CONSTRUCTION OF A 2-D FERMI LIQUID

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The temperature zero renormalized perturbation expansions of a class of interacting many-fermion models in two space dimensions have nonzero radius of convergence. The models have "asymmetric" Fermi surfaces and short range interactions. One consequence of the convergence is the existence of a discontinuity in the particle number density at the Fermi surface. Here we describe the main results and highlight some of the strategy of the construction.

1. The Results

The concept of a Fermi liquid was introduced by L. D. Landau in [16, 17, 18] and has become the generally accepted explanation for the unexpected success of the independent electron approximation. An elementary sketch of Landau's well known physical arguments can be found in [4, pp 345–351]. More thorough and technical discussions are presented in [1, pp 154-203] and [22].

Roughly speaking, at temperature zero, the single particle excitations of a noninteracting Fermi gas become (almost stable) 'quasi–particles' in a Fermi liquid. The quasi–particle spectrum has the 'same structure' as the noninteracting single particle excitation spectrum and the quasi–particle density function $n(\mathbf{k})$ still has a jump at the 'Fermi surface'. The quasi–particle interaction at temperature zero is encoded in Landau's f–function $f(\mathbf{k}_F, \mathbf{k}'_F)$.

It is well known that there are a number of potential instabilities that can drive an interacting Fermi gas away from the Fermi liquid state. See, for example, [21, §1.2,4.5]. One of the most celebrated is the BCS instability for the formation of Cooper pairs leading to superconductivity in 2 and 3 dimensions. This is a potential instability for any time reversal invariant system [15, 19].

Another important instability is the Luttinger instability. There are solvable models in one space dimension that exhibit qualitatively different behavior from that of a three dimensional Landau Fermi liquid. In particular, the quasi–particle density function $n(\mathbf{k})$ is continuous across the 'Fermi surface' but has infinite slope there. These systems are called Luttinger liquids. For a rigorous treatment of Luttinger liquids in one dimension, see [5] and the references therein.

Anderson [2, 3] suggested that a two dimensional Fermi gas should exhibit behavior

similar to a one dimensional Luttinger liquid. [11, Theorem I.4] rigorously shows that this is not the case for a specific class of models. In particular, we show that, at temperature zero, the density function $n(\mathbf{k})$ has a jump discontinuity across the Fermi surface [11, Theorem I.5]. The existence of the Landau f-function and its basic regularity properties follow directly from [11, Theorem I.7]. For results concerning Fermi liquids at strictly positive temperature, see [7, 8, 23, 24].

The class of models that we consider is somewhat unusual in that the Fermi surface survives the turning on of all sufficiently weak short range interactions. To motivate the class, consider a gas of fermions with prescribed, strictly positive, density, together with a crystal lattice of magnetic ions. The fermions interact with each other through a two-body potential. The lattice provides periodic scalar and vector background potentials. As well, the ions can oscillate, generating phonons and then the fermions interact with the phonons. At the present time our result is restricted to $d = 2$ space dimensions. But we believe that the difficulties preventing the extension to $d = 3$ are technical rather than physical. Indeed, there has already been some progress in this direction [20, 6].

To start, turn off the fermion–fermion and fermion–phonon interactions. Then we have a gas of independent fermions, each with Hamiltonian

$$H_0 = \tfrac{1}{2m} \left(i\boldsymbol{\nabla} + \mathbf{A}(\mathbf{x})\right)^2 + U(\mathbf{x})$$

We assume that the vector and scalar potentials $\mathbf{A}$, U are periodic with respect to some lattice Γ in $\mathbb{R}^2$. Note that it is the magnetic potential, and not just the magnetic field, that is assumed to be periodic. This forces the magnetic field to have mean zero. By convention, bold face characters are two component vectors. Because the Hamiltonian commutes with lattice translations it is possible to simultaneously diagonalize the Hamiltonian and the generators of lattice translations. Call the eigenvalues and eigenvectors $\varepsilon_\nu(\mathbf{k})$ and $\phi_{\nu,\mathbf{k}}(\mathbf{x})$ respectively. They obey

$$\begin{aligned} H_0\phi_{\nu,\mathbf{k}}(\mathbf{x}) &= \varepsilon_\nu(\mathbf{k})\phi_{\nu,\mathbf{k}}(\mathbf{x}) \\ \phi_{\nu,\mathbf{k}}(\mathbf{x}+\boldsymbol{\gamma}) &= e^{i<\mathbf{k},\boldsymbol{\gamma}>}\phi_{\nu,\mathbf{k}}(\mathbf{x}) \qquad\qquad \forall\, \boldsymbol{\gamma} \in \Gamma \end{aligned} \tag{1}$$

The crystal momentum $\mathbf{k}$ runs over $\mathbb{R}^2/\Gamma^\#$ where

$$\Gamma^\# = \left\{\ \mathbf{b} \in \mathbb{R}^2 \ \middle|\ <\mathbf{b},\boldsymbol{\gamma}> \in 2\pi\mathbb{Z} \text{ for all } \boldsymbol{\gamma} \in \Gamma\ \right\}$$

is the dual lattice to Γ. The band index $\nu \in \mathbb{N}$ just labels the eigenvalues for boundary condition $\mathbf{k}$ in increasing order. When $\mathbf{A} = U = 0$, $\varepsilon_\nu(\mathbf{k}) = \frac{1}{2m}(\mathbf{k} - \mathbf{b}_{\nu,\mathbf{k}})^2$ for some $\mathbf{b}_{\nu,\mathbf{k}} \in \Gamma^\#$.

In the grand canonical ensemble, the Hamiltonian H is replaced by $H - \mu N$ where N is the number operator and the chemical potential μ is used to control the density of the gas. At very low temperature, which is the physically interesting domain, only those pairs $\nu, \mathbf{k}$ for which $\varepsilon_\nu(\mathbf{k}) \approx \mu$ are important. To keep things as simple as possible, we assume that $\varepsilon_\nu(\mathbf{k}) \approx \mu$ only for one value ν_0 of ν and we fix an ultraviolet cutoff so that we consider only those crystal momenta in a region B for which $|\varepsilon_{\nu_0}(\mathbf{k}) - \mu|$ is smaller than some fixed small constant. We denote $E(\mathbf{k}) = \varepsilon_{\nu_0}(\mathbf{k}) - \mu$.

