A Functional Integral Representation for Many Boson Systems II: Correlation Functions

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Abstract. We derive functional integral representations for the partition function and correlation functions of many Boson systems for which the configuration space consists of finitely many points.

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I. Introduction

We are developing a set of tools and techniques for analyzing the large distance/infrared behaviour of a gas of bosons as the temperature tends to zero. In [I], we developed functional integral representations for the partition function of a many–boson system on a finite configuration space X with a repulsive two particle potential $v(\mathbf{x},\mathbf{y})$. Let H be the Hamiltonian, N the number operator, β the inverse temperature and μ the chemical potential. The main result, Theorem III.13, of [I] is

$$\mathrm{Tr}\, e^{-\beta(\mathrm{H}-\mu N)} = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \Big[d\mu_{\mathrm{R}(p)}(\phi_\tau^*,\phi_\tau)\ e^{-\int d\mathbf{y}\ [\phi_\tau^*(\mathbf{y})-\phi_{\tau-\varepsilon}^*(\mathbf{y})]\phi_\tau(\mathbf{y})}\ e^{-\varepsilon K(\phi_{\tau-\varepsilon}^*,\phi_\tau)} \Big] \tag{I.1}$$

with the conventions[1] that $\varepsilon = \frac{\beta}{p}$, $\phi_0 = \phi_\beta$ and $\mathcal{T}_p = \big\{\ \tau = q\frac{\beta}{p}\ \big|\ q = 1, \cdots, p\ \big\}$. The "classical" $\mathrm{H}-\mu N$ is

$$\begin{aligned} K(\alpha^*,\phi) = \iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\mathrm{h}(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) - \mu \int d\mathbf{x}\ \alpha(\mathbf{x})^*\phi(\mathbf{x}) \\ + \tfrac{1}{2} \iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\alpha(\mathbf{y})^* v(\mathbf{x},\mathbf{y})\, \phi(\mathbf{x})\phi(\mathbf{y}) \end{aligned}$$

where h is the single particle operator. For each $r > 0$,

$$d\mu_r(\,\phi^*,\,\phi) = \prod_{\mathbf{x}\in X} \Big[\tfrac{d\phi^*(\mathbf{x})\wedge d\phi(\mathbf{x})}{2\pi\imath}\ \chi_r(|\phi(\mathbf{x})|) \Big]$$

where χ_r is the characteristic function of the closed interval $[0,r]$. In [I, Theorem III.13], we need the hypothesis that the integration radius $\mathrm{R}(p)$ obeys

$$\lim_{p\to\infty}\ p\ e^{-\frac{1}{2}\mathrm{R}(p)^2}\ =\ 0 \qquad \text{and} \qquad \mathrm{R}(p) < p^{\frac{1}{24|X|}} \tag{I.2}$$

In [I], we outlined our motivation for deriving the function integral representation (I.1). We wish to use functional integrals as a starting point for analyzing the long distance behaviour of a many boson system. Such an analysis begins by directly extracting detailed properties of the ultraviolet limit $p \to \infty$ from the finite dimensional integrals in (I.1). These detailed properties would, in turn, provide a suitable starting point for an analysis of the thermodynamic limit and the temperature zero limit. The restrictions (I.2) on the domain of integration in (I.1) are not well suited for such a program. This is particularly obvious for the $|X|$ dependent second condition in (I.2). In Theorem II.2, we prove a representation for the partition function, similar to (I.1), but with functional integrals that are better suited to this program.

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(1) We also use the convention that $\int d\mathbf{x} = \sum_{\mathbf{x}\in X}$.

The choice of integration domain in Theorem II.2 is motivated by the following considerations. For two particle potentials that are repulsive in the sense that

$$\lambda_0 = \lambda_0(v) = \inf \left\{ \textstyle\int d\mathbf{x}d\mathbf{y}\ \rho(\mathbf{x})v(\mathbf{x},\mathbf{y})\rho(\mathbf{y}) \;\middle|\; \int d\mathbf{x}\ \rho(\mathbf{x})^2 = 1,\ \rho(\mathbf{x}) \ge 0 \text{ for all } \mathbf{x} \in X \right\} > 0 \tag{I.3}$$

the real part of the exponent of the integrand of (I.1) is, roughly speaking, dominated by

$$\begin{aligned} -\sum_{\tau\in\mathcal{T}_p} \Big\{ \tfrac{1}{2} \int d\mathbf{x}\ \big|\phi_\tau(\mathbf{x}) - \phi_{\tau-\varepsilon}(\mathbf{x})\big|^2 + \tfrac{\varepsilon}{2} \int d\mathbf{x}\, d\mathbf{y}\ \big|\phi_\tau(\mathbf{x})\big|^2 v(\mathbf{x},\mathbf{y}) \big|\phi_\tau(\mathbf{y})\big|^2 \Big\} \\ \le -\tfrac{1}{2} \sum_{\tau\in\mathcal{T}_p} \Big\{ \max_{\mathbf{x}\in X} \big|\phi_\tau(\mathbf{x}) - \phi_{\tau-\varepsilon}(\mathbf{x})\big|^2 + \varepsilon\lambda_0 \max_{\mathbf{x}\in X} \big|\phi_\tau(\mathbf{x})\big|^4 \Big\} \end{aligned}$$

Contributions to the integral of (I.1) coming from the part of the domain of integration where, for some τ and $\mathbf{x}$, $\big|\phi_\tau(\mathbf{x}) - \phi_{\tau-\varepsilon}(\mathbf{x})\big| \gg 1$ or $\big|\phi_\tau(\mathbf{x})\big| \gg \frac{1}{\sqrt[4]{\varepsilon\lambda_0}}$ will be extremely small. Consequently, we ought to restrict the domain of integration to be something like

$$\left\{ \big(\phi_\tau(\mathbf{x})\big)_{\substack{\tau\in\mathcal{T}_p\\ \mathbf{x}\in X}} \;\middle|\; \big|\phi_\tau(\mathbf{x}) - \phi_{\tau-\varepsilon}(\mathbf{x})\big| \le p_0(\varepsilon),\ \big|\phi_\tau(\mathbf{x})\big| \le \tfrac{1}{\sqrt[4]{\varepsilon\lambda_0}} p_0(\varepsilon),\ \tau \in \mathcal{T}_p,\ \mathbf{x} \in X \right\}$$

for some function, $p_0(\varepsilon)$, that grows slowly as $\varepsilon \to 0$.

To study the long distance behaviour of a many boson system, one needs to study correlation functions. By definition, an n–point correlation function at inverse temperature β is (up to a sign) an expression of the form

$$\frac{\mathrm{Tr}\ e^{-\beta(\mathrm{H}-\mu N)}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)}{\mathrm{Tr}\ e^{-\beta(\mathrm{H}-\mu N)}}$$

Here $\psi^{(\dagger)}$ refers to either ψ or $\psi^\dagger$ and

$$\psi^{(\dagger)}(\tau,\mathbf{x}) = e^{(\mathrm{H}-\mu N)\tau}\psi^{(\dagger)}(\mathbf{x})e^{-(\mathrm{H}-\mu N)\tau}$$

The time–ordering operator $\mathbb{T}$ orders the product $\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)$ with smaller times to the right. In the case of equal times, $\psi^\dagger$'s are placed to the right of ψ's. Theorem II.2 and (I.1) give functional integral representations for the denominator. The "times" β_j appearing in the numerator need not be rational multiples of β. Therefore in the functional integral representations for the correlation functions we replace the set $\mathcal{T}_p$ of allowed times by a partition $P = \{ \tau_\ell \mid 0 \le \ell \le p \}$ of the interval $[0,\beta]$ that contains the points $\beta_1,\ \beta_2,\ \cdots,\ \beta_n$. The analogs for correlation functions of (I.1) and Theorem II.2 are Theorems III.5 and III.7, respectively.

II. Another Integral Representation of the Partition Function

Let h be a single particle operator on X and $v(\mathbf{x}_1, \mathbf{x}_2)$ a real, symmetric, pair potential which is repulsive in the sense of (I.3). Throughout this section, except where otherwise noted, we write

$$K = K(\mathrm{h}, v, X, \mu) = \mathrm{H}_0(\mathrm{h}, X) + \mathrm{V}(v, X) - \mu N$$

where, as in [I, Propostion II.14], $\mathrm{H}_0(\mathrm{h}, X)$ is the second quantized free Hamiltonian with single particle operator h, $\mathrm{V}(v, X)$ is the second quantized interaction and N is the number operator. In this section, we prove a variant of the functional integral representation of [I, Theorem III.13] that is better adapted to a rigorous renormalization group analysis. Recall, from [I, Theorem III.1], that

$$\mathrm{Tr}\; e^{-\beta K} \;=\; \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \Big[d\mu_{\mathrm{R}(p)}(\phi_\tau^*, \phi_\tau)\; e^{-\int d\mathbf{y}\; |\phi_\tau(\mathbf{y})|^2} \Big] \prod_{\tau\in\mathcal{T}_p} \big\langle\, \phi_{\tau-\varepsilon} \,\big|\, e^{-\varepsilon\, K} \,\big|\, \phi_\tau \,\big\rangle$$

with the conventions that $\varepsilon = \frac{\beta}{p}$ and $\phi_0 = \phi_{p\varepsilon} = \phi_\beta$. Further recall, from [I, Proposition III.6], that

$$\big\langle\, \alpha \,\big|\, e^{-\varepsilon K} \,\big|\, \phi \,\big\rangle = e^{F(\varepsilon, \alpha^*, \phi)}$$

and, from [I, Lemma III.9], that

$$F(\varepsilon, \alpha^*, \phi) = \sum_{n=1}^{\infty} \int_{X^{2n}} d^n\tilde{\mathbf{x}}\, d^n\tilde{\mathbf{y}}\; e^{n\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^{n} \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \text{ where } \begin{array}{l} \tilde{\mathbf{x}} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \\ \tilde{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_n) \end{array}$$

