r,1}^i(\mathbf{k})\, \psi^{\le i}(k)\bar\psi^{\le i}(k) = -\mathbf{L} \int \Big(V(\psi^{\le i+1}) - V(\psi^{\le i}) \Big)\, d\mu_{C^{i+1}}$$

where V is the initial quartic interaction. The counterterm $\delta\mathcal{V}_r$ does not appear on the right hand side since there is at least one contraction. Thus $e_{r,l=1}^i$ is determined for all i. Iterating (II.23a) determines $e_{r,l}^i$ for $l > 1$, without the need for any estimates.

With this choice of δe_r, $\mathcal{W}^j$ becomes

$$\mathcal{W}^j(\psi^{\leq j}) = \sum_{i=j}^{0} \mathbf{R}\mathcal{V}^i(\psi^{\leq j}) - \sum_{i=r}^{j-1} \int \frac{d^3k}{(2\pi)^3}\, LV_1^i(\mathbf{k},\lambda)\, \psi^{\leq j}(k)\bar{\psi}^{\leq j}(k)\,. \tag{II.24}$$

Now, scales below j also appear on the right hand side of (II.24). That means that the estimates (II.16,17) for V_q^{j-1} contain scales below j, too. So we can no longer proceed by induction on scale. Nevertheless, there is another way to construct δe_r which allows one to prove the bounds on V_q^{j-1} by induction (on iteration steps, which play a rôle similar to the order of perturbation theory).

Lemma II.5 *Let $r > -\infty$ be the infrared cutoff and let $\underline{\mathcal{U}}$ be a vector of effective potentials $\underline{\mathcal{U}} = (\mathcal{U}^0, \mathcal{U}^{-1}, \cdots, \mathcal{U}^r)$. Define a sequence of vectors $\underline{\mathcal{U}}^k = (\mathcal{U}^{k,0}, \mathcal{U}^{k,-1}, \cdots, \mathcal{U}^{k,r})$ by*

$$\underline{\mathcal{U}}^0(\psi^{\leq 0}) = \big(-\lambda\mathcal{V}(\psi^{\leq 0}),\, 0\,,\, \cdots\,,\, 0\big)$$

$$\mathcal{U}^{k+1,0} = -\lambda\mathcal{V} \quad \forall\, k \geq 0$$

and for all $r+1 \leq j \leq 0$, $k \geq 0$

$$\mathcal{U}^{k+1,j-1}(\psi^{\leq j-1}) = Q_4 \log \frac{1}{I_j^k} \int e^{\sum_{i=j}^{0} \mathbf{R}\mathcal{U}^{k,i}(\psi^{\leq j};\psi^{\leq j-1}) - \sum_{i=r}^{j-1} \mathbf{L}\mathcal{U}^{k,i}(\psi^{\leq j};\psi^{\leq j-1})}\, d\mu_{C^j}\,, \tag{II.25}$$

$$I_j^k = \int e^{\sum_{i=j}^{0} \mathbf{R}\mathcal{U}^{k,i}(\psi^j) - \sum_{i=r}^{j-1} \mathbf{L}\mathcal{U}^{k,i}(\psi^j)}\, d\mu_{C^j}$$

where we have used the abbreviation $\mathcal{U}^{k,i}(\psi^{\leq j};\psi^{\leq j-1}) = \mathcal{U}^{k,i}(\psi^{\leq j}) - \mathcal{U}^{k,i}(\psi^{\leq j-1})$. Here $\mathbf{L}$ is the localization operator (II.22) and $\mathbf{R} = 1 - \mathbf{L}$.

Let n be an arbitrary natural number and P_n be the operator which projects out the n^{th} order coefficient with respect to λ. Then one has

$$\forall\, s \geq 0 \qquad\qquad P_n\, \underline{\mathcal{U}}^{2n+s} = P_n\, \underline{\mathcal{U}}^{2n-1}$$

and, if $\mathcal{V}^j$ are the potentials defined in (II.21) such that $\mathcal{W}^r = \sum_{i=r}^{-1} \mathcal{V}^i + \delta\mathcal{V}_r$,

$$\forall\, r \leq j \leq -1 \qquad\qquad P_n\, \mathcal{U}^{2n,j} = P_n\, \mathcal{V}^j$$

where we used $\underline{\mathcal{U}}^{2n}$ instead of $\underline{\mathcal{U}}^{2n-1}$ since it makes the formulas shorter. In particular, the coefficients up to $n^{\rm th}$ order of the the solution δe_r of the equations (II.23) are given by

$$\forall\, 1\le l\le n \qquad\qquad \delta e_{r,l}=\sum_{i=r}^{-1}\delta e^i_{r,l}=-\sum_{i=r}^{-1}LU^{2n,i}_{1,l}$$

if

$$\mathcal{U}^{2n,i}(\psi)=\sum_{\substack{q=1\\ q\neq 2}}^{\infty}\sum_{l=1}^{\infty}\lambda^l\int d\underline{\xi}\,U^{2n,i}_{q,l}(\xi_1,\cdots,\xi_{2q})\,\psi(\xi_1)\cdots\psi(\xi_{2q})\,.$$