When the fermion–fermion and fermion–phonon interactions are turned on, the models at temperature zero are characterized by the Euclidean Green's functions, formally defined

by

$$G_{2n}(p_1,\cdots,q_n)\delta(\Sigma p_i-\Sigma q_i)=\left\langle\prod_{i=1}^{n}\psi_{p_i}\bar\psi_{q_i}\right\rangle=\frac{\int\,\big(\prod_{i=1}^{n}\psi_{p_i}\bar\psi_{q_i}\big)\,e^{\mathcal{A}(\psi,\bar\psi)}\,\prod_{k,\sigma}d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)}\,\prod_{k,\sigma}d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}}\tag{2}$$

The action

$$\mathcal{A}(\psi,\bar\psi)=-\int dk\;\big(ik_0-E(\mathbf{k})\big)\bar\psi_k\psi_k\;+\;\mathcal{V}(\psi,\bar\psi)\tag{3}$$

The interaction $\mathcal{V}$ will be specified shortly. We prefer to split $\mathcal{A}=\mathcal{Q}+\mathcal{V}$ where $\mathcal{Q}=-\int dk\;\big(ik_0-E(\mathbf{k})\big)\bar\psi_k\psi_k$ and write

$$\begin{aligned}\big\langle f(\psi,\bar\psi)\big\rangle&=\frac{\int\,f(\psi,\bar\psi)\,e^{\mathcal{A}(\psi,\bar\psi)}\,\prod_{k,\sigma}d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)}\,\prod_{k,\sigma}d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}}\\&=\frac{\int\,f(\psi,\bar\psi)\,e^{\mathcal{V}(\psi,\bar\psi)}\,d\mu_C(\psi,\bar\psi)}{\int e^{\mathcal{V}(\psi,\bar\psi)}\,d\mu_C(\psi,\bar\psi)}\end{aligned}$$

where $d\mu_C$ is the Grassmann gaussian "measure" with covariance

$$C(k)=\tfrac{1}{ik_0-E(\mathbf{k})}$$

We have here dropped some factors of 2π. We will continue to routinely drop various unimportant constants through this article.

We now take some time to explain (2). The fermion fields are vectors

$$\psi_k=\begin{bmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{bmatrix}\qquad\bar\psi_k=\begin{bmatrix}\bar\psi_{k,\uparrow}\;\bar\psi_{k,\downarrow}\end{bmatrix}$$

whose components $\psi_{k,\sigma},\bar\psi_{k,\sigma},\ k=(k_0,\mathbf{k})\in\mathbb{R}\times\mathrm{B},\ \sigma\in\{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other.

$$\overset{(-)}{\psi}_{k,\sigma}\overset{(-)}{\psi}_{p,\tau}=-\overset{(-)}{\psi}_{p,\tau}\overset{(-)}{\psi}_{k,\sigma}$$

We have deliberately chosen $\bar\psi$ to be a row vector and ψ to be a column vector so that

$$\bar\psi_k\psi_p=\bar\psi_{k,\uparrow}\psi_{p,\uparrow}+\bar\psi_{k,\downarrow}\psi_{p,\downarrow}\qquad\psi_k\bar\psi_p=\begin{bmatrix}\psi_{k,\uparrow}\bar\psi_{p,\uparrow}\;\psi_{k,\uparrow}\bar\psi_{p,\downarrow}\\ \psi_{k,\downarrow}\bar\psi_{p,\uparrow}\;\psi_{k,\downarrow}\bar\psi_{p,\downarrow}\end{bmatrix}$$

In the argument $k=(k_0,\mathbf{k})$, the last d components $\mathbf{k}$ are to be thought of as a crystal momentum and the first component k_0 as the dual variable to an imaginary time. Hence the $\sqrt{-1}$ in $ik_0-E(\mathbf{k})$. Our ultraviolet cutoff restricts $\mathbf{k}$ to B. In the full model, $\mathbf{k}$ is replaced by $(\nu,\mathbf{k})$ with ν summed over $\mathbb{N}$ and $\mathbf{k}$ integrated over $\mathbb{R}^d/\Gamma^{\#}$. On the other hand, the ultraviolet cutoff does not restrict k_0 at all. It still runs over $\mathbb{R}$. So we could equally well express the model in terms of a Hamiltonian acting on a Fock space. We find the functional integral notation more efficient, so we use it. The relationship between the position space field $\psi(x)$, with $x=(t,\mathbf{x})$ running over (imaginary)time×space, and the momentum space field ψ_k is really given, in our single band approximation, by

$$\psi(x)=\int dk\;e^{ik_0t}\overline{\phi_{\nu_0,\mathbf{k}}(\mathbf{x})}\psi_k$$

We find it convenient to use a conventional Fourier transform, so we work in a "pseudo" space–time and instead define

$$\psi(x) = \int dk\ e^{ik\cdot x}\psi_k$$

Under suitable conditions on $\phi_{\nu_0,\mathbf{k}}(\mathbf{x})$, it is easy to go from the pseudo space–time $\psi(x)$ to the real one.

For a simple two–body fermion–fermion interaction, with no phonon interaction,

$$\mathcal{V} = -\tfrac{1}{2}\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \int dt d\mathbf{x} d\mathbf{y}\ u(\mathbf{x}-\mathbf{y})\bar\psi_\sigma(t,\mathbf{x})\psi_\sigma(t,\mathbf{x})\bar\psi_\tau(t,\mathbf{y})\psi_\tau(t,\mathbf{y})$$

The general spin independent form of the interaction is

$$\begin{aligned}\mathcal{V}(\psi,\bar\psi) &= \tfrac{1}{2}\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \int \prod_{i=1}^4 dx_i\, V(x_1,x_2,x_3,x_4)\bar\psi_\sigma(x_1)\psi_\sigma(x_3)\bar\psi_\tau(x_2)\psi_\tau(x_4)\\ &= \tfrac{1}{2}\int \prod_{i=1}^4 dk_i\ \delta(k_1+k_2-k_3-k_4)\ \bar\psi_{k_1}\psi_{k_3}\ \langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\ \bar\psi_{k_2}\psi_{k_4}\end{aligned}$$

Spin independence is imposed purely for notational convenience. It plays no role. The function $V(x_1,x_2,x_3,x_4)$, or equivalently $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$, can implement both the fermion–fermion and fermion–phonon interactions. Its precise value does not concern us. We just assume

Hypothesis 1.1. *The interaction is weak and short range. That is, V is sufficiently near the origin in $\mathfrak{V}$, which is a Banach space of fairly short range, translation invariant functions $V(x_1,x_2,x_3,x_4)$. See [11, Theorem I.4] for $\mathfrak{V}$'s precise norm.*

For some results, we also assume that V is "k_0–reversal real"

$$V(Rx_1,Rx_2,Rx_3,Rx_4) = \overline{V(x_1,x_2,x_3,x_4)} \tag{4}$$

where $R(x_0,\mathbf{x}) = (x_0,-\mathbf{x})$ and "bar/unbar exchange invariant"

$$V(-x_2,-x_1,-x_4,-x_3) = V(x_1,x_2,x_3,x_4) \tag{5}$$

If V corresponds to a two–body interaction $u(\mathbf{x}_1-\mathbf{x}_3)$ with a real–valued Fourier transform, then V obeys (4) and (5).