In [I, Theorem III.13], we approximated $e^{\varepsilon\mu} F_1(\mathbf{x}, \mathbf{y})$, which is the kernel of the operator $e^{-\varepsilon(\mathrm{h}-\mu)}$, by $\mathbb{1} - \varepsilon(\mathrm{h} - \mu)$. Now, more generally, we approximate it by $j(\varepsilon, \mathbf{x}, \mathbf{y})$ where we only assume that there is a constant c_j such that

$$\big\| j(\varepsilon) - e^{-\varepsilon(\mathrm{h}-\mu)} \big\|_{1,\infty} \le c_j \varepsilon^2 \tag{II.1}$$

where, as in [I], for any operator A on $L^2(X)$ with kernel $A(\mathbf{x}, \mathbf{y})$, the norm

$$\|A\|_{1,\infty} = \max\Big\{ \max_{\mathbf{x}\in X} \int d\mathbf{y}\; |A(\mathbf{x}, \mathbf{y})|, \max_{\mathbf{y}\in X} \int d\mathbf{x}\; |A(\mathbf{x}, \mathbf{y})| \Big\}$$

For fields, we use the norms

$$\|\alpha\| = \Big[\sum_{\mathbf{x}\in X} |\alpha(\mathbf{x})|^2 \Big]^{1/2} \qquad \text{and} \qquad |\alpha|_X = \max_{\mathbf{x}\in X} |\alpha(\mathbf{x})|$$

In [I, Theorem III.13], the domain of integration restricted each field $\big|\phi_\tau\big|_X \le \mathrm{R}(p) < p^{\frac{1}{24|X|}}$. Now we relax that condition to $\big|\phi_\tau\big|_X \le \mathrm{R}_\varepsilon$, with R_ε satisfying Hypothesis II.1, below. In addition, the new domain of integration will restrict each "time derivative" $\big|\phi_{\tau+\varepsilon} - \phi_\tau\big|_X \le p_0(\varepsilon)$ with p_0 satisfying

Hypothesis II.1 *Let* $\mathrm{R}_\varepsilon > 0$ *and* $p_0(\varepsilon) \ge \ln\frac{1}{\varepsilon}$ *be decreasing functions of* ε *defined for all* $0 < \varepsilon \le 1$. *Assume that*

$$\mathrm{R}_\varepsilon \ge \tfrac{1}{\sqrt[4]{\varepsilon}} p_0(\varepsilon) \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} \sqrt{\varepsilon}\,\mathrm{R}_\varepsilon = 0$$

Theorem II.2 *Let* R_ε *and* $p_0(\varepsilon)$ *obey Hypothesis II.1 and* $j(\varepsilon)$ *obey (II.1). Let* $\beta > 0$. *Then, with the conventions that* $\varepsilon = \frac{\beta}{p}$ *and* $\phi_0 = \phi_{p\varepsilon}$,

$$\mathrm{Tr}\; e^{-\beta\, K} = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \Big[d\mu_{\mathrm{R}_\varepsilon}(\phi_\tau^*, \phi_\tau)\; \zeta_\varepsilon(\phi_{\tau-\varepsilon}, \phi_\tau)\; e^{\mathcal{A}(\varepsilon, \phi_{\tau-\varepsilon}^*, \phi_\tau)} \Big]$$

where

$$\begin{aligned}\mathcal{A}(\varepsilon, \alpha^*, \phi) &= -\tfrac{1}{2}\|\alpha\|^2 + \iint_{X^2} d\mathbf{x}d\mathbf{y}\;\; \alpha(\mathbf{x})^* j(\varepsilon; \mathbf{x}, \mathbf{y})\phi(\mathbf{y}) - \tfrac{1}{2}\|\phi\|^2 \\ &\qquad\qquad - \tfrac{\varepsilon}{2} \iint_{X^2} d\mathbf{x}d\mathbf{y}\;\; \alpha(\mathbf{x})^*\alpha(\mathbf{y})^*\, v(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{x})\phi(\mathbf{y}) \\ \mathcal{T}_p &= \big\{\; \tau = q\varepsilon \;\big|\; q = 1, \cdots, p \;\big\}\end{aligned}$$

and $\zeta_\varepsilon(\alpha, \phi)$ *is the characteristic function of* $|\alpha - \phi|_X \le p_0(\varepsilon)$.

The proof of Theorem II.2, which comes at the end of this section, is similar in spirit to that of [I, Theorem III.13], but uses, in place of [I, Example III.15],

Example II.3 For each $\varepsilon > 0$, set

$$\mathcal{I}_\varepsilon(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} e^{F(\varepsilon, \alpha^*, \phi)} = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \left\langle\, \alpha \;\middle|\; e^{-\varepsilon K} \;\middle|\; \phi \,\right\rangle$$

and use $*_\varepsilon$ to denote the convolution

$$(\mathcal{I} *_r \mathcal{J})(\alpha, \gamma) = \int \mathcal{I}(\alpha, \phi)\mathcal{J}(\phi, \gamma)\; d\mu_r(\phi^*, \phi)$$

of [I, Definition III.14], with $r = \mathrm{R}_\varepsilon$. Then

$$\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \left\langle\, \alpha \;\middle|\; e^{-\varepsilon K} \mathrm{I}_{\mathrm{R}_\varepsilon} e^{-\varepsilon K} \mathrm{I}_{\mathrm{R}_\varepsilon} \cdots \mathrm{I}_{\mathrm{R}_\varepsilon} e^{-\varepsilon K} \;\middle|\; \phi \,\right\rangle$$

with q factors of $e^{-\varepsilon K}$. Also set

$$\delta\mathcal{I}_\varepsilon(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} e^{F(\varepsilon, \alpha^*, \phi)} \big\{ e^{-\mathcal{F}_1(\varepsilon, \alpha^*, \phi)} \zeta_\varepsilon(\alpha, \phi) - 1 \big\}$$

where $\mathcal{F}_1$ was defined, for $|\alpha|_X,\ |\phi|_X < \Big[8e^{\varepsilon(\|\mathrm{h}\|_{1,\infty} + \mu + v_0)}\, \varepsilon \|v\|_{1,\infty} \Big]^{-1/2}$, in [I, Proposition III.6]. Here $v_0 = \max_{\mathbf{x}\in X} |v(\mathbf{x}, \mathbf{x})|$.

The principal difference between the proofs of Theorem II.2 and [I, Theorem III.13] is that is in the latter we simply bounded each integral by the supremum of its integrand multiplied by its volume of integration while in the former we use a field dependent, integrable, bound on the integrand. This demands relatively fine bounds on $\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha, \phi)$ and $\delta\mathcal{I}_\varepsilon(\alpha, \phi)$, which we prove in the next subsection.

Bounds on $\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)$ and $\delta\mathcal{I}_\varepsilon(\alpha,\phi)$

Set $\lambda_0=\lambda_0(v)$ as in (I.3). By [I, Proposition II.7],

$$K \ge \lambda_0' N - \mu N + \tfrac{1}{2}\big(\lambda_0 \tfrac{N}{|X|} - v_0\big)N = \tfrac{\lambda_0}{2}\big(\tfrac{N^2}{|X|} - \nu N\big) \tag{II.2}$$

where $\nu = \frac{2}{\lambda_0}\max\big\{0, v_0+\mu-\lambda_0'\big\}$. Here λ_0' is the smallest eigenvalue of h.

Lemma II.4 *The functionals $\mathcal{I}_\varepsilon(\alpha,\phi)$ and $\delta\mathcal{I}_\varepsilon(\alpha,\phi)$ of Example II.3 obey the following.*

(a) For any $\gamma>0$ and $q\in\mathbb{N}$,

$$\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| \le c_1 e^{-\frac{1}{2}\min\{1,q\varepsilon\lambda_0\gamma\}t}$$

where

$$t=\tfrac{1}{2}\big(\|\alpha\|^2+\|\phi\|^2\big) \qquad c_1 = e^{q\varepsilon\lambda_0(\nu+\gamma)^2|X|}$$

(b) For any $q\in\mathbb{N}$,

$$\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| \le c_2\big(\tfrac{1}{q\varepsilon\lambda_0}e^{-c_3 q\varepsilon t^2} + e^{-\frac{t}{8}}\big)$$

where

$$t=\tfrac{1}{2}\big(\|\alpha\|^2+\|\phi\|^2\big) \qquad c_2 = 65e^{(1+q\varepsilon\lambda_0\nu^2)|X|} \qquad c_3=\tfrac{\lambda_0}{40|X|}$$