Altogether, if W^r_q are the kernels of the effective potential without four legged subgraphs defined in Theorem II.1, we have

$$\sum_{l=1}^{n}\lambda^l W^r_{q,l}=\sum_{l=1}^{n}\sum_{i=r}^{0}\lambda^l U^{2n,i}_{q,l}-\delta_{q,1}\sum_{l=1}^{n}\sum_{i=r}^{-1}\lambda^l LU^{2n,i}_{1,l}\,. \tag{II.26}$$

Remark. There is an analogous version of this lemma for the construction of the full effective potential, including four legged subgraphs. The only modifications to be made are to omit the operator Q_4 in the definition of $\mathcal{U}^{k+1,j-1}$ and to define $\mathcal{U}^{k+1,0}=-\lambda\mathcal{V}$ for all $k\ge 0$.

By (II.26) we have to bound $U^{2n,i}_{q,l}$ and $LU^{2n,i}_{1,l}$. But (II.25) expresses $\underline{\mathcal{U}}^{k+1}$ in terms of $\underline{\mathcal{U}}^{k}$. That is, we do a single scale expansion in (II.25) and then proceed by induction on k.

The single scale expansion is done as above. It is generated by the integration by parts formula Lemma II.4. We Taylor expand the fields at the beginning of each step of partial integration and use sectors of length $M^{\frac{j}{2}}$. We apply the Combinatorial Tree Lemma II.3 to control the various sums produced in the expansion. Sector sums are estimated by the Sector Counting Lemma II.3' of [FMRT1]. The result is

$$\begin{aligned}
&|\!|\!|U^{k+1,j-1}_q|\!|\!|_{\emptyset,\Sigma_j,\le n}\le\sum_{\gamma\in\mathcal{P}_{n,q}}|\!|\!|K^{k+1,j-1}_\gamma|\!|\!|_{\emptyset,\Sigma_j,\le n}\le \qquad\qquad\qquad (\mathrm{II.27})\\
&\sup_{\gamma\in\mathcal{P}_{n,q}}\Big\{c_2^{\sum_v q_v}c_1^{\sum_v q_v-q}\prod_{v\in V\setminus V_2}\Big(M^{\frac{j}{2}(q_v-2)}\sum_{j_v=j}^{-1}M^{\frac{j_v+1}{2}(2q_v-3)}|\!|\!|U^{k,j_v}_{q_v}|\!|\!|_{\emptyset,\Sigma_{j_v+1},\le n}\Big)\times\\
&\prod_{v\in V_{2,R}}\Big(\sum_{j_v=j}^{-1}|\!|\!|\,|x|\,U^{j_v}_1|\!|\!|_{\emptyset,\le n}\Big)\prod_{v\in V_{2,L}}\Big(\sum_{j_v=r}^{j-1}|\!|\!|U^{k,j_v}_1|\!|\!|_{\emptyset,\le n}M^{-j}\Big)M^{-\frac{j}{2}(q-2)}M^{-\frac{j}{2}(2q-3)}\Big\}\,.
\end{aligned}$$

We now briefly explain the power counting of the renormalized and counterterm two legged vertices. If a renormalized vertex of scale j_v contributes to $K_\gamma^{k+1,j-1}$, one obtains a factor† $\sup_{k\in \mathrm{supp}C^j(k)} |RU_1^{k,j_v}(k)|$ which is estimated as follows

$$\begin{aligned} |RU_1^{k,j_v}(k)| &= |U_1^{k,j_v}(k) - LU_1^{k,j_v}(k)| = |U_1^{k,j_v}(k_0,\mathbf{k}) - U_1^{k,j_v}(0,\pi_F\mathbf{k})| \\ &\le |\partial_{k_0} U_1^{k,j_v}(\tilde{k})|\cdot|k_0| + |\nabla_{\mathbf{k}} U_1^{k,j_v}(\tilde{\tilde{k}})|\cdot|\mathbf{k}-\pi_F\mathbf{k}| \\ &\le |||\, |x_0| U_1^{k,j_v} |||_\emptyset \cdot |k_0| + |||\, |\mathbf{x}| U_1^{k,j_v} |||_\emptyset \cdot |\mathbf{k}-\pi_F\mathbf{k}| \\ &\le 2 |||\, |x| U_1^{k,j_v} |||_\emptyset \cdot M^j \qquad \text{on the support of } C^j(k)\,. \end{aligned}$$