Our goal is to prove that perturbation expansions for various objects converge. These objects depend on both $E(\mathbf{k})$ and V and are **not** smooth in V when $E(\mathbf{k})$ is held fixed. However, we can recover smoothness in V by a change of variables. To do so, we split $E(\mathbf{k}) = e(\mathbf{k}) - \delta e(V,\mathbf{k})$ into two parts and choose $\delta e(V,\mathbf{k})$ to satisfy an implicit renormalization condition. This is called renormalization of the dispersion relation. Define the proper self energy $\Sigma(p)$ for the action $\mathcal{A}$ by the equation

$$\Big(ip_0 - e(\mathbf{p}) - \Sigma(p)\Big)^{-1}\ \delta(p-q)\ =\ \frac{\int \psi_p\bar\psi_q\ \ e^{\mathcal{A}(\psi,\bar\psi)}\ \prod d\psi_{k,\sigma}d\bar\psi_{k,\sigma}}{\int\ e^{\mathcal{A}(\psi,\bar\psi)}\ \prod d\psi_{k,\sigma}d\bar\psi_{k,\sigma}}$$

The counterterm $\delta e(V,\mathbf{k})$ is chosen so that $\Sigma\big(0,\mathbf{p}\big)$ vanishes on the Fermi surface $F = \big\{\ \mathbf{p}\ \big|\ e(\mathbf{p}) = 0\ \big\}$. We take $e(\mathbf{k})$ and V, rather than the more natural, $E(\mathbf{k})$ and V as input data. The counterterm δe will be an output of our main theorem. It will lie in a suitable

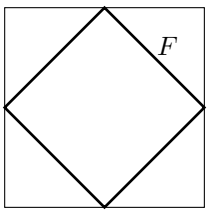
Banach space $\mathcal{E}$. While the problem of inverting the map $e \mapsto E = e - \delta e$ is reasonably well understood on a perturbative level [13], our estimates are not yet good enough to do so nonperturbatively. Our main hypotheses are imposed on $e(\mathbf{k})$.

Hypothesis 1.2. *The dispersion relation $e(\mathbf{k})$ is a real-valued, sufficiently smooth, function. We further assume that*

- *(a) the Fermi curve $F = \{ \mathbf{k} \in \mathbb{R}^2 \mid e(\mathbf{k}) = 0 \}$ is a simple closed, connected, convex curve with nowhere vanishing curvature.*
- *(b) $\nabla e(\mathbf{k})$ does not vanish on F.*
- *(c) For each $\mathbf{q} \in \mathbb{R}^2$, F and $-F + \mathbf{q}$ have low degree of tangency. (F is "strongly asymmetric".) Here $-F + q = \{ -\mathbf{k} + \mathbf{q} \mid \mathbf{k} \in F \}$.*

Again, for the details, see [11, Hypothesis I.12].

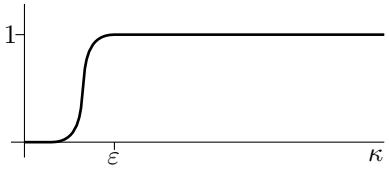
It is the strong asymmetry condition, Hypothesis 1.2.c, that makes this class of models somewhat unusual and permits the system to remain a Fermi liquid when the interaction is turned on. If $\mathbf{A} = 0$ then, taking the complex conjugate of (1), we see that $\varepsilon_\nu(-\mathbf{k}) = \varepsilon_\nu(\mathbf{k})$ so that Hypothesis 1.2.c is violated for $\mathbf{q} = 0$. Hence the presence of a nonzero vector potential $\mathbf{A}$ is essential. We shall say more about the role of strong asymmetry later. For now, we just mention one model that violates these hypotheses, not only for technical reasons but because it exhibits different physics. It is the Hubbard model at half filling, whose Fermi surface looks like



This Fermi curve is not smooth, violating Hypothesis 1.2.b, has zero curvature almost everywhere, violating Hypothesis 1.2.a, and is invariant under $\mathbf{k} \to -\mathbf{k}$ so that $F = -F$, violating Hypothesis 1.2.c with $\mathbf{q} = 0$.

To give a rigorous definition of (I.2) one must introduce cutoffs and then take the limit in which the cutoffs are removed. To impose an infrared cutoff in the spatial directions one could put the system in a finite periodic box $\mathbb{R}^d / L\Gamma$. To impose an ultraviolet cutoff in the spatial directions one may put the system on a lattice. By also imposing infrared and ultraviolet cutoffs in the temporal direction, we could arrange to start from a finite dimensional Grassmann algebra. We choose not to do so. Our goal is to prove that formal renormalized perturbation expansions converge. The coefficients in those expansions are well–defined even without a finite volume cutoff. So we choose to start with x running over all $\mathbb{R}^3$. We impose a (permanent) ultraviolet cutoff through a smooth compactly supported function $U(\mathbf{k})$. This keeps $\mathbf{k}$ permanently bounded. We impose a (temporary) infrared

cutoff through a function $\nu_\varepsilon\big(k_0^2 + e(\mathbf{k})^2\big)$ where $\nu_\varepsilon(\kappa)$ looks like



When $\varepsilon > 0$ and $\nu_\varepsilon\big(k_0^2 + e(\mathbf{k})^2\big) > 0$, $|ik_0 - e(\mathbf{k})|$ is at least of order ε. The coefficients of the perturbation expansion (either renormalized or not) of the cutoff Euclidean Green's functions

$$G_{2n;\varepsilon}(x_1, \sigma_1, \cdots, y_n, \tau_n) = \left\langle \prod_{i=1}^{n} \psi_{\sigma_i}(x_i) \bar{\psi}_{\tau_i}(y_i) \right\rangle_\varepsilon$$

where

$$\langle f \rangle_\varepsilon = \frac{\int f(\psi, \bar{\psi})\, e^{\mathcal{V}(\psi,\bar{\psi})}\, d\mu_{C_\varepsilon}(\psi, \bar{\psi})}{\int e^{\mathcal{V}(\psi,\bar{\psi})}\, d\mu_{C_\varepsilon}(\psi, \bar{\psi})} \qquad \text{with} \qquad C_\varepsilon(k; \delta e) = \tfrac{U(\mathbf{k})\nu_\varepsilon(k_0^2 + e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k}) + \delta e(\mathbf{k})}$$

are well–defined. Our main result is

Theorem 1.1. ***[11, Theorem I.4]***

Assume that $d = 2$ and that $e(\mathbf{k})$ fulfils Hypothesis 1.2. There is

- *a nontrivial open ball $\mathcal{B} \subset \mathfrak{V}$, centered on the origin, and*
- *an analytic function $V \in \mathcal{B} \mapsto \delta e(V) \in \mathcal{E}$, that vanishes for $V = 0$,*

such that:

- *For any $\varepsilon > 0$ and $n \in \mathbb{N}$, the formal Taylor series for the Green's functions $G_{2n;\varepsilon}$ converges to an analytic function on $\mathcal{B}$.*
- *As $\varepsilon \to 0$, $G_{2n;\varepsilon}$ converges uniformly, in $x_1, \cdots, y_n$ and $V \in \mathcal{B}$, to a translation invariant, spin independent, particle number conserving function G_{2n} that is analytic in V.*

If, in addition, V is k_0–reversal real, as in (4), then $\delta e(\mathbf{k}; V)$ is real for all $\mathbf{k}$.