(c) Let $\beta>0$ and assume that $q\in\mathbb{N}$ and $\varepsilon>0$ are such that $0<q\varepsilon\le\beta$. If R_ε is large enough (depending only on ν and $|X|$), then there is a constant const *(depending only on $|X|$, β, λ_0 and ν) such that*

$$\Big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi) - e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2}\big\langle\, \alpha \,\big|\, e^{-q\varepsilon K} \,\big|\, \phi\,\big\rangle\Big| \le \mathrm{const}\,\tfrac{1}{\varepsilon}\big(e^{-\frac{1}{4}\mathrm{R}_\varepsilon^2} + e^{-\frac{\lambda_0}{54|X|}\mathrm{R}_\varepsilon^4 q\varepsilon}\big)$$

Proof: (a) Recalling that $\mathrm{P}^{(n)}$ denotes projection onto $\mathcal{B}_n(X)$,

$$\begin{aligned}\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| &\le e^{-\frac{1}{2}\|\alpha\|^2}e^{-\frac{1}{2}\|\phi\|^2}\sum_{n=0}^\infty \big\|\mathrm{P}^{(n)}|\alpha\rangle\big\|\ \big\|\mathrm{P}^{(n)}e^{-\varepsilon K}\mathrm{I}_{\mathrm{R}_\varepsilon}e^{-\varepsilon K}\mathrm{I}_{\mathrm{R}_\varepsilon}\cdots\mathrm{I}_{\mathrm{R}_\varepsilon}e^{-\varepsilon K}\big\|\ \big\|\mathrm{P}^{(n)}|\phi\rangle\big\| \\ &\le e^{-\frac{1}{2}\|\alpha\|^2}e^{-\frac{1}{2}\|\phi\|^2}\sum_{n=0}^\infty \tfrac{\|\alpha\|^n}{\sqrt{n!}}e^{-\frac{1}{2}q\varepsilon\lambda_0(\frac{n^2}{|X|}-\nu n)}\tfrac{\|\phi\|^n}{\sqrt{n!}}\end{aligned}$$

Observe that

$$\tfrac{n^2}{|X|} - \nu n - 2\gamma n = \tfrac{1}{|X|}\big(n-\tfrac{\nu+2\gamma}{2}|X|\big)^2 - \tfrac{1}{4}(\nu+2\gamma)^2|X| \ge -(\nu+\gamma)^2|X| \tag{II.3}$$

Using that $\|\alpha\|\,\|\phi\| \le \frac{1}{2}\big(\|\alpha\|^2 + \|\phi\|^2\big) = t$,

$$\begin{aligned}\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| &\le e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-q\varepsilon\lambda_0\gamma n}e^{-\frac{1}{2}q\varepsilon\lambda_0(\frac{n^2}{|X|}-\nu n-2\gamma n)} \le e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-q\varepsilon\lambda_0\gamma n}e^{q\varepsilon\lambda_0(\nu+\gamma)^2|X|}\\ &= e^{-(1-e^{-q\varepsilon\lambda_0\gamma})t}e^{q\varepsilon\lambda_0(\nu+\gamma)^2|X|}\end{aligned}$$

If $q\varepsilon\lambda_0\gamma \le 1$, then, by the alternating series test $e^{-q\varepsilon\lambda_0\gamma} \le 1-q\varepsilon\lambda_0\gamma+\frac{1}{2}(q\varepsilon\lambda_0\gamma)^2 \le 1-\frac{1}{2}q\varepsilon\lambda_0\gamma$, which implies that

$$\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| \le e^{-\frac{1}{2}q\varepsilon\lambda_0\gamma t}e^{q\varepsilon\lambda_0(\nu+\gamma)^2|X|}$$

If $q\varepsilon\lambda_0\gamma \ge 1$, then

$$\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| \le e^{-\frac{1}{2}t}e^{q\varepsilon\lambda_0(\nu+\gamma)^2|X|}$$

(b) As

$$\tfrac{n^2}{2|X|} - \nu n = \tfrac{1}{2|X|}\big(n-\nu|X|\big)^2 - \tfrac{1}{2}\nu^2|X| \ge -\nu^2|X|$$

we have, since $\|\mathrm{I}_{\mathrm{R}_\varepsilon}\| \le 1$ and $\|\mathrm{P}^{(n)}e^{-\varepsilon K}\| \le e^{-\varepsilon\lambda_0(\frac{n^2}{|X|}-\nu n)}$,

$$\begin{aligned}\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| &\le e^{-\frac{1}{2}\|\alpha\|^2}e^{-\frac{1}{2}\|\phi\|^2}\sum_{n=0}^{\infty}\big\|\mathrm{P}^{(n)}|\alpha\rangle\big\|\;\big\|\mathrm{P}^{(n)}e^{-\varepsilon K}\mathrm{I}_{\mathrm{R}_\varepsilon}e^{-\varepsilon K}\mathrm{I}_{\mathrm{R}_\varepsilon}\cdots\mathrm{I}_{\mathrm{R}_\varepsilon}e^{-\varepsilon K}\big\|\;\big\|\mathrm{P}^{(n)}|\phi\rangle\big\|\\ &\le e^{-\frac{1}{2}\|\alpha\|^2}e^{-\frac{1}{2}\|\phi\|^2}\sum_{n=0}^{\infty}\tfrac{\|\alpha\|^n}{\sqrt{n!}}e^{-\frac{1}{2}q\varepsilon\lambda_0(\frac{n^2}{|X|}-\nu n)}\tfrac{\|\phi\|^n}{\sqrt{n!}}\\ &\le e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2}e^{-\frac{1}{2}q\varepsilon\lambda_0(\frac{n^2}{2|X|}-\nu n)}\\ &\le e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2}e^{q\varepsilon\lambda_0\nu^2|X|}\end{aligned}$$

If $t \le e$, it suffices to use the bound $\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| \le e^{-t}e^te^{q\varepsilon\lambda_0\nu^2|X|} = e^{q\varepsilon\lambda_0\nu^2|X|}$ since $65e^1e^{-\frac{e}{8}} \ge 1$ implies that $c_2e^{-\frac{t}{8}} \ge e^{q\varepsilon\lambda_0\nu^2|X|}$ for all $t \le e$. So we may assume that $t > e$. Similarly, if $\frac{q\varepsilon\lambda_0}{|X|} \ge 1$, we may use the bound

$$\begin{aligned}\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| &\le e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-\frac{1}{4}n^2}e^{q\varepsilon\lambda_0\nu^2|X|} \le e^{-t}\sum_{n=0}^{\infty}\tfrac{(t/e)^n}{n!}e^{-\frac{1}{4}n^2+n}e^{q\varepsilon\lambda_0\nu^2|X|}\\ &\le e^{-t}\sum_{n=0}^{\infty}\tfrac{(t/e)^n}{n!}e^{1+q\varepsilon\lambda_0\nu^2|X|} = e^{1+q\varepsilon\lambda_0\nu^2|X|}e^{-(1-\frac{1}{e})t}\end{aligned}$$

So we may also assume that $\frac{q\varepsilon\lambda_0}{|X|} \le 1$. For any $m > 0$ (not necessarily integer)

$$\begin{aligned}
\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| &\le e^{-t}\sum_{n\le 4m|X|}\tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2}e^{q\varepsilon\lambda_0\nu^2|X|} + e^{-t}\sum_{n\ge 4m|X|}\tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2}e^{q\varepsilon\lambda_0\nu^2|X|} \\
&\le e^{q\varepsilon\lambda_0\nu^2|X|}e^{-t}(4m|X|+1)\sup_n \tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2} + e^{-t}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}e^{-q\varepsilon\lambda_0 mn}e^{q\varepsilon\lambda_0\nu^2|X|} \\
&= e^{q\varepsilon\lambda_0\nu^2|X|}e^{-t}(4m|X|+1)\sup_n \tfrac{t^n}{n!}e^{-\frac{q\varepsilon\lambda_0}{4|X|}n^2} + e^{-(1-e^{-q\varepsilon\lambda_0 m})t}e^{q\varepsilon\lambda_0\nu^2|X|}
\end{aligned}$$

Choose the m specified in Lemma II.5 below with ε replaced by $\frac{q\varepsilon\lambda_0}{4|X|}$. Applying that Lemma gives

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le e^{q\varepsilon\lambda_0\nu^2|X|}e^{-t}(4m|X|+1)2e^{(m+t)/2} + e^{-(1-e^{-q\varepsilon\lambda_0 m})t}e^{q\varepsilon\lambda_0\nu^2|X|}$$

where m is the unique solution to

$$\tfrac{q\varepsilon\lambda_0}{2|X|}m + \ln m + \tfrac{1}{2m} = \ln t$$

with $m \ge 1$. Since

$$\ln m \le \ln t - \tfrac{q\varepsilon\lambda_0}{2|X|}m \iff m \le te^{-\frac{q\varepsilon\lambda_0}{2|X|}m}$$

we have

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le 2e^{q\varepsilon\lambda_0\nu^2|X|}(4m|X|+1)e^{-(1-e^{-\frac{q\varepsilon\lambda_0}{2|X|}m})t/2} + e^{-(1-e^{-q\varepsilon\lambda_0 m})t}e^{q\varepsilon\lambda_0\nu^2|X|}$$

We treat the two terms,

$$\begin{aligned}
T1 &= 2e^{q\varepsilon\lambda_0\nu^2|X|}(4m|X|+1)e^{-(1-e^{-\frac{q\varepsilon\lambda_0}{2|X|}m})t/2} \\
T2 &= e^{-(1-e^{-q\varepsilon\lambda_0 m})t}e^{q\varepsilon\lambda_0\nu^2|X|}
\end{aligned}$$

separately.