A counterterm vertex LU_1^{k,j_v} is simply estimated by

$$\sup_{k\in \mathrm{supp}C^j(k)} |LU_1^{k,j_v}(k)| \le \sup_k |U_1^{k,j_v}(0,\pi_F\mathbf{k})| \le |||U_1^{k,j_v}|||_\emptyset\,.$$

Assigning an M^{-j} to each counterterm vertex as in (II.27), one can say that there is an additional M^j for each two legged vertex. Furthermore, by conservation of momentum, there are no sector sums coming with a two legged vertex. Therefore we obtain (compare the discussion following (II.15))

$$M^{\frac{3}{2}j(\sum_v q_v - q - \sum_v 1 + 1)} M^{-j(\sum_v 1 - 1)} \prod_{v\in V\setminus V_2} M^{-\frac{j-(j_v+1)}{2}(2q_v-3)} \prod_{v\in V_2} M^j =$$

$$\prod_{v\in V\setminus V_2} \left(M^{\frac{j}{2}(q_v-2)} M^{\frac{j_v+1}{2}(2q_v-3)}\right) M^{-\frac{j}{2}(q-2)} M^{-\frac{j}{2}(2q-3)}\,,$$

which is the power counting of (II.27).

If some legs of $U_q^{k+1,j-1}$ are integrated against test functions, one obtains

$$|||U_q^{k+1,j-1}|||_{S,\Sigma_j,\le n} \le \sum_{\gamma\in\mathcal{P}_{n,q}} |||K_\gamma^{k+1,j-1}|||_{S,\Sigma_j,\le n} \le$$

$$\sup_{\gamma\in\mathcal{P}_{n,q}} \Bigg\{ c_2^{\sum_v q_v - |S|} c_1^{\sum_v q_v - q} \prod_{v\in V_{int}\setminus V_2} \left(M^{\frac{j}{2}(q_v-2)} \sum_{j_v=j}^{-1} M^{\frac{j_v+1}{2}(2q_v-3)} |||U_{q_v}^{k,j_v}|||_{\emptyset,\Sigma_{j_v+1},\le n} \right) \times$$

$$\prod_{v\in V_{ext}\setminus V_2} \left(M^{\frac{j}{2}(q_v-\frac{1}{2}|S_v|)} \sum_{j_v=j}^{-1} M^{\frac{j_v+1}{2}(2q_v-|S_v|)} |||\, U_{q_v}^{k,j_v} |||_{S_v,\Sigma_{j_v+1},\le n} \right) \times$$

† One has to be a little careful in going to momentum space, since the Taylor operators $(x-c)\cdot\nabla_x$ break translation invariance, it still comes down to the above estimate.

$$\prod_{v\in V_{2,R}} \left(\sum_{j_v=j}^{-1} \| |x| U_1^{k,j_v} \|_{\emptyset,\le n} \right) \prod_{v\in V_{2,C}} \left(\sum_{j_v=r}^{j-1} \| U_1^{k,j_v} \|_{\emptyset,\le n} M^{-j} \right) \times$$

$$M^{-\frac{j}{2}(q-\frac{1}{2}|S|)} M^{-\frac{j}{2}(2q-|S|)} \prod_{f \text{ at } V_{2,ext}} \|f\|_{L^1} \Bigg\} . \tag{II.28}$$

which may be compared to (II.17).