Theorem 1.2. ***[11, Theorem I.5]***

Under the hypotheses of Theorem 1.1 and the assumption that $V \in \mathcal{B}$ obeys the symmetries (4) and (5), the Fourier transform

$$\begin{aligned}\hat{G}_2(k_0, \mathbf{k}) &= \int dx_0 d^d\mathbf{x}\; e^{\imath(-k_0 x_0 + \mathbf{k}\cdot\mathbf{x})}\, G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big) \\ &= \int dx_0 d^d\mathbf{x}\; e^{\imath(-k_0 x_0 + \mathbf{k}\cdot\mathbf{x})}\, G_2\big((0,0,\downarrow),(x_0,\mathbf{x},\downarrow)\big) \\ &= \tfrac{1}{ik_0 - e(\mathbf{k}) - \Sigma(k)} \text{ when } U(\mathbf{k}) = 1\end{aligned}$$

of the two–point function exists and is continuous, except on the Fermi surface (precisely, except when $k_0 = 0$ and $e(\mathbf{k}) = 0$). The momentum distribution function

$$n(\mathbf{k}) = \lim_{\tau \to 0+} \int \tfrac{dk_0}{2\pi}\; e^{\imath k_0 \tau}\; \hat{G}_2(k_0, \mathbf{k})$$

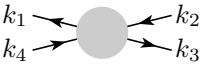
is continuous except on the Fermi surface F. If $\bar{\mathbf{k}} \in F$, then $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k})$ *and* $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k})$ *exist and obey*

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k}) = 1 + O\big(V\big) > \tfrac{1}{2}$$

Theorem 1.3. ***[11, Theorem I.7]***

Let

$$\hat{G}_4(k_1,k_2,k_3,k_4) =$$



(spin dropped from notation) be the Fourier transform of the four–point function and

$$\hat{G}_4^A(k_1,k_2,k_3,k_4) = \hat{G}_4(k_1,k_2,k_3,k_4) \prod_{\ell=1}^{4} \tfrac{1}{\hat{G}_2(k_\ell)}$$

its amputation by the physical propagator. Under the hypotheses of Theorem 1.2, $\hat{G}_4^A$ has a decomposition

$$\begin{aligned}\hat{G}_4^A(k_1,k_2,k_3,k_4) = N(k_1,k_2,k_3,k_4) \\ + \tfrac{1}{2} L\big(\tfrac{k_1+k_2}{2}, \tfrac{k_3+k_4}{2}, k_2-k_1\big) \\ - \tfrac{1}{2} L\big(\tfrac{k_3+k_2}{2}, \tfrac{k_1+k_4}{2}, k_2-k_3\big)\end{aligned}$$

with

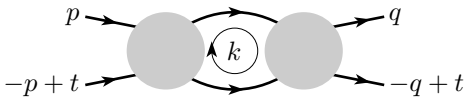
- *N continuous*
- *$L(q_1,q_2,t)$ continuous except at $t=0$*
- *$\lim_{t_0\to 0} L(q_1,q_2,t)$ continuous*
- *$\lim_{\mathbf{t}\to 0} L(q_1,q_2,t)$ continuous*

Think of L as a particle–hole ladder

$$L(q_1,q_2,t) =$$

2. Blocking the Cooper Channel

We now discuss further the role of the geometric conditions of Hypothesis 1.2 in blocking the Cooper channel. When you turn on the interaction V, the system itself effectively replaces V by more complicated "effective interaction". The (dominant) contribution



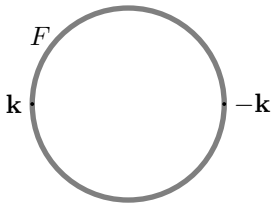
to the strength of the effective interaction between two particles of total momentum $t = p_1 + p_2 = q_1 + q_2$ is

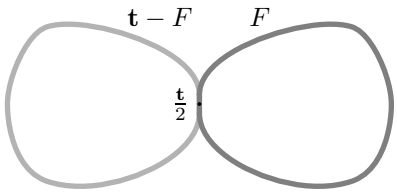
$$\int dk\, \frac{\text{stuff}}{[ik_0 - e(\mathbf{k})][i(-k_0+t_0) - e(-\mathbf{k}+\mathbf{t})]}$$

Note that

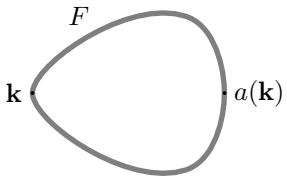
$$\begin{aligned}
[ik_0 - e(\mathbf{k})] = 0 &\iff k_0 = 0,\ e(\mathbf{k}) = 0 \\
&\iff k_0 = 0,\ \mathbf{k} \in F \\
[i(-k_0 + t_0) - e(-\mathbf{k} + \mathbf{t})] = 0 &\iff k_0 = t_0,\ e(-\mathbf{k} + \mathbf{t}) = 0 \\
&\iff k_0 = t_0, \mathbf{k} \in \mathbf{t} - F
\end{aligned}$$

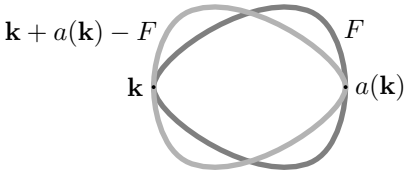
We can transform $\frac{1}{ik_0 - e(\mathbf{k})}$ locally to $\frac{1}{ik_0 - k_1}$ by a simple change of variables. Thus $\frac{1}{ik_0 - e(\mathbf{k})}$ is locally integrable, but is not locally L^2. So the strength of the effective interaction diverges when the total momentum t obeys $t_0 = 0$ and $F = \mathbf{t} - F$, because then the singular locus of $\frac{1}{ik_0 - e(\mathbf{k})}$ coincides with the singular locus of $\frac{1}{i(-k_0+t_0) - e(-\mathbf{k}+\mathbf{t})}$. This always happens when $F = -F$ (for example, when F is a circle) and $t = 0$. Similarly the strength of the effective interaction diverges when F has a flat piece and $\mathbf{t}/2$ lies in that flat piece, as in the figure on the right below. On the other hand, when F is strongly asymmetric, F and $\mathbf{t} - F$ always





intersect only at isolated points. A "worst" case is illustrated below. There the antipode, $a(\mathbf{k})$, of $\mathbf{k} \in F$, is the unique point of F, different from $\mathbf{k}$, such that the tangents to F at $\mathbf{k}$ and $a(\mathbf{k})$ are parallel.