Case 1: First term, $\frac{q\varepsilon\lambda_0 m}{2|X|} \ge 1$

In this case $1 - e^{-\frac{q\varepsilon\lambda_0}{2|X|}m} \ge 1 - e^{-1} \ge \frac{1}{2}$ and, since $m < t$,

$$\begin{aligned}
\left|T1\right| &\le 2e^{q\varepsilon\lambda_0\nu^2|X|}(4m|X|+1)e^{-t/4} \le 2e^{q\varepsilon\lambda_0\nu^2|X|}(4t|X|+1)e^{-t/4} \\
&\le 64e^{q\varepsilon\lambda_0\nu^2|X|}|X|(1+\tfrac{t}{8})e^{-t/4} \ \le 64e^{(1+q\varepsilon\lambda_0\nu^2)|X|}e^{-t/8}
\end{aligned}$$

Case 2: First term, $\frac{q\varepsilon\lambda_0 m}{2|X|} \le 1$

In this case $1 - e^{-\frac{q\varepsilon\lambda_0}{2|X|}m} \ge \frac{q\varepsilon\lambda_0 m}{4|X|}$ since $1 - e^{-\alpha} \ge \frac{\alpha}{2}$ for all $0 \le \alpha \le 1$. Hence

$$\left|T1\right| \le 2e^{q\varepsilon\lambda_0\nu^2|X|}(4m|X|+1)e^{-q\varepsilon\lambda_0 mt/(8|X|)}$$

Since

$$\ln m = \ln t - \tfrac{q\varepsilon\lambda_0 m}{2|X|} - \tfrac{1}{2m} \ge \ln t - \tfrac{3}{2} \ \Rightarrow\ m \ge \tfrac{t}{5}$$

we have

$$\begin{aligned}
\big|T1\big| &\le 2e^{q\varepsilon\lambda_0\nu^2|X|}(4m|X|+1)e^{-\frac{q\varepsilon\lambda_0}{40|X|}t^2} \\
&\le 2e^{q\varepsilon\lambda_0\nu^2|X|}\big(8\tfrac{|X|^2}{q\varepsilon\lambda_0}+1\big)e^{-\frac{q\varepsilon\lambda_0}{40|X|}t^2} \\
&\le 2e^{q\varepsilon\lambda_0\nu^2|X|}\tfrac{16}{q\varepsilon\lambda_0}\big(\tfrac{|X|^2}{2}+|X|\big)e^{-\frac{q\varepsilon\lambda_0}{40|X|}t^2} \\
&\le 32e^{(1+q\varepsilon\lambda_0\nu^2)|X|}\tfrac{1}{q\varepsilon\lambda_0}e^{-\frac{q\varepsilon\lambda_0}{40|X|}t^2}
\end{aligned}$$

Case 3: Second term, $q\varepsilon\lambda_0 m \ge 1$
In this case $1 - e^{-q\varepsilon\lambda_0 m} \ge 1 - e^{-1} \ge \frac{1}{2}$ and

$$\big|T2\big| \le e^{-t/2}e^{q\varepsilon\lambda_0\nu^2|X|}$$

Case 4: Second term, $q\varepsilon\lambda_0 m \le 1$
Now $1 - e^{-q\varepsilon\lambda_0 m} \ge \frac{1}{2}q\varepsilon\lambda_0 m$ so that

$$\big|T2\big| \le e^{-\frac{1}{2}q\varepsilon\lambda_0 mt}e^{q\varepsilon\lambda_0\nu^2|X|} \le e^{-\frac{1}{10}q\varepsilon\lambda_0 t^2}e^{q\varepsilon\lambda_0\nu^2|X|} \le e^{q\varepsilon\lambda_0\nu^2|X|}\tfrac{1}{q\varepsilon\lambda_0}e^{-\frac{q\varepsilon\lambda_0}{40|X|}t^2}$$

since we again have $m \ge \frac{t}{5}$, as in Case 2.

(c)

Introduce the local notation

$$A_i = \begin{Bmatrix} e^{-\varepsilon K} & \text{if } i \text{ is odd} \\ \mathrm{I}_{\mathrm{R}_\varepsilon} & \text{if } i \text{ is even}\end{Bmatrix} \qquad\qquad B_i = \begin{Bmatrix} e^{-\varepsilon K} & \text{if } i \text{ is odd} \\ \mathbb{1} & \text{if } i \text{ is even}\end{Bmatrix}$$

so that

$$\Big(e^{-\varepsilon K}\,\mathrm{I}_{\mathrm{R}_\varepsilon}\Big)^{q-1} e^{-\varepsilon K} = \prod_{i=1}^{2q-1} A_i \qquad \text{and} \qquad e^{-q\varepsilon K} = \prod_{i=1}^{2q-1} B_i$$

For any $n \in \mathbb{N}$,

$$\begin{aligned}
&\Big|\Big\langle \alpha \Big| \Big(e^{-\varepsilon K}\,\mathrm{I}_{\mathrm{R}_\varepsilon}\Big)^{q-1} e^{-\varepsilon K} \ - \ e^{-q\varepsilon K} \Big| \phi \Big\rangle\Big| \\
&\qquad \le \Big|\Big\langle \alpha \Big| \Big(\prod_{i=1}^{2q-1} A_i \ - \ \prod_{i=1}^{2q-1} B_i\Big)\mathrm{P}_n \Big| \phi \Big\rangle\Big| && \text{(II.4)} \\
&\qquad\qquad + \Big|\Big\langle \alpha \Big| \prod_{i=1}^{2q-1} A_i(\mathbb{1}-\mathrm{P}_n) \Big| \phi \Big\rangle\Big| + \Big|\Big\langle \alpha \Big| e^{-q\varepsilon K}(\mathbb{1}-\mathrm{P}_n) \Big| \phi \Big\rangle\Big| && \text{(II.5)}
\end{aligned}$$

Consider the first line, (II.4). Observe that $\big\|\mathrm{I}_R\big\| \le 1$, by part (c) of [I, Theorem II.26] and $K \ge -\frac{1}{2}\lambda_0\nu^2|X|$, by (II.3) with $\gamma = 0$. Hence

$$\|A_i\|\,,\ \|B_i\| \le \begin{cases} e^{\varepsilon\lambda_0\nu^2|X|} & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even}\end{cases}$$

Since $A_\ell - B_\ell = \mathrm{I}_{\mathrm{R}_\varepsilon} - \mathbb{1}$ for $\ell = 2,\ 4,\ \cdots,\ 2q-2$ and is zero otherwise, we have, for all $n\,,\, q \in \mathbb{N}\,$,

$$\Big\|\Big(\prod_{i=1}^{2q-1} A_i \;-\; \prod_{i=1}^{2q-1} B_i\Big)\mathrm{P}_n\Big\| \le \sum_{\ell=1}^{2q-1} \Big\|\prod_{i=1}^{\ell-1} A_i\,\big(A_\ell - B_\ell\big)\prod_{i=\ell+1}^{2q-1} B_i\,\mathrm{P}_n\Big\|$$
$$\le\ (q-1)e^{q\varepsilon\,\lambda_0\nu^2|X|}\,\big\|\big(\mathrm{I}_{\mathrm{R}_\varepsilon} - \mathbb{1}\big)\mathrm{P}_n\big\|$$
$$\le\ q e^{q\varepsilon\,\lambda_0\nu^2|X|}\,|X|\,2^{n+1}\,e^{-\mathrm{R}_\varepsilon^2/2}$$

by part (d) of [I, Theorem II.26]. Consequently,

$$e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\Big|\Big\langle\,\alpha\,\Big|\,\Big(\Big(e^{-\varepsilon K}\,\mathrm{I}_{\mathrm{R}_\varepsilon}\Big)^{q-1} e^{-\varepsilon K} - e^{-q\varepsilon K}\Big)\mathrm{P}_n\,\Big|\,\phi\,\Big\rangle\Big| \le q e^{q\varepsilon\,\lambda_0\nu^2|X|}\,|X|\,2^{n+1}\,e^{-\mathrm{R}_\varepsilon^2/2} \tag{II.6}$$

Now consider the second line, (II.5). For all $m\ge 1$, $K\big|_{\mathcal{B}_m} \ge \frac12\lambda_0\big(\frac{m}{|X|}-\nu\big)m$ and

$$\Big\|\prod_{i=1}^{2q-1} A_i\Big|_{\mathcal{B}_m}\Big\|\ \ ,\ \ \Big\|e^{-q\varepsilon K}\Big|_{\mathcal{B}_m}\Big\|\ \le\ e^{-\frac12 q\varepsilon\,\lambda_0(\frac{m}{|X|}-\nu)\,m}$$

and it follows that

$$\Big|\Big\langle\,\alpha\,\Big|\,\prod_{i=1}^{2q-1} A_i\big|_{\mathcal{B}_m}\,\Big|\,\phi\,\Big\rangle\Big|\ \ ,\ \ \Big|\Big\langle\,\alpha\,\Big|\,e^{-q\varepsilon K}\big|_{\mathcal{B}_m}\,\Big|\,\phi\,\Big\rangle\Big|\ \le\ e^{-\frac12 q\varepsilon\,\lambda_0(\frac{m}{|X|}-\nu)\,m}\,e^{\frac12\|\alpha\|^2+\frac12\|\phi\|^2}$$