We are now in a position to state the induction hypothesises on $\underline{\mathcal{U}}^k$. They are

$$\| U_q^{k,j-1} \|_{\emptyset,\Sigma_j,\le n} \le |\lambda|^{\frac{q}{2}} M^{-\frac{j}{2}(3q-5)} \tag{II.29}$$

$$\| |x|^w U_1^{k,j-1} \|_{\emptyset,\le n} \le |\lambda|^{\frac{1}{2}} M^{-wj} M^{\frac{3}{2}j}, \qquad w \in \{0,1\} \tag{II.30}$$

$$\| U_q^{k,j-1} \|_{S,\Sigma_j,\le n} \le \binom{2q}{|S|}^{-1} p(j)^{|S|} |\lambda|^{\frac{q}{2}} M^{-\frac{j}{2}(3q-\frac{3}{2}|S|-\frac{1}{4})} \prod_{k\in S} \Big(\|f_k\|_{L^1} + \|f_k\|_{L^\infty} \Big) \tag{II.31}$$

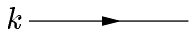
Using (II.27,28), they are verified for $\underline{\mathcal{U}}^{k+1}$. In particular, (II.29-30) hold for $k = 2n$. Then we can apply (II.26) of Lemma II.5 to conclude

$$\begin{aligned}
\int \prod_{i=1}^{2q} d\xi_i \, |f_i(\xi_i)| \, |W_q^r(\xi_1, \cdots, \xi_{2q})| &= \|W_q^r\|_{\{1,\cdots,2q\}} \\
&\le \sum_{i=r}^{-1} \|U_q^{2n,i}\|_{\{1,\cdots,2q\}} + \delta_{q,1} \sum_{i=r}^{-1} \|LU_q^{2n,i}\|_{\{1,2\}} \\
&= \sum_{i=r}^{-1} \|U_q^{2n,i}\|_{\{1,\cdots,2q\},\Sigma_{i+1}} + \delta_{q,1} \sum_{i=r}^{-1} \|LU_q^{2n,i}\|_{\{1,2\},\Sigma_{i+1}} \\
&\le 4\, c_3^{2q} \, |\lambda|^{\frac{q}{2}} \prod_{k=1}^{2q} \Big(\|f_k\|_{L^1} + \|f_k\|_{L^\infty} \Big) .
\end{aligned}$$

This completes our discussion of Theorem II.1.

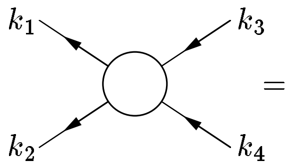
§III Four Legged Diagrams

Spin plays no role in this section. So we supress it. Feynman diagrams in this model have lines



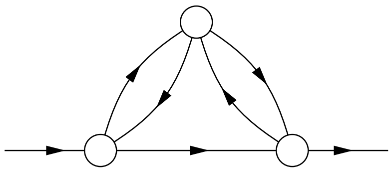
$$= \frac{1}{ik_0 - e(\mathbf{k})}$$

and vertices

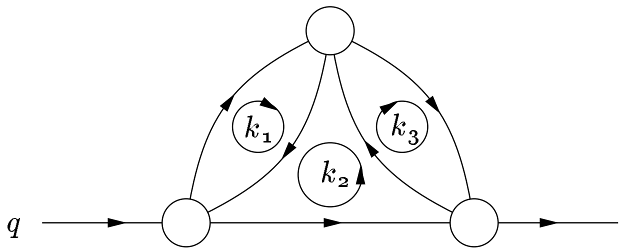


$$= (2\pi)^{d+1}\delta(k_1 + k_2 - k_3 - k_4)\lambda \left\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \right\rangle$$

For example



is one graph contributing to the proper self energy. This is a three loop graph. Choosing the loops as in



we see that the value of this graph is

$$\begin{aligned}\int \text{đ}k_1\, \text{đ}k_2\, \text{đ}k_3\ & \frac{1}{i(k_1)_0 - e(\mathbf{k}_1)}\ \frac{1}{i(k_1+k_2)_0 - e(\mathbf{k}_1 + \mathbf{k}_2)} \\ & \frac{1}{i(k_2+k_3)_0 - e(\mathbf{k}_2 + \mathbf{k}_3)}\ \frac{1}{i(k_3)_0 - e(\mathbf{k}_3)}\ \frac{1}{i(k_2+q)_0 - e(\mathbf{k}_2 + \mathbf{q})} \\ & \left\langle k_1 + k_2, k_3 | \mathrm{V} | k_1, k_2 + k_3 \right\rangle \left\langle k_1, k_2 + q | \mathrm{V} | q, k_1 + k_2 \right\rangle \left\langle k_2 + k_3, q | \mathrm{V} | k_2 + q, k_3 \right\rangle\end{aligned}$$

It is not clear that this integral converges. The domain of integration is compact, because of the ultraviolet cutoff, but the integrand is singular.