For strongly asymmetric Fermi curves, $\frac{1}{[ik_0 - e(\mathbf{k})][i(-k_0+t_0) - e(-\mathbf{k}+\mathbf{t})]}$ remains locally integrable in k for each fixed $\mathbf{t}$ and strength of the effective interaction remains bounded.

3. Power Counting and Nonperturbative Bounds

The proofs of Theorems 1.1, 1.2 and 1.3 are quite technical. The whole construction is given in a series of papers [10]. In the first paper, [11], of the series, the main difficulties and our strategies to overcome them are described. Here, we concentrate on one aspect, namely the need to use both position space and momentum space arguments and the problems created by the interplay between them.

The Green's functions G_{2n} are constructed using a multiscale analysis and renormalization. The multiscale analysis is introduced by choosing a parameter $M > 1$ and decomposing momentum space into a family of shells, with the j^{th} shell consisting of those momenta k obeying $|ik_0 - e(\mathbf{k})| \approx \frac{1}{M^j}$. Correspondingly, we write the covariance as a telescoping series

$C(k) = \sum_{j=0}^{\infty} C^{(j)}(k)$ where, for $j \geq 1$,

$$C^{(j)} = C_{M^{-j}} - C_{M^{-j+1}}$$

is the "covariance at scale j". By construction $C^{(j)}(k)$ vanishes unless $\sqrt{k_0^2 + e(\mathbf{k})^2}$ is of order M^{-j}, and $\|C^{(j)}(k)\|_{L^\infty} \approx M^j$.

We consider, for each j, the effective interaction at scale j

$$\mathcal{W}_j(\phi, \bar\phi, \psi, \bar\psi) = \log \tfrac{1}{Z_j} \int e^{\phi J\zeta + \mathcal{V}(\psi+\zeta, \bar\psi+\bar\zeta)} d\mu_{C_{M^{-j}}}(\zeta, \bar\zeta)$$

where the source term $\phi J\zeta = \int dx\ \bar\phi(x)\zeta(x) + \bar\psi(x)\zeta(x)$ and the partition function Z_j is chosen so that $\mathcal{W}_j(0,0,0,0) = 0$. The coefficients in the expansion of $\mathcal{W}_j(\phi, \bar\phi, 0, 0)$ in powers of ϕ are the Euclidean Green's functions $G_{2n;M^{-j}}$. The effective interactions are controlled using the recursion relation

$$\mathcal{W}_{j+1}(\phi, \bar\phi, \psi, \bar\psi) = \log \tfrac{1}{Z} \int e^{\phi J\zeta + \mathcal{W}_j(\phi, \bar\phi, \psi+\zeta, \bar\psi+\bar\zeta)} d\mu_{C^{(j)}}(\zeta, \bar\zeta) \tag{6}$$

The recursion relation (6) is the renormalization group map. The main difficulties in controlling it already arise when $\phi = \bar\phi = 0$. We fix a scale j and consider the passage from $W_j(0,0,\psi,\bar\psi)$ to $W_{j+1}(0,0,\psi,\bar\psi)$.

Expand

$$\begin{aligned}
\mathcal{W}_j(0,0,\psi,\bar\psi) &= \sum_{n\geq 0} \int dx_1 \cdots dx_n\, dy_1 \cdots dy_n\ w_{2n}(x_1,\cdots,x_n,y_1,\cdots,y_n)\, \bar\psi(x_1)\cdots\bar\psi(x_n)\, \psi(y_1)\cdots\psi(y_n) \\
&= \sum_{n\geq 0} \int dp_1 \cdots dp_n\, dq_1 \cdots dq_n\, \delta(p_1 + \cdots + p_n - q_1 - \cdots - q_n) \\
&\qquad\qquad \hat w_{2n}(p_1,\cdots,p_n,q_1,\cdots,q_n)\, \bar\psi_{p_1} \cdots \bar\psi_{p_n}\, \psi_{q_1} \cdots \psi_{q_n}
\end{aligned}$$

with the position space kernels $w_{2n}(x_1,\cdots,x_n,y_1,\cdots,y_n)$ translation invariant and antisymmetric in the x and y variables separately. Similarly, write

$$\begin{aligned}
\mathcal{W}_{j+1}(0,0,\psi,\bar\psi) &= \sum_{n\geq 0} \int dx_1 \cdots dx_n\, dy_1 \cdots dy_n\ w'_{2n}(x_1,\cdots,x_n,y_1,\cdots,y_n)\, \bar\psi(x_1)\cdots\bar\psi(x_n)\, \psi(y_1)\cdots\psi(y_n) \\
&= \sum_{n\geq 0} \int dp_1 \cdots dp_n\, dq_1 \cdots dq_n\, \delta(p_1 + \cdots + p_n - q_1 - \cdots - q_n) \\
&\qquad\qquad \hat w'_{2n}(p_1,\cdots,p_n,q_1,\cdots,q_n)\, \bar\psi_{p_1} \cdots \bar\psi_{p_n}\, \psi_{q_1} \cdots \psi_{q_n}
\end{aligned}$$

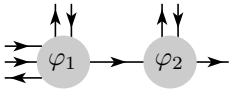
Then w'_{2n} can be written as a sum of values of connected directed Feynman graphs with vertices $w_2, w_4, \cdots$ and propagator $C^{(j)}$. See [14, Chapter 3].

By power counting we mean finding simple j–dependent bounds on appropriate norms of w_{2n} such that all diagrams contributing to w'_{2n} fulfil analogous bounds with j replaced

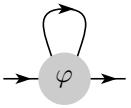
by $j+1$. Also, two and four–legged vertices should remain bounded as $j \to \infty$.[a] The choice of an appropriate system of norms is an important issue in our construction. To treat all orders in perturbation theory, but without worrying about convergence of the series, the supremum norm in momentum space $\|\hat{w}_{2n}\|_{L^\infty}$ seems to be the most convenient norm.

To illustrate power counting, observe that every diagram can be built by successively applying one of the two following procedures.

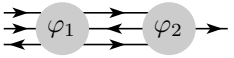
(i) Contraction, that is connecting two disjoint vertices φ_1 and φ_2 by one line to form a new diagram Γ_c.

$\Gamma_c =$ 

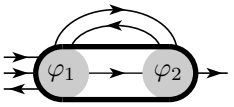
(ii) Tadpole formation, that is connecting one outgoing leg to one incoming leg of a single vertex φ to form a new diagram Γ_t.

$\Gamma_t =$ 

For example, let φ_1 be a six–legged vertex and φ_2 a four–legged vertex. The diagram

$\Gamma =$ 

can be constructed by first contracting once to yield the graph Γ_c above and then forming two tadpole lines.