If we impose the stronger condition $m\ge n$ with $\frac{n}{|X|}\ge 2\nu$, the last inequality becomes

$$\Big|\Big\langle\,\alpha\,\Big|\,\prod_{i=1}^{2q-1} A_i\big|_{\mathcal{B}_m}\,\Big|\,\phi\,\Big\rangle\Big|\ \ ,\ \ \Big|\Big\langle\,\alpha\,\Big|\,e^{-q\varepsilon K}\big|_{\mathcal{B}_m}\,\Big|\,\phi\,\Big\rangle\Big|\ \le\ e^{-\frac{\lambda_0}{4|X|}\,m^2 q\varepsilon}\,e^{\frac12\|\alpha\|^2+\frac12\|\phi\|^2}$$

Now, we have, if n is large enough (depending only on ν and $|X|$)

$$e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\Big|\Big\langle\,\alpha\,\Big|\,\prod_{i=1}^{2q-1} A_i(\mathbb{1}-\mathrm{P}_n)\,\Big|\,\phi\,\Big\rangle\Big|\ \ ,\ \ e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\Big|\Big\langle\,\alpha\,\Big|\,e^{-q\varepsilon K}(\mathbb{1}-\mathrm{P}_n)\,\Big|\,\phi\,\Big\rangle\Big|$$
$$\le \sum_{m>n} e^{-\frac{\lambda_0}{4|X|}\,m^2 q\varepsilon} \qquad \le \int_n^\infty ds\ e^{-\frac{\lambda_0}{4|X|}\,s^2 q\varepsilon}$$
$$\le e^{-\frac{\lambda_0}{6|X|}\,n^2 q\varepsilon}\int_0^\infty ds\ e^{-\frac{\lambda_0}{12|X|}\,s^2 q\varepsilon} \quad = \sqrt{\frac{6|X|}{\lambda_0 q\varepsilon}}\,e^{-\frac{\lambda_0}{6|X|}\,n^2 q\varepsilon}\int_0^\infty ds\ e^{-\frac12 s^2} \tag{II.7}$$
$$= \sqrt{\frac{6|X|}{\lambda_0 q\varepsilon}}\,e^{-\frac{\lambda_0}{6|X|}\,n^2 q\varepsilon}\,\frac12\sqrt{2\pi} \qquad \le 4\sqrt{\frac{|X|}{\lambda_0 q\varepsilon}}\,e^{-\frac{\lambda_0}{6|X|}\,n^2 q\varepsilon}$$

Choosing $n = \frac13\mathrm{R}_\varepsilon^2$ (so that $2^n \le e^{\frac14\mathrm{R}_\varepsilon^2}$) and adding (II.6) and twice (II.7) gives

$$\Big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi) - e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\,\big\langle\,\alpha\,\big|\,e^{-q\varepsilon K}\,\big|\,\phi\,\big\rangle\Big|$$
$$\le 2q e^{q\varepsilon\,\lambda_0\nu^2|X|}\,|X|\,e^{-\mathrm{R}_\varepsilon^2/4} + 8\sqrt{\frac{|X|}{\lambda_0 q\varepsilon}}\,e^{-\frac{\lambda_0}{54|X|}\,\mathrm{R}_\varepsilon^4 q\varepsilon}$$

■

Lemma II.5 *Let $0 < \varepsilon \le \frac{1}{4}$ and $t \ge e$. Then*

$$\sup_{n\ge 1} e^{-\varepsilon n^2} \tfrac{t^n}{n!} \le 2e^{(m+t)/2}$$

where m is the unique solution to

$$2\varepsilon m + \ln m + \tfrac{1}{2m} = \ln t$$

with $m \ge 1$.

Proof: Recall that Stirling's formula [AS, 6.1.38] states that for each real $n > 0$, there is a $0 < \theta < 1$ such that

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{\theta}{12n}}$$

In particular, for $n \ge 1$,

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \le n! \le \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+1}$$

Hence

$$e^{-\varepsilon n^2} \tfrac{t^n}{n!} \le \tfrac{1}{\sqrt{2\pi}} \tfrac{1}{n^{\frac{1}{2}}} e^n e^{-\varepsilon n^2} \left(\tfrac{t}{n}\right)^n = \tfrac{1}{\sqrt{2\pi}} e^{n-\varepsilon n^2 + n\ln t - n\ln n - \frac{1}{2}\ln n} \tag{II.8}$$

Observe that, for $n \ge 1$,

$$\begin{aligned} \tfrac{d}{dn}\left[n - \varepsilon n^2 + n\ln t - n\ln n - \tfrac{1}{2}\ln n\right] &= 1 - 2\varepsilon n + \ln t - \ln n - 1 - \tfrac{1}{2n} \\ &= \ln t - 2\varepsilon n - \ln n - \tfrac{1}{2n} \end{aligned}$$

Since $\frac{d}{dn}\left[2\varepsilon n + \ln n + \frac{1}{2n}\right] = 2\varepsilon + \frac{1}{n} - \frac{1}{2n^2} > 0$ for all $n \ge 1$ and $\ln t \ge 2\varepsilon + \frac{1}{2}$, the equation

$$2\varepsilon m + \ln m + \tfrac{1}{2m} = \ln t$$

has a unique solution $m \ge 1$. This solution obeys

$$2\varepsilon m^2 + m\ln m + \tfrac{1}{2} = m\ln t \;\Rightarrow\; e^{2\varepsilon m^2} = e^{-1/2}\left(\tfrac{t}{m}\right)^m$$

The derivative of the last exponent in (II.8) is positive for $n < m$ and negative for $n > m$. Hence the last exponent of (II.8) takes its maximum value at $n = m$ and

$$\begin{aligned} e^{-\varepsilon n^2} \tfrac{t^n}{n!} &\le \tfrac{1}{\sqrt{2\pi}} \tfrac{1}{m^{1/2}} e^m e^{-\varepsilon m^2} \left(\tfrac{t}{m}\right)^m = \tfrac{e^{1/4}}{\sqrt{2\pi}} \tfrac{1}{m^{1/2}} e^m \left(\tfrac{t}{m}\right)^{m/2} = \tfrac{e^{1/4}}{\sqrt{2\pi}} \tfrac{1}{m^{1/4}} e^{m/2} \left(\tfrac{e^m t^m}{m^{m+1/2}}\right)^{1/2} \\ &\le \tfrac{e^{1/4}}{\sqrt{2\pi}} \tfrac{1}{m^{1/4}} e^{m/2} \left(e\sqrt{2\pi} \tfrac{t^m}{m!}\right)^{1/2} \le 2e^{m/2} \left(\tfrac{t^m}{m!}\right)^{1/2} \le 2e^{(m+t)/2} \end{aligned}$$

■

For the rest of this section, except where otherwise specified, all constants may depend on $|X|$, v, $\|\mathrm{h}\|_{1,\infty}$, c_j, β and μ. They may not depend on ε or p.

Lemma II.6 *Let $\delta\mathcal{I}_\varepsilon(\alpha,\phi)$ be as in Example II.3. There are constants* const *and C_R such that, for all sufficiently small ε and all $|\alpha|_X,\ |\phi|_X \le \frac{C_R}{\sqrt{\varepsilon}}$,*

$$\big|\delta\mathcal{I}_\varepsilon(\alpha,\phi)\big| \le \mathrm{const}\,\Big\{\varepsilon^2\big(1+|\alpha|_X^6+|\phi|_X^6\big)e^{-\frac{1}{4}\|\alpha-\phi\|^2-\frac{1}{8}\varepsilon\lambda_0\left(\|\alpha\|_{\ell^4}^4+\|\phi\|_{\ell^4}^4\right)}+e^{-\frac{1}{4}p_0(\varepsilon)^2}\Big\}$$

Proof: By [I, Corollary III.7],

$$\begin{aligned}
&\mathrm{Re}\big[-\tfrac{1}{2}\|\alpha\|^2-\tfrac{1}{2}\|\phi\|^2+F(\varepsilon,\alpha^*,\phi)\big]\\
&\quad=\mathrm{Re}\Big[-\tfrac{1}{2}\|\alpha\|^2-\tfrac{1}{2}\|\phi\|^2+\int_X d\mathbf{x}\ \alpha(\mathbf{x})^*\phi(\mathbf{x})-\varepsilon K(\alpha^*,\phi)+\mathcal{F}_0(\varepsilon,\alpha^*,\phi)\Big]\\
&\quad=-\tfrac{1}{2}\|\alpha-\phi\|^2+\mathrm{Re}\Big[-\varepsilon\iint_{X^2} d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\mathrm{h}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})+\varepsilon\mu\int_X d\mathbf{x}\ \alpha(\mathbf{x})^*\phi(\mathbf{x})\\
&\qquad\qquad-\tfrac{\varepsilon}{2}\iint_{X^2} d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\alpha(\mathbf{y})^*\,v(\mathbf{x},\mathbf{y})\,\phi(\mathbf{x})\phi(\mathbf{y})+\mathcal{F}_0(\varepsilon,\alpha^*,\phi)\Big]
\end{aligned}$$