To check for convergence one does "naive power counting" bounds. In field theory propagator singularities occur at points. Then power counting just comes down to some simple dimensional analysis. Here there are singularities on curves, like $(k_1)_0 = 0,\ \mathbf{k}_1 \in F$. We have to have a simple yet precise way of measuring whether the integrand is large a lot. To do so we decompose the propagator

$$\begin{aligned} C(k) &= \frac{1}{ik_0 - e(\mathbf{k})} \\ &= \sum_{j=-\infty}^{0} C^{(j)} \end{aligned}$$

where

$$C^{(j)}(k) = \frac{1}{ik_0 - e(\mathbf{k})} \chi\big(2^j \le |ik_0 - e(\mathbf{k})| < 2^{j+1}\big)$$

Note, the perhaps bizarre, convention that j is negative. As j tends to *minus* infinity, 2^j approaches zero and, on the support of $C^{(j)}$, $|ik_0 - e(\mathbf{k})|$ approaches zero. Naive power counting just uses

Lemma III.1 *Let d be arbitrary and Hypothesis I be satisfied. Then*

a)

$$\|C^{(j)}\|_\infty = \sup_k |C^{(j)}(k)| \le 2^{-j}$$

b)

$$\|C^{(j)}\|_1 = \int đk\ |C^{(j)}(k)| \le \mathrm{const}\, 2^j$$

Proof: Part a) is obvious because, by construction, $|ik_0 - e(\mathbf{k})| \ge 2^j$ on the support of $C^{(j)}(k)$.

For part b) observe that

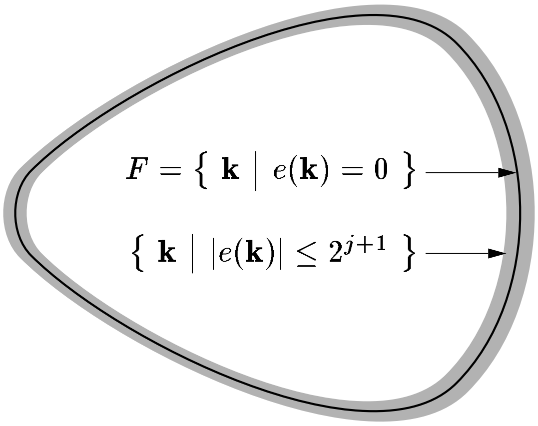
$$\begin{aligned} \mathrm{vol}\left\{ k = (k_0, \mathbf{k}) \;\middle|\; C^{(j)}(k) \ne 0 \right\} &\le \mathrm{vol}\left\{ k_0 \;\middle|\; |k_0| \le 2^{j+1} \right\} \mathrm{vol}\left\{ \mathbf{k} \in B \;\middle|\; |e(\mathbf{k})| \le 2^{j+1} \right\} \\ &\le 2^{j+2}\mathrm{vol}\left\{ \mathbf{k} \in B \;\middle|\; |e(\mathbf{k})| \le 2^{j+1} \right\} \end{aligned}$$

The set $\left\{ \mathbf{k} \in B \;\middle|\; |e(\mathbf{k})| \le 2^{j+1} \right\}$ consists of a shell of thickness $O(2^j)$ around F and hence has volume bounded by $\mathrm{const}\, 2^j$ so that

$$\mathrm{vol}\left\{ k = (k_0, \mathbf{k}) \;\middle|\; C^{(j)}(k) \ne 0 \right\} \le \mathrm{const}\, 2^{2j} \tag{III.1}$$

and

$$\|C^{(j)}\|_1 = \int đk \ |C^{(j)}(k)| \le \sup_k |C^{(j)}(k)| \mathrm{vol}\left\{\ k = (k_0, \mathbf{k}) \ \middle|\ C^{(j)}(k) \ne 0\ \right\}$$
$$\le \mathrm{const}\, 2^j$$



■

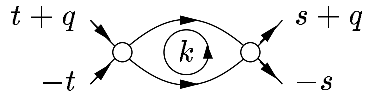
We remark that the smoothness condition $\nabla e(\mathbf{k}) \ne 0$ of Hypothesis I was used to get the volume bound (III.1). The corresponding volume for the Hubbard model at half filling is $|j|2^{2j}$ which leads to $\|C^{(j)}\|_1 \le \mathrm{const}\, |j| 2^j$.