If, in the contraction described above, φ_1 has $2r$ legs and φ_2 has $2s$ legs, then Γ_c has $2(r+s-1)$ legs and

$$\begin{aligned}&\hat{\Gamma}_c(p_1,\cdots,p_{r+s-1},q_1,\cdots,q_{r+s-1}) \\ &\quad = \hat{\varphi}_1(p_1,\cdots,p_r,q_1,\cdots,q_{r-1},k)\; C^{(j)}(k)\; \hat{\varphi}_2(k,p_{r+1},\cdots,p_{r+s-1},q_r,\cdots,q_{r+s-1})\end{aligned} \tag{7}$$

with $k = p_1+\cdots+p_r-q_1-\cdots-q_{r-1} = q_r+\cdots+q_{r+s-1}-p_{r+1}-\cdots-p_{r+s-1}$. Therefore

$$\|\hat{\Gamma}_c\|_{L^\infty} \le c\; \|\hat{\varphi}_1\|_{L^\infty}\; \|\hat{\varphi}_2\|_{L^\infty} \qquad \text{with} \qquad c = \|C^{(j)}(k)\|_{L^\infty} \approx M^j$$

If, in the tadpole formation described above, φ has $2n$ legs, then Γ_t has $2(n-1)$ legs and

$$\hat{\Gamma}_t(p_1,\cdots,p_{n-1},q_1,\cdots,q_{n-1}) = \int dk\; \hat{\varphi}(p_1,\cdots,p_{n-1},k,q_1,\cdots,q_{n-1},k)\; C^{(j)}(k)$$

Therefore

$$\|\hat{\Gamma}_t\|_{L^\infty} \le b\; \|\hat{\varphi}\|_{L^\infty} \qquad \text{with} \qquad b = \|C^{(j)}(k)\|_{L^1} \approx M^{-j}$$

[a] To achieve this, special effects like renormalization and the special geometry of the Fermi surface, mentioned above, are also used.

since $C^{(j)}(k)$ is supported in a region having volume of order M^{-2j} so that $\|C^{(j)}(k)\|_{L^1} \approx M^{-2j}\|C^{(j)}(k)\|_{L^\infty} \approx M^{-j}$.

In general, if one has a system of norms $\| \cdot \|$ for vertices having arbitrarily many legs, we call c a contraction bound with respect to those norms if, in the contraction described above,

$$\|\Gamma_c\| \leq c \, \|\varphi_1\| \, \|\varphi_2\|$$

Similarly, we call b a tadpole bound if, in the tadpole formation described above,

$$\|\Gamma_t\| \leq b \, \|\varphi\|$$

Standard power counting can be phrased in this language as follows. If one assumes that

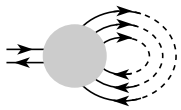
$$\|w_{2n}\| = O\big(\tfrac{1}{cb^{n-1}}\big) \quad \text{for all } n \tag{8}$$

then every graph contributing to w'_{2n} is again of order $\frac{1}{cb^{n-1}}$. For example, if such a graph Γ has two vertices, w_{2n_1} and w_{2n_2} then there are $r = n_1 + n_2 - n$ connecting lines and the norm of Γ is bounded by

$$\begin{aligned} cb^{n_1+n_2-n-1}\|w_{2n_1}\|\|w_{2n_2}\| &= O\big(cb^{n_1+n_2-n-1}\tfrac{1}{cb^{n_1-1}}\tfrac{1}{cb^{n_2-1}}\big) \\ &= O\big(\tfrac{1}{cb^{n-1}}\big) \end{aligned}$$

A general graph may be bounded by building it up one vertex at a time. In the case of the supremum norm in momentum space, $c \approx M^j$ and $b \approx M^{-j}$ so that $\frac{1}{cb^{n-1}} \approx M^{j(n-2)}$.

Problems with the convergence of the perturbation expansion can arise when one builds diagrams from one vertex by forming a large number n of tadpoles.



There are $n!$ choices for connecting the outgoing arrows to the incoming arrows. For each choice one gets a diagram whose norm can be bounded by $b^n\|\varphi\|$. The estimate $n!\, b^n\|\varphi\|$ is too weak to prove convergence for the series for W_{j+1}. However, due to the Pauli exclusion principle, the sum of all $n!$ diagrams obtained in this way has a bound of order $\sqrt{n!}\, b^n\|\varphi\|$.

All known strategies for implementing the Pauli exclusion principle, that is, to exploit the antisymmetry of the kernels, use position space variables. One way of implementing the Pauli exclusion principle is to observe, [9, §2.10, Exercise 4], that the Fourier transform $\hat{f}(k_1, \cdots, k_n)$ of an antisymmetric function $f(x_1, \cdots, x_n)$ fulfils a better bound than the standard estimate

$$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

To see this, note that

$$\begin{aligned}\hat{f}(k_1,\cdots,k_n) &= \int dx_1\cdots dx_n\ e^{\imath k_1\cdot x_1}\cdots e^{\imath k_n\cdot x_n} f(x_1,\cdots,x_n)\\ &= \int dx_1\cdots dx_n\ e^{\imath k_1\cdot x_1}\cdots e^{\imath k_n\cdot x_n}\tfrac{1}{n!}\sum_{\pi\in S_n}\operatorname{sgn}\pi\ f(x_{\pi(1)},\cdots,x_{\pi(n)})\\ &= \tfrac{1}{n!}\int dx_1\cdots dx_n\ f(x_1,\cdots,x_n)\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\ e^{\imath k_1\cdot x_{\sigma(1)}}\cdots e^{\imath k_n\cdot x_{\sigma(n)}}\\ &= \tfrac{1}{n!}\int dx_1\cdots dx_n\ f(x_1,\cdots,x_n)\ D(k_1,\cdots k_n;x_1\cdots x_n)\end{aligned}$$

where

$$D(k_1,\cdots k_n;x_1\cdots x_n)=\det\left[e^{\imath k_i\cdot x_j}\right]_{i,j=1,\cdots,n}$$

The Euclidean length of each column of this matrix is $\sqrt{n}$. Therefore, by Hadamard's estimate, $\left\|D\right\|_{L^\infty}\le(\sqrt{n})^n$ so that

$$\|\hat{f}\|_{L^\infty}\le\tfrac{n^{n/2}}{n!}\|f\|_{L^1}\le\tfrac{\text{const}^n}{\sqrt{n!}}\|f\|_{L^1}$$

This suggests that the L^1 norm in position space — or rather, to break translation invariance

$$\|w_{2n}\|_{1,\infty}=\max_{x_1}\int dx_2\cdots x_n\ |w_{2n}(x_1,\cdots,x_{2n})|$$

— should be used as the principal norm for estimating the effective interaction. Whenever one encounters a situation in which one forms all possible tadpoles between n incoming and n outgoing legs of a kernel φ, one could take the partial Fourier transform of φ with respect to the variables associated to the legs involved in tadpole formation. The argument sketched above gives a factor $\left(\frac{\text{const}^n}{\sqrt{n!}}\right)^2$ relating the norm of the partial Fourier transform to $\|\varphi\|_{1,\infty}$. The number $n!$ of such tadpole terms is compensated for by this factor. In this way one shows that iterated tadpole formation does not create a convergence problem if one uses the $\|\ \cdot\ \|_{1,\infty}$ norm.