For any $\alpha,\phi\in L^2(X)$, we have

$$\begin{aligned}
\varepsilon\mu\ \mathrm{Re}\int d\mathbf{x}\ \alpha(\mathbf{x})^*\phi(\mathbf{x}) &\le \tfrac{1}{2}\varepsilon\mu\big[\|\alpha\|^2+\|\phi\|^2\big]\\
&=\tfrac{1}{2}\varepsilon\int d\mathbf{x}\ \big(\tfrac{4\mu}{\sqrt{\lambda_0}}\ \tfrac{\sqrt{\lambda_0}}{4}|\alpha(\mathbf{x})|^2+\tfrac{4\mu}{\sqrt{\lambda_0}}\ \tfrac{\sqrt{\lambda_0}}{4}|\phi(\mathbf{x})|^2\big)\\
&\le \tfrac{8\mu^2}{\lambda_0}|X|\,\varepsilon+\tfrac{1}{64}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4+\|\phi\|_{\ell^4}^4\big)
\end{aligned}$$

and

$$\begin{aligned}
&-\varepsilon\ \mathrm{Re}\iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\mathrm{h}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\\
&\qquad=-\varepsilon\iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\mathrm{h}(\mathbf{x},\mathbf{y})\alpha(\mathbf{y})+\varepsilon\ \mathrm{Re}\iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\mathrm{h}(\mathbf{x},\mathbf{y})(\alpha-\phi)(\mathbf{y})\\
&\qquad\le \varepsilon\|\mathrm{h}\|\,\|\alpha\|\|\alpha-\phi\|\\
&\qquad\le \tfrac{1}{2}\varepsilon^{3/2}\|\mathrm{h}\|\,\|\alpha\|^2+\tfrac{1}{2}\varepsilon^{1/2}\|\mathrm{h}\|\,\|\alpha-\phi\|^2\\
&\qquad\le \tfrac{\|\mathrm{h}\|}{\lambda_0}|X|\,\varepsilon^{3/2}+\tfrac{\|\mathrm{h}\|}{16}\varepsilon^{3/2}\lambda_0\|\alpha\|_{\ell^4}^4+\tfrac{1}{2}\varepsilon^{1/2}\|\mathrm{h}\|\,\|\alpha-\phi\|^2
\end{aligned}$$

and

$$
\begin{aligned}
&- \tfrac{\varepsilon}{2}\ \mathrm{Re} \iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\alpha(\mathbf{y})^*\, v(\mathbf{x},\mathbf{y})\, \phi(\mathbf{x})\phi(\mathbf{y}) \\
&= -\tfrac{1}{4}\varepsilon \iint d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\alpha(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \alpha(\mathbf{y})^*\alpha(\mathbf{y}) - \tfrac{1}{4}\varepsilon \iint d\mathbf{x}d\mathbf{y}\ \phi(\mathbf{x})^*\phi(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})^*\phi(\mathbf{y}) \\
&\quad + \tfrac{1}{4}\varepsilon\, \mathrm{Re} \iint d\mathbf{x}d\mathbf{y}\ \big[\alpha(\mathbf{x})^*(\alpha-\phi)(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \alpha(\mathbf{y})^*\phi(\mathbf{y}) + \alpha^*(\mathbf{x})\alpha(\mathbf{x}) v(\mathbf{x},\mathbf{y}) \alpha^*(\mathbf{y})(\alpha-\phi)(\mathbf{y})\big] \\
&\quad - \tfrac{1}{4}\varepsilon\, \mathrm{Re} \iint d\mathbf{x}d\mathbf{y}\ \big[(\alpha-\phi)^*(\mathbf{x})\phi(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \alpha(\mathbf{y})^*\phi(\mathbf{y}) + \phi^*(\mathbf{x})\phi(\mathbf{x}) v(\mathbf{x},\mathbf{y}) (\alpha-\phi)^*(\mathbf{y})\phi(\mathbf{y})\big] \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha^*\alpha\|^2 + \|\phi^*\phi\|^2\big) + \tfrac{1}{4}\varepsilon\|v\|\Big[\|\alpha^*(\alpha-\phi)\|\,\|\alpha^*\phi\| + \|\alpha^*(\alpha-\phi)\|\,\|\alpha^*\alpha\| \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \|\phi^*(\alpha-\phi)\|\,\|\alpha^*\phi\| + \|\phi^*(\alpha-\phi)\|\,\|\phi^*\phi\|\Big] \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \tfrac{1}{4}\varepsilon\|v\|\,\|\alpha-\phi\|\Big[|\alpha|_X\big(\|\alpha^*\phi\| + \|\alpha^*\alpha\|\big) + |\phi|_X\big(\|\alpha^*\phi\| + \|\phi^*\phi\|\big)\Big] \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \tfrac{C_R}{4}\sqrt{\varepsilon}\|v\|\,\|\alpha-\phi\|\Big[\|\alpha^*\phi\| + \|\alpha^*\alpha\| + \|\alpha^*\phi\| + \|\phi^*\phi\|\Big] \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \tfrac{C_R}{4}\sqrt{\varepsilon}\|v\|\,\|\alpha-\phi\|\Big[2\|\alpha\|_{\ell^4}\|\phi\|_{\ell^4} + \|\alpha\|_{\ell^4}^2 + \|\phi\|_{\ell^4}^2\Big] \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \tfrac{C_R}{2}\sqrt{\varepsilon}\|v\|\,\|\alpha-\phi\|\|\alpha\|_{\ell^4}^2 + \tfrac{C_R}{2}\sqrt{\varepsilon}\|v\|\,\|\alpha-\phi\|\|\phi\|_{\ell^4}^2 \\
&\le -\tfrac{1}{4}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \big(2C_R\tfrac{\|v\|}{\sqrt{\lambda_0}}\big)^2\,\|\alpha-\phi\|^2 + \tfrac{1}{32}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) \\
&= -\tfrac{7}{32}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \big(2C_R\tfrac{\|v\|}{\sqrt{\lambda_0}}\big)^2\,\|\alpha-\phi\|^2
\end{aligned}
$$

and, by [I, Corollary III.7],

$$
\begin{aligned}
|\mathcal{F}_0(\alpha^*,\phi)| &\le c_0\varepsilon^2|X|\big(|\alpha|_X^2 + |\phi|_X^2 + \|v\|_{1,\infty}^2|\alpha|_X^6 + \|v\|_{1,\infty}^2|\phi|_X^6\big) \\
&\le c_0\varepsilon^2|X|\big(1 + \tfrac{1}{2}\|\alpha\|_{\ell^4}^4 + \tfrac{1}{2}\|\phi\|_{\ell^4}^4\big) + c_0C_R^2\varepsilon\|v\|_{1,\infty}^2|X|\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big)
\end{aligned}
$$

All together, if ε is small enough and $2C_R\frac{\|v\|}{\sqrt{\lambda_0}} \le \frac{1}{3}$ and $c_0C_R^2\|v\|_{1,\infty}^2|X| \le \frac{1}{96}\lambda_0$, then there is a constant const such that

$$
\mathrm{Re}\,\big[-\tfrac{1}{2}\|\alpha\|^2 - \tfrac{1}{2}\|\phi\|^2 + F(\varepsilon,\alpha^*,\phi)\big] \le -\tfrac{1}{4}\|\alpha-\phi\|^2 - \tfrac{3}{16}\varepsilon\lambda_0\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big) + \mathrm{const}\,\varepsilon
$$

and

$$
\big|\mathcal{I}_\varepsilon(\alpha,\phi)\big| \le e^{-\frac{1}{4}\|\alpha-\phi\|^2 - \frac{3}{16}\varepsilon\lambda_0\left(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\right) + \mathrm{const}\,\varepsilon} \tag{II.9}
$$

Similarly, by [I, Proposition III.6],

$$
\begin{aligned}
|\mathcal{F}_1(\alpha^*,\phi)| &\le c_0\varepsilon^2|X|\big(|\alpha|_X^2 + |\phi|_X^2 + \|v\|_{1,\infty}^2|\alpha|_X^6 + \|v\|_{1,\infty}^2|\phi|_X^6\big) \\
&\le c_0\varepsilon^2|X|\big(1 + \tfrac{1}{2}\|\alpha\|_{\ell^4}^4 + \tfrac{1}{2}\|\phi\|_{\ell^4}^4\big) + c_0C_R^2\varepsilon\|v\|_{1,\infty}^2|X|\big(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\big)
\end{aligned}
$$

so that, if $c_0C_R^2\|v\|_{1,\infty}^2|X| \le \frac{1}{32}\lambda_0$ and ε is sufficiently small,

$$
\begin{aligned}
\Big|e^{-\mathcal{F}_1(\varepsilon,\alpha^*,\phi)}\Big| &\le \mathrm{const}\, e^{\frac{1}{16}\varepsilon\lambda_0\left(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\right)} \\
\Big|e^{-\mathcal{F}_1(\varepsilon,\alpha^*,\phi)} - 1\Big| &\le \mathrm{const}\,\varepsilon^2\big(1 + |\alpha|_X^6 + |\phi|_X^6\big)e^{\frac{1}{16}\varepsilon\lambda_0\left(\|\alpha\|_{\ell^4}^4 + \|\phi\|_{\ell^4}^4\right)}
\end{aligned}
$$

and

$$\left|\delta\mathcal{I}_\varepsilon(\alpha,\phi)\right| \le e^{\mathrm{Re}\,[-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2+F(\varepsilon,\alpha^*,\phi)]}\left|\left\{e^{-\mathcal{F}_1(\varepsilon,\alpha^*,\phi)}-1\right\}+e^{-\mathcal{F}_1(\varepsilon,\alpha^*,\phi)}\left\{\zeta_\varepsilon(\alpha,\phi)-1\right\}\right|$$
$$\le \mathrm{const}\left\{\varepsilon^2\big(1+|\alpha|_X^6+|\phi|_X^6\big)+\left\{\zeta_\varepsilon(\alpha,\phi)-1\right\}\right\}e^{-\frac{1}{4}\|\alpha-\phi\|^2-\frac{1}{8}\varepsilon\lambda_0\left(\|\alpha\|_{\ell^4}^4+\|\phi\|_{\ell^4}^4\right)}$$

which yields the desired bound. ■

Lemma II.7 *Let $\beta>0$ and assume that $q\in\mathbb{N}$ and $\varepsilon>0$ are such that $0<q\varepsilon\le\beta$. Let $\mathcal{I}_\varepsilon(\alpha,\phi)$ be as in Example II.3 and R_ε obey Hypothesis II.1. Then there are constants a_1, a_2 and a_3 such that*