The analog of Lemma III.1 for the infrared Φ_4^4 model is $\|C^{(j)}\|_\infty \le 2^{-2j}$, $\|C^{(j)}\|_1 \le \mathrm{const}\, 2^{2j}$. The replacement $j \to 2j$ can be viewed simply as a change of units. So it is not too surprising that Lemma III.1 implies [FT2, FMRT1] that models satisfying Hypotheses I and IV obey bounds typical of strictly renormalizable models in the infrared regime. Two legged are linearly divergent and must be renormalized. Four legged subdiagrams are marginal and all other subdiagrams are convergent. As is normal for infrared models, the two legged counterterm is finite and the marginality of four legged subdiagrams does not require a counterterm. The four legged subdiagrams are divergent only for certain exceptional momenta and then only logarithmically divergent. These logarithmic singularities are integrable and hence do not prevent diagrams from being well defined. But they can cause the values of diagrams containing many four legged subdiagrams to be anomalously large through

$$\int đk \ln^n |k| \sim n!$$

Normally, under these circumstances one of two possibilities occur. The renormalization group flow of the four point function is either asymptotically free or is to a nontrivial fixed point and is acompanied by some interesting physics, like mass generation or symmetry breaking. We shall now see that under Hypotheses I-IV, the bounds which give marginality of four legged subdiagrams are not saturated. Four legged subdiagrams are in fact convergent. The models behave more like superrenormalizable models than strictly renormalizable ones.

To be concrete, we'll first do the naive power counting bound explicitly on one simple, but very important, graph – the particle-particle bubble



If the total momentum entering from the left is q, the value of this graph is

$$B(s,t,q) = \int đk\ C(-k+q)C(k)\,\langle -k+q,k|\mathrm{V}|t+q,-t\rangle\,\langle s+q,-s|\mathrm{V}|-k+q,k\rangle$$

Decomposing the two propagators into scales and then bounding the integral by the supremum of the integrand times the volume of the support of the integrand, we have

$$\begin{aligned}
|B(s,t,q)| &= \left| \sum_{j_1,j_2\le 0} \int_{\mathcal{B}} đk\ C_{j_1}C_{j_2}\,\langle -k+q,k|\mathrm{V}|t+q,-t\rangle\,\langle s+q,-s|\mathrm{V}|-k+q,k\rangle \right| \\
&\le \sum_{j_1,j_2\le 0} \|V\|_\infty^2 2^{-j_1-j_2}\mathrm{vol}\big\{\ k\in\mathcal{B}\ \big|\ |ik_0-e(\mathbf{k})|\le 2^{j_1+1}, |i(-k+q)_0-e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\} \\
&\le \sum_{j_1,j_2\le 0} \|V\|_\infty^2 2^{-j_1-j_2}2^{\min\{j_1,j_2\}}\mathrm{vol}\big\{\ \mathbf{k}\in \mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}\quad |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\}
\end{aligned} \tag{III.2}$$

Even without using Hypotheses II and III we can bound the volume

$$\begin{aligned}
&\mathrm{vol}\big\{\ \mathbf{k}\in \mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}\quad |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\} \\
&\le \min\Big\{\mathrm{vol}\big\{\ \mathbf{k}\in \mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}\ \big\}, \mathrm{vol}\big\{\ \mathbf{k}\in \mathrm{B}\ \big|\ |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_1+1}\ \big\}\Big\} \\
&\le \mathrm{const}\ \min\big\{2^{j_2},2^{j_2}\big\} = \mathrm{const}\,2^{\min\{j_1,j_2\}}
\end{aligned} \tag{III.3}$$

This gives

$$\begin{aligned}
\sup_{s,t,q}|B(s,t,q)| &\le \sum_{j_1,j_2\le 0} \mathrm{const}\ \|V\|_\infty^2 2^{-j_1-j_2}2^{2\min\{j_1,j_2\}} \\
&= \sum_{j_1,j_2\le 0} \mathrm{const}\ \|V\|_\infty^2 2^{-|j_1-j_2|} \\
&= \sum_{j_1\le 0} \mathrm{const}\ \|V\|_\infty^2
\end{aligned}$$

which diverges logarithmically. Recall that 2^j has the units of energy.