However, there are at least two obstacles which prevent us from using this position space norm. One is that special effects like the suppression of the Cooper channel due to the special geometry of the Fermi surface, sketched in §2, is naturally seen in momentum space. The other is that the contraction bound for this norm is worse than for the supremum norm in momentum space. In position space, the contraction (7) is

$$\begin{aligned}&\Gamma_c(x_1,\cdots,x_{r+s-1},y_1,\cdots,y_{r+s-1})\\ &\quad=\int dudu'\ \varphi_1(x_1,\cdots,x_r,y_1,\cdots,y_{r-1},u)\ \check{C}^{(j)}(u-u')\ \varphi_2(u',x_{r+1},\cdots,y_{r+s-1})\end{aligned}$$

where $\check{C}^{(j)}(x)$ is the inverse Fourier transform of $C^{(j)}(k)$. It follows that

$$\|\Gamma_c\|_{1,\infty}\le\left\|\check{C}^{(j)}(x)\right\|_{L^1}\ \|\varphi_1\|_{1,\infty}\ \|\varphi_2\|_{1,\infty}$$

A naive computation, given in the next paragraph, gives a bound on $\|\check{C}^{(j)}(x)\|_{L^1}$ that is of order M^{2j}. A more refined argument, sketched in the [11, §II.7] gives a more realistic bound

of order $M^{3j/2}$. In any event, $M^{3j/2} \gg \|C^{(j)}(k)\|_{L^\infty}$ and naive power counting in position space does not coincide with power counting in momentum space. It is easily seen that $\|\check{C}^{(j)}(x)\|_{L^\infty}$ is a tadpole bound for the $\|\cdot\|_{1,\infty}$ norm. Clearly, $\|\check{C}^{(j)}(x)\|_{L^\infty} \le \|C^{(j)}(k)\|_{L^1}$, so that we again have a tadpole bound of order $\frac{1}{M^j}$. Substituting $c = O\big(M^{3j/2}\big)$ and $b = O\big(\frac{1}{M^j}\big)$ into (8) yields the requirement that $\|w_{2n}\|_{1,\infty}$ be order $M^{j(n-\frac{5}{2})}$. In particular the norm of the four point function would have to decrease like $\frac{1}{\sqrt{M^j}}$ as j increased. This is absurd, since the original interaction V is, at each scale, the dominant part of the four point function.

We sketch the standard calculation that gives the naive bound on $\|\check{C}^{(j)}(x)\|_{L^1}$. For a multi index $\delta = (\delta_0, \delta_1, \delta_2)$ of non negative integers write $|\delta| = \delta_0+\delta_1+\delta_2$ and $x^\delta = x_0^{\delta_0} x_1^{\delta_1} x_2^{\delta_2}$. Then, integrating by parts $|\delta|$ times,

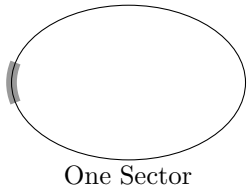
$$\big(\tfrac{x}{M^j}\big)^\delta |\check{C}^{(j)}(x)| \le \tfrac{1}{M^{j|\delta|}} \|\tfrac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)\|_1 = O(\tfrac{1}{M^j})$$

since the support of $\frac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)$ has volume of order $\frac{1}{M^{2j}}$ and $\|\frac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)\|_{L^\infty}$ is of order $M^{j(|\delta|+1)}$. Therefore

$$\Big(1 + \big(\tfrac{x_0}{M^j}\big)^2\Big)\Big(1 + \big(\tfrac{x_1}{M^j}\big)^2\Big)\Big(1 + \big(\tfrac{x_2}{M^j}\big)^2\Big) |\check{C}^{(j)}(x)| = O(\tfrac{1}{M^j})$$

Dividing by $\prod_{\nu=0,1,2} \Big(1 + \big(\frac{x_\nu}{M^j}\big)^2\Big)$ and integrating over $\mathbb{R}^3$ gives the bound $\|\check{C}^{(j)}(x)\|_{L^1} = O(M^{2j})$.

To overcome the mismatch between position and momentum space we use, as in [12], a hybrid of the supremum norm in momentum space and the L^1 norm in position space: We cover the support of $C^{(j)}(k)$ by a union of "sectors"



One Sector

that are short enough to not feel the curvature of the Fermi curve. Then we write $\hat{w}_{2n}(k_1, \cdots, k_{2n})$ as a sum of terms for which each variable k_i is supported in a single sector and apply the position space L^1 norm to each term. In this way one constructs a norm which allows one to implement the Pauli exclusion principle and has good power counting properties in two space dimensions. For details, and for the obstacle hindering the extension of this procedure to three space dimension, see [11, §II.8].

It is an intriguing question whether it is possible to implement the Pauli exclusion principle by an argument which only uses momentum space variables. The would almost surely dramatically simplify the proof of Theorems 1.1–1.3.

4. A Model Problem

In this section, we formulate an elementary question about permutations that may be connected with implementing the Pauli exclusion principle in momentum space.

Let $p_1, \cdots, p_n, s_1, \cdots, s_n$ be real numbers. For a permutation $\pi \in S_n$, set

$$\varepsilon(\pi; \vec{p}, \vec{s}) = \begin{cases} \operatorname{sgn} \pi & \text{if} \quad p_{\pi(1)} \le s_1,\ p_{\pi(1)} + p_{\pi(2)} \le s_2,\ \ldots,\ p_{\pi(1)} + \cdots + p_{\pi(n)} \le s_n \\ 0 & \text{otherwise} \end{cases}$$

Question 4.1. Is there a constant K such that, for all natural numbers n and all $\vec{p} = (p_1, \cdots, p_n)$, $\vec{s} = (s_1, \cdots, s_n)$ in $\mathbb{R}^n$,

$$\Big| \sum_{\pi \in S_n} \varepsilon(\pi; \vec{p}, \vec{s}) \Big| \le K^n \sqrt{n!}$$

The following slight variation of Question 4.1 is directly related to cancellations between Fermionic diagrams.