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le a_1\big(e^{-a_2\|\alpha-\phi\|^2}+e^{-a_3p_0(\varepsilon)^2}\big)$$

Proof: Let

$$c_2 = 65e^{(1+\beta\lambda_0\nu^2)|X|},\qquad c_3=\tfrac{\lambda_0}{40|X|}$$

If either $\|\alpha\|$ of $\|\phi\|$ is larger than $\frac{C_R}{\sqrt{q\varepsilon}}$ then, $t\ge\frac{C_R^2}{2q\varepsilon}$ and, by part (b) of Lemma II.4,

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le c_2\big(\tfrac{1}{\lambda_0 q\varepsilon}e^{-c_3q\varepsilon t^2}+e^{-\frac{t}{8}}\big)\le \tfrac{2c_2}{\lambda_0C_R^2}te^{-\frac{1}{2}c_3C_R^2t}+c_2e^{-\frac{t}{8}}\le c_4e^{-c_5\|\alpha-\phi\|^2}$$

where

$$c_4=c_2\big(\tfrac{2}{\lambda_0C_R^2}\tfrac{4}{c_3C_R^2}+1\big)\qquad c_5=\min\left\{\tfrac{1}{32},\tfrac{1}{16}c_3C_R^2\right\}$$

Here we used that

$$\|\alpha-\phi\|^2\le 2\|\alpha\|^2+2\|\phi\|^2=4t$$

On the other hand if both $\|\alpha\|$ and $\|\phi\|$ are smaller than $\frac{C_R}{\sqrt{q\varepsilon}}$, then by (II.9), with ε replaced by $q\varepsilon$, and part (c) of Lemma II.4,

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le e^{-\frac{1}{4}\|\alpha-\phi\|^2+\mathrm{const}\,q\varepsilon}+\mathrm{const}\,\tfrac{1}{\varepsilon}\big(e^{-\frac{1}{4}\mathrm{R}_\varepsilon^2}+e^{-\frac{\lambda_0}{54|X|}\mathrm{R}_\varepsilon^4q\varepsilon}\big)$$
$$\le e^{-\frac{1}{4}\|\alpha-\phi\|^2+\mathrm{const}\,q\varepsilon}+\mathrm{const}\,\tfrac{1}{\varepsilon}\left\{e^{-\frac{1}{4}p_0(\varepsilon)^2}+e^{-\frac{\lambda_0}{54|X|}p_0(\varepsilon)^4}\right\}$$
$$\le c_7\big(e^{-\frac{1}{4}\|\alpha-\phi\|^2}+e^{-c_6p_0(\varepsilon)^2}\big)$$

In both cases,

$$\left|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\right| \le a_1\big(e^{-a_2\|\alpha-\phi\|^2}+e^{-a_3p_0(\varepsilon)^2}\big)$$

with

$$a_1=\max\{c_4,c_7\},\qquad a_2=c_5,\qquad a_3=c_6$$

■

Proof of Theorem II.2

Lemma II.8 *Under the notation and hypotheses of Theorem II.2 there are constants C and $\kappa > 0$ such that*

$$\int \big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\,\delta\mathcal{I}_\varepsilon(\phi,\gamma)\big| e^{-\kappa Q\varepsilon\|\gamma\|}\ d\mu_{\mathrm{R}_\varepsilon}(\phi^*,\phi) d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma) \le C\Big\{e^{-\kappa\min\{1,(Q+q+1)\varepsilon\}\|\alpha\|} + e^{-\kappa p_0(\varepsilon)^2}\Big\}\min\Big\{\sqrt{\varepsilon},\, \tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\Big\}$$

for all $0 < \varepsilon < 1$, $0 \le q \le \frac{\beta}{\varepsilon}$ and $Q \ge 0$.

Proof: By Lemmas II.4, II.6 and II.7 and the bounds

$$\begin{aligned} |\gamma|_X^6 &\le 2^6\big(|\phi|_X^6 + |\gamma-\phi|_X^6\big) \\ e^{-\delta\|\alpha\|^2} &\le e^\delta e^{-\delta\|\alpha\|} \\ e^{-\delta\|\alpha\|_{\ell^4}^4} &\le e^{\delta|X|} e^{-\delta\|\alpha\|^2} \le e^{\delta(|X|+1)}e^{-\delta\|\alpha\|} \end{aligned}$$

(for all $\delta > 0$) we have

$$\begin{aligned} \big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\big| &\le \tilde{C} e^{-\kappa q\varepsilon\|\alpha\|}\big(e^{-2\kappa\|\alpha-\phi\|} + e^{-2\kappa p_0(\varepsilon)^2}\big)\min\Big\{1, \tfrac{1}{q\varepsilon}e^{-\kappa q\varepsilon\|\phi\|_{\ell^4}^4} + e^{-\kappa\|\phi\|}\Big\} \\ \big|\delta\mathcal{I}_\varepsilon(\phi,\gamma)\big| &\le \tilde{C}\Big\{\varepsilon^2\big(1 + |\phi|_X^6 + |\gamma-\phi|_X^6\big)e^{-\kappa\varepsilon\|\phi\|}e^{-3\kappa\|\phi-\gamma\|}e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4} + e^{-3\kappa p_0(\varepsilon)^2}\Big\} \\ &\le \tilde{C} e^{-\kappa\varepsilon\|\phi\|}\big(e^{-2\kappa\|\phi-\gamma\|} + e^{-2\kappa p_0(\varepsilon)^2}\big)\,\varepsilon^2\big(1 + |\phi|_X^6 e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4} + |\gamma-\phi|_X^6 e^{-\kappa\|\phi-\gamma\|}\big) \end{aligned} \tag{II.10}$$

For the last inequality of (II.10), we used that, for all $|\phi|_X \le \mathrm{R}_\varepsilon$,

$$e^{-\kappa\varepsilon\|\phi\|}\,\varepsilon^2\big(1 + |\phi|_X^6 e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4} + |\gamma-\phi|_X^6 e^{-\kappa\|\phi-\gamma\|}\big) \ge e^{-\kappa\varepsilon \mathrm{R}_\varepsilon\sqrt{|X|}}\varepsilon^2 \ge \mathrm{const}\, e^{-\kappa p_0(\varepsilon)^2}$$

First use (twice) that, for $Q\varepsilon \le 1$ (if $Q\varepsilon > 1$, replace $Q\varepsilon$ by 1),

$$\begin{aligned} q\kappa\varepsilon\|\alpha\| + \kappa\|\alpha-\phi\| + \kappa Q\varepsilon\|\phi\| &\ge q\kappa\varepsilon\|\alpha\| + \kappa Q\varepsilon\|\alpha-\phi\| + \kappa Q\varepsilon\|\phi\| \\ &\ge \kappa(Q+q)\varepsilon\|\alpha\| \end{aligned} \tag{II.11}$$

to prove that

$$\begin{aligned} &e^{-\kappa q\varepsilon\|\alpha\|}\big(e^{-2\kappa\|\alpha-\phi\|} + e^{-2\kappa p_0(\varepsilon)^2}\big)e^{-\kappa\varepsilon\|\phi\|}\big(e^{-2\kappa\|\phi-\gamma\|} + e^{-2\kappa p_0(\varepsilon)^2}\big)e^{-\kappa Q\varepsilon\|\gamma\|} \\ &\quad\le e^{-\kappa q\varepsilon\|\alpha\|}\big(e^{-2\kappa\|\alpha-\phi\|} + e^{-2\kappa p_0(\varepsilon)^2}\big)\big(e^{-\kappa(Q+1)\varepsilon\|\phi\|}e^{-\kappa\|\phi-\gamma\|} + e^{-2\kappa p_0(\varepsilon)^2}\big) \\ &\quad\le e^{-\kappa\min\{1,(Q+q+1)\varepsilon\}\|\alpha\|}e^{-\kappa\|\alpha-\phi\|}e^{-\kappa\|\phi-\gamma\|} + 3e^{-2\kappa p_0(\varepsilon)^2} \end{aligned} \tag{II.12}$$