In the event that $e(\mathbf{k}) = e(-\mathbf{k})$, violating Hypothesis III, and $\mathbf{q} = 0$ we have

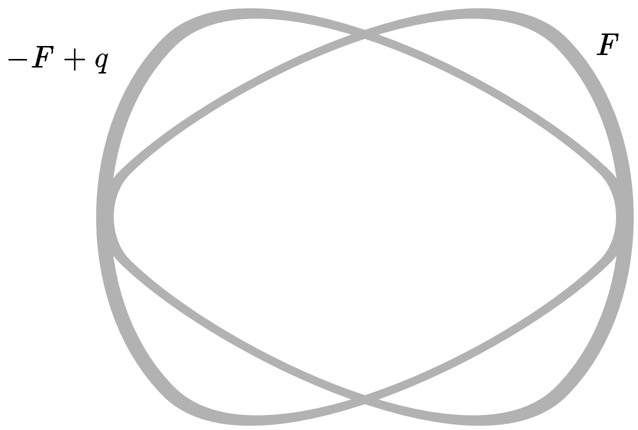
$$\begin{aligned}
&\mathrm{vol}\left\{\ \mathbf{k} \in \mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1} \quad |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \right\} \\
&= \mathrm{vol}\left\{\ \mathbf{k} \in \mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{\min\{j_1,j_2\}+1}\ \right\} \\
&= O\left(2^{\min\{j_1,j_2\}+1}\right)
\end{aligned}$$

and (III.3) is saturated. In this case $q = 0$ really is an exceptional momentum for $B(q)$ which really does have a logarithmic singularity at $q = 0$.

We now turn on Hypotheses II,III and show that then the above bound is not saturated and that four legged subgraphs are really convergent so that the model really acts superrenormalizable. By Hypotheses III and analyticity (or even with just Hypothesis II if $\mathbf{q} \neq 0$) the Fermi curve F can only meet the reflected translated Fermi curve $-F + \mathbf{q}$ transversely or with a tangency of some finite order. Hence there is an $\epsilon > 0$ such that

$$\mathrm{vol}\left\{\ \mathbf{k} \in \mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1} \quad |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \right\} \le \mathrm{const}\, 2^{\min\{j_1,j_2\}} 2^{\epsilon \max\{j_1,j_2\}} \tag{III.3a}$$

Here const $2^{\min\{j_1,j_2\}}$ is the thickness of each component of the intersection of the two shells and const $2^{\epsilon\max\{j_1,j_2\}}$ is a bound on the length of each component.



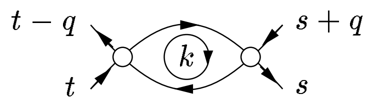
Substituting (III.3a) into (III.2) gives

$$\sup_{s,t,q} |B(s,t,q)| \le \sum_{j_1,j_2 \le 0} \mathrm{const}\, \|V\|_\infty^2 2^{-j_1-j_2} 2^{2\min\{j_1,j_2\}} 2^{\epsilon\max\{j_1,j_2\}}$$

$$
\begin{aligned}
&= \sum_{j_1, j_2 \leq 0} \text{const } \|V\|_\infty^2 2^{-|j_1 - j_2|} 2^{\epsilon \max\{j_1, j_2\}} \\
&= \sum_{j \leq 0} \text{const } \|V\|_\infty^2 2^{\epsilon j} \\
&< \infty
\end{aligned}
$$

When Hypothesis III is turned on the particle-particle bubble becomes uniformly bounded.

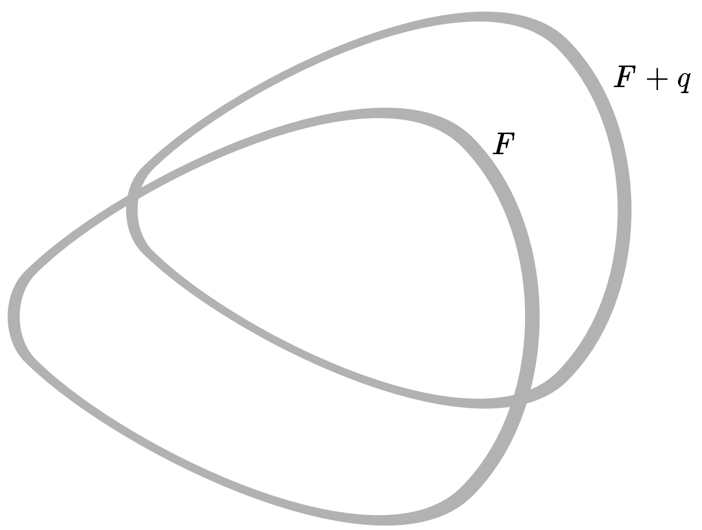
Of course the particle-particle bubble is just one graph. As a second example we consider the second most important graph in our class of models – the particle hole bubble



The value of this bubble is

$$B_2(s, t, q) = \int đk \; C(k+q) C(k) \langle t - q, k | \mathrm{V} | k, t \rangle \langle s, k | \mathrm{V} | k + q, s + q \rangle$$

When Hypothesis II is satisfied and when q is bounded away from zero, we can apply the same argument as in the particle-particle bubble, now using the fact that shells around F and $F + q$ have small intersections.