Question 4.2. For all natural numbers n and all $\vec{p} = (p_1, \cdots, p_n)$, $\vec{q} = (q_1, \cdots, q_n)$ and $\vec{s} = (s_1, \cdots, s_n)$ in $\mathbb{R}^n$ and all pairs of permutations $\pi, \pi' \in S_n$, set

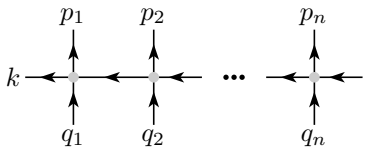
$$\varepsilon(\pi, \pi'; \vec{p}, \vec{q}, \vec{s}) = \begin{cases} \operatorname{sgn} \pi \ \operatorname{sgn} \pi' & \text{if} \quad p_{\pi(1)} + \cdots + p_{\pi(\ell)} - q_{\pi'(1)} - \cdots - q_{\pi'(\ell)} \le s_\ell \\ & \qquad \text{for all } \ell = 1, \cdots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Is there a constant C such that, for all n and all $\vec{p}$, $\vec{q}$, $\vec{s}$,

$$\Big| \sum_{\pi, \pi' \in S_n} \varepsilon(\pi, \pi'; \vec{p}, \vec{q}, \vec{s}) \Big| \le C^n n!$$

Observe that $p_{\pi(1)} + \cdots + p_{\pi(n)} - q_{\pi'(1)} - \cdots - q_{\pi'(n)} = p_1 + \cdots + p_n - q_1 - \cdots - q_n \le s_n$ either for every pair π, π' of permutations or for no such pair.

To illustrate how Question 4.2 is connected with cancellations between fermionic diagrams, let $C(k)$ be a covariance that depends on one real variable k and is Schwartz class. We consider a kernel that is obtained from n four–legged vertices by iterated contraction. Each vertex has the kernel $\delta(p_1 + p_2 - q_1 - q_2)$ in momentum space.



The (amputated) value of this diagram is a momentum conserving delta function times

$$\begin{aligned} &\hat{\varphi}(k, p_1, \cdots, p_n, q_1, \cdots, q_n) \qquad (9) \\ &\quad = C(k + p_1 - q_1) C(k + p_1 + p_2 - q_1 - q_2) \cdots C(k + p_1 + \cdots + p_{n-1} - q_1 - \cdots - q_{n-1}) \end{aligned}$$

Denote by $S(k)$ the sum (with appropriate fermionic signs) of all diagrams obtained from φ by forming tadpoles from all legs p_i to all legs q_j. That is

$$S(k) = \sum_{\pi \in S_n} \operatorname{sgn} \pi \int dp_1 \cdots dp_n \ \hat{\varphi}(k, p_1, \cdots, p_n, p_{\pi(1)}, \cdots, p_{\pi(n)}) \ C(p_1) \cdots C(p_n)$$

There are $n!$ terms in the sum. Applying the tadpole estimate term by term gives

$$\|S(k)\|_{L^\infty} \le n! \ \|C(k)\|_{L^1}^n \|\hat{\varphi}\|_{L^\infty} \qquad (10)$$

Equation (9) immediately gives $\|\hat\varphi\|_{L^\infty} \le \|C(k)\|_{L^\infty}^{n-1}$ and we get the "perturbative estimate"

$$\|S(k)\|_{L^\infty} \le n!\ \|C(k)\|_{L^1}^n \|C(k)\|_{L^\infty}^{n-1}$$

The Fourier transform of Equation (9) also shows that, in position space,

$$\|\varphi\|_{1,\infty} \le \|\check C(x)\|_{L^1}^{n-1} \tag{11}$$

Let

$$\hat\varphi^{\rm ant} = \tfrac{1}{n!^2} \sum_{\pi,\pi'\in S_n} \operatorname{sgn}\pi\ \operatorname{sgn}\pi'\ \hat\varphi(k, p_{\pi(1)}, \cdots, p_{\pi(n)}, q_{\pi'(1)}, \cdots, q_{\pi'(n)}, k+\Sigma p_i - \Sigma q_i)$$

be the antisymmetrization of $\hat\varphi$ in its p and q variables separately. $S(k)$ is also the sum of all diagrams obtained from $\varphi^{\rm ant}$ by forming tadpoles from all p–legs to all q–legs. The argument concerning the Fourier transforms of antisymmetric functions, described in the last section, shows that

$$\|\hat\varphi^{\rm ant}\|_{L^\infty} \le \tfrac{\text{const}^{2n}}{n!}\ \|\varphi^{\rm ant}\|_{1,\infty} \le \tfrac{\text{const}^{2n}}{n!}\ \|\varphi\|_{1,\infty} \tag{12}$$

Combining (10), with $\hat\varphi$ replaced by $\hat\varphi^{\rm ant}$, (12) and (11) gives the "nonperturbative estimate"

$$\|S(k)\|_{L^\infty} \le \text{const}^{2n}\ \|C(k)\|_{L^1}^n\ \|\check C(x)\|_{L^1}^{n-1} \tag{13}$$

However, this argument uses position as well as momentum space.

If the answer to Question 4.2 were positive, one could get an estimate like (13) without passing to position space. Recall that

$$\hat\varphi^{\rm ant} = \tfrac{1}{n!^2} \sum_{\pi,\pi'\in S_n} \operatorname{sgn}\pi\ \operatorname{sgn}\pi'\ \prod_{\ell=1}^{n-1} C(k + p_{\pi(1)} + \cdots + p_{\pi(\ell)} - q_{\pi'(1)} - \cdots - q_{\pi'(\ell)})$$

For each $r = p_{\pi(1)} + \cdots + p_{\pi(\ell)} - q_{\pi'(1)} - \cdots - q_{\pi'(\ell)}$,

$$C(k+r) = \int dt\ \delta(k+r-t)\, C(t) = \int dt\ \theta(t-k-r)\, C'(t)$$

where θ is the Heavyside step function. Consequently,

$$\hat\varphi^{\rm ant} = \tfrac{1}{n!^2} \int dt_1 \cdots dt_{n-1} \Theta(t_1, \cdots, t_{n-1}; k, \vec p, \vec q)\ C'(t_1) \cdots C'(t_{n-1}) \tag{14}$$

where

$$\Theta(t_1, \cdots, t_{n-1}; k, \vec p, \vec q) = \sum_{\pi,\pi'\in S_n} \operatorname{sgn}\pi \operatorname{sgn}\pi' \prod_{\ell=1}^{n-1} \theta\big(t_\ell - k - (p_{\pi(1)} + \cdots + p_{\pi(\ell)} - q_{\pi'(1)} - \cdots - q_{\pi'(\ell)})\big)$$

If the answer to Question 4.2 were positive, it would follow, setting $s_\ell = t_\ell - k$, that

$$\|\Theta\|_{L^\infty} \le C^n n!$$

It would then follow from (14) that

$$\|\hat\varphi^{\rm ant}\|_{L^\infty} \le \tfrac{C^n}{n!}\ \|C'(k)\|_{L^1}^{n-1} \tag{15}$$

Again, combining (10), with $\hat\varphi$ replaced by $\hat\varphi^{\rm ant}$, (15) and (11) would give the "nonperturbative estimate"

$$\|S(k)\|_{L^\infty} \le C^n\ \|C(k)\|_{L^1}^n\ \|C'(k)\|_{L^1}^{n-1}$$

by a purely momentum space argument.

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