As an illustration of what happens when q is small, consider $q = 0$. Then, changing variables to

$$x = k_0$$

$$y = e(\mathbf{k})$$

and some angular variable and performing the integral over the angular variable, we have

$$\begin{aligned} B_2(s,t,0) &= \int đk \ \frac{1}{[ik_0 - e(\mathbf{k})]^2} \ \langle t,k|\mathrm{V}|k,t\rangle \ \langle s,k|\mathrm{V}|k,s\rangle \\ &= \int dxdy \ \frac{1}{[ix-y]^2} I(x,y) \end{aligned}$$

with $I(x,y)$ being some C^∞ function. Making the further change of variables to polar coordinates

$$\begin{aligned} B_2(s,t,0) &= \int drd\theta \ \frac{r}{i[re^{i\theta}]^2} I(r\cos\theta, r\sin\theta) \\ &= \int drd\theta \ \frac{r}{i[re^{i\theta}]^2} [I(0,0) + O(r)] \end{aligned}$$

The potentially logarithmically divergent term

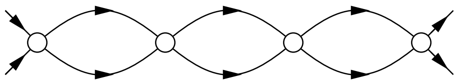
$$\int drd\theta \ \frac{1}{ire^{2i\theta}} I(0,0)$$

vanishes because

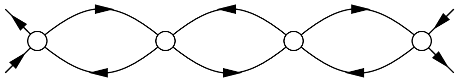
$$\int_0^{2\pi} d\theta \ e^{in\theta} = 0$$

for all nonzero integers n. Hence $B_2(s,t,0)$ is bounded.

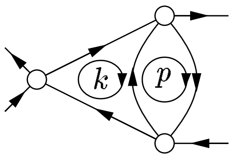
Higher order graphs fall into two categories. There are strings of bubbles, like



and



that can be treated as above. And there are graphs like



which have overlapping loops. Because the k-loop and the p-loop share a line, all three propagators $C^{(j_1)}(k)C^{(j_2)}(p)C^{(j_3)}(p-k)$ appear in the integrand. The supports properties of these propagators constrains the domain of integration to

$$\left\{ \ (\mathbf{k},\mathbf{p}) \in \mathrm{B} \times \mathrm{B} \ \middle| \ |e(\mathbf{k})| \le 2^{j_1+1}, \ |e(\mathbf{p})| \le 2^{j_2+1}, \ |e(\mathbf{p}-\mathbf{k})| \le 2^{j_3+1} \ \right\}$$

The third condition gives some "volume improvement" over naive power counting. See [FKLT1, FKST1]. So all four legged subdiagrams have convergent, rather than marginal, power counting.

References

[FKLT1] J. Feldman, H. Knörrer, D. Lehmann and E. Trubowitz, in preparation.

[FKST1] J. Feldman, H. Knörrer, M. Salmhofer and E. Trubowitz, Renormalization Theory for Many-Fermion Systems: One-Band Models with Non-Nested Fermi Surfaces, in preparation.

[FMRT1] J. Feldman, J. Magnen, V. Rivassseau and E. Trubowitz, An Infinite Volume Expansion for Many Fermion Green's Functions, Helvetica Physica Acta, **65** (1992) 679-721.

[FMRT2] J. Feldman, J. Magnen, V. Rivassseau and E. Trubowitz, Fermionic Many-Body Models, in *Mathematical Quantum Theory I: Field Theory and Many-Body Theory*, J. Feldman, R. Froese and L. Rosen eds, CRM Proceedings & Lecture Notes.

[FMRT3] J. Feldman, J. Magnen, V. Rivassseau and E. Trubowitz, Two Dimensional Many Fermion Systems as Vector Models, Europhysics Letters, **24** (1993) 521-526.

[FMRT4] J. Feldman, J. Magnen, V. Rivassseau and E. Trubowitz, An Intrinsic 1/N Expansion for Many Fermion Systems, Europhysics Letters, **24** (1993) 437-442.

[FT1] J. Feldman and E. Trubowitz, Perturbation Theory for Many Fermion Systems, Helvetica Physica Acta **63** (1990) 156-260.

[FT2] J. Feldman and E. Trubowitz, The Flow of an Electron-Phonon System to the Superconducting State, Helvetica Physica Acta, **64** (1991) 214-357.