Next combine

$$\begin{aligned} |\phi|_X^6\big(\tfrac{1}{q\varepsilon}e^{-\kappa q\varepsilon\|\phi\|_{\ell^4}^4} + e^{-\kappa\|\phi\|}\big) &\le \frac{1}{(\kappa q\varepsilon)^{\frac{3}{2}}q\varepsilon}\big(\kappa q\varepsilon\|\phi\|_{\ell^4}^4\big)^{3/2}e^{-\kappa q\varepsilon\|\phi\|_{\ell^4}^4} + \frac{1}{\kappa^6}\big(\kappa\|\phi\|\big)^6 e^{-\kappa\|\phi\|} \\ &\le C_\kappa\big(1 + \tfrac{1}{(q\varepsilon)^{5/2}}\big) \end{aligned}$$

and

$$|\phi|_X^6 e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4} \le \tfrac{1}{(\kappa\varepsilon)^{3/2}}\big(\kappa\varepsilon\|\phi\|_{\ell^4}^4\big)^{3/2} e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4}$$
$$\le C_\kappa \tfrac{1}{\varepsilon^{3/2}}$$

and

$$|\gamma-\phi|_X^6\, e^{-\kappa\|\phi-\gamma\|} \le C_\kappa$$

to give

$$\begin{aligned}\big(1+|\phi|_X^6 e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4}+|\gamma-\phi|_X^6 e^{-\kappa\|\phi-\gamma\|}\big)\min\Big\{1,\tfrac{1}{q\varepsilon}e^{-\kappa q\varepsilon\|\phi\|_{\ell^4}^4}+e^{-\kappa\|\phi\|}\Big\}&\\ \le C_\kappa\Big[3+\min\big\{\tfrac{1}{\varepsilon^{3/2}},\tfrac{1}{(q\varepsilon)^{5/2}}\big\}\Big]&\end{aligned} \tag{II.13}$$

By (II.10), (II.12) and (II.13)

$$\begin{aligned}&\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\delta\mathcal{I}_\varepsilon(\phi,\gamma)\big|e^{-\kappa Q\varepsilon\|\gamma\|}\\ &\le \tilde C^2 C_\kappa\Big[3\varepsilon^2+\min\big\{\sqrt{\varepsilon},\tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\big\}\Big]\big(e^{-\kappa\min\{1,(Q+q+1)\varepsilon\}\|\alpha\|}e^{-\kappa\|\alpha-\phi\|}e^{-\kappa\|\phi-\gamma\|}+3e^{-2\kappa p_0(\varepsilon)^2}\big)\end{aligned}$$

Hence

$$\begin{aligned}&\int\big|\mathcal{I}_\varepsilon^{*_\varepsilon q}(\alpha,\phi)\delta\mathcal{I}_\varepsilon(\phi,\gamma)\big|e^{-\kappa Q\varepsilon\|\gamma\|}\ d\mu_{\mathrm{R}_\varepsilon}(\phi^*,\phi)d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma)\\ &\quad\le\ \tilde C^2 C_\kappa\Big[3\varepsilon^2+\min\big\{\sqrt{\varepsilon},\tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\big\}\Big]\Big(D^2e^{-\kappa\min\{1,(Q+q+1)\varepsilon\}\|\alpha\|}+3\big(\pi\mathrm{R}_\varepsilon^2\big)^{2|X|}e^{-2\kappa p_0(\varepsilon)^2}\Big)\end{aligned}$$

with

$$D=\int e^{-\kappa\|\gamma\|}\ d\mu(\gamma^*,\gamma)$$

For $q\le\frac{\beta}{\varepsilon}$ we have $\varepsilon^2\le\frac{\beta^{5/2}}{q^{5/2}\sqrt{\varepsilon}}$ and the bound follows. ■

Proof of Theorem II.2: Expand

$$(\mathcal{I}_\varepsilon+\delta\mathcal{I}_\varepsilon)^{*_\varepsilon p}-\mathcal{I}_\varepsilon^{*_\varepsilon p}=\sum_{r=1}^{p}\ \sum_{\substack{q_1,\cdots,q_{r+1}\ge0\\ q_1+\cdots+q_{r+1}=p-r}}\mathcal{I}_\varepsilon^{*_\varepsilon q_1}*\delta\mathcal{I}_\varepsilon*\mathcal{I}_\varepsilon^{*_\varepsilon q_2}*\delta\mathcal{I}_\varepsilon*\cdots*\mathcal{I}_\varepsilon^{*_\varepsilon q_r}*\delta\mathcal{I}_\varepsilon*\mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}$$

Hence

$$\begin{aligned}&\Big|\int\big[(\mathcal{I}_\varepsilon+\delta\mathcal{I}_\varepsilon)^{*_\varepsilon p}-\mathcal{I}_\varepsilon^{*_\varepsilon p}\big](\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)\Big|\\ &\quad\le\sum_{r=1}^{p}\ \sum_{\substack{q_1,\cdots,q_{r+1}\ge0\\ q_1+\cdots+q_{r+1}=p-r}}\int d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)\ \sup_{|\gamma|_X\le\mathrm{R}_\varepsilon}\big|\mathcal{I}_\varepsilon^{*_\varepsilon q_1}*\delta\mathcal{I}_\varepsilon*\cdots*\mathcal{I}_\varepsilon^{*_\varepsilon q_r}*\delta\mathcal{I}_\varepsilon*\mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}(\alpha,\gamma)\big|\end{aligned}\tag{II.14}$$

We now prove by backwards induction that, for each $r \ge s \ge 0$,

$$\sup_{|\gamma|_X \le \mathrm{R}_\varepsilon} \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_r} * \delta\mathcal{I}_\varepsilon * \mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}(\alpha,\gamma) \right|$$
$$\le (3C)^{r-s+1} \prod_{\ell=s+1}^{r} \min\left\{ \sqrt{\varepsilon}, \tfrac{1}{q_\ell^{5/2}\sqrt{\varepsilon}} \right\}$$
$$\int d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma) \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_s} * \delta\mathcal{I}_\varepsilon(\alpha,\gamma) \right| \left\{ e^{-\kappa \min\{p_s\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2} \right\} \tag{II.15}$$

with

$$p_s = q_{s+1} + \cdots + q_{r+1} + (r-s)$$

and C the constant of Lemma II.8. When $s=0$, $\mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_s} * \delta\mathcal{I}_\varepsilon(\alpha,\gamma)$ is the kernel of the identity operator.

Consider the initial case, $s=r$. By (II.10),

$$\sup_{|\gamma|_X \le \mathrm{R}_\varepsilon} \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_r} * \delta\mathcal{I}_\varepsilon * \mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}(\alpha,\gamma) \right|$$
$$\le \int d\mu_{\mathrm{R}_\varepsilon}(\alpha'^*,\alpha')$$
$$\sup_{|\gamma|_X \le \mathrm{R}_\varepsilon} \left| \left(\mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_r} * \delta\mathcal{I}_\varepsilon \right)(\alpha,\alpha')\ \mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}(\alpha',\gamma) \right|$$
$$\le C \int d\mu_{\mathrm{R}_\varepsilon}(\alpha'^*,\alpha') \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_r} * \delta\mathcal{I}_\varepsilon(\alpha,\alpha') \right| e^{-\kappa q_{r+1}\varepsilon\|\alpha'\|}$$

which provides the induction hypothesis for $s=r$. Now assume that the induction hypothesis holds for s. Observe that

$$\int d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma) \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_s} * \delta\mathcal{I}_\varepsilon(\alpha,\gamma) \right| \left\{ e^{-\kappa \min\{p_s\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2} \right\}$$
$$\le \int d\mu_{\mathrm{R}_\varepsilon}(\alpha'^*,\alpha') d\mu_{\mathrm{R}_\varepsilon}(\phi^*,\phi) d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma) \left\{ e^{-\kappa \min\{p_s\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2} \right\}$$
$$\left| \left(\mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_{s-1}} * \delta\mathcal{I}_\varepsilon \right)(\alpha,\alpha')\ \mathcal{I}_\varepsilon^{*_\varepsilon q_s}(\alpha',\phi)\ \delta\mathcal{I}_\varepsilon(\phi,\gamma) \right| \tag{II.16}$$

By Lemma II.8, with $q = q_s$,

$$\int d\mu_{\mathrm{R}_\varepsilon}(\phi^*,\phi) d\mu_{\mathrm{R}_\varepsilon}(\gamma^*,\gamma) \left\{ e^{-\kappa \min\{p_s\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2} \right\} \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_s}(\alpha',\phi)\ \delta\mathcal{I}_\varepsilon(\phi,\gamma) \right|$$
$$\le C \left\{ \left[e^{-\kappa \min\{(p_s+q_s+1)\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2} \right] + e^{-\kappa p_0(\varepsilon)^2} \left[e^{-\kappa \min\{(q_s+1)\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2} \right] \right\}$$
$$\min\left\{ \sqrt{\varepsilon}, \tfrac{1}{q_s^{5/2}\sqrt{\varepsilon}} \right\}$$
$$\le 3C \left\{ e^{-\kappa \min\{p_{s-1}\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2} \right\} \min\left\{ \sqrt{\varepsilon}, \tfrac{1}{q_s^{5/2}\sqrt{\varepsilon}} \right\} \tag{II.17}$$

Here we used Lemma II.8 with $Q\varepsilon = \min\{p_s\varepsilon, 1\}$ for the first term in the curly bracket and with $Q = 0$ for the second term in the curly bracket. Inserting this result into (II.16) and then applying the inductive hypothesis (II.15) yields (II.15) with s replaced by $s-1$ and γ replaced by α'. In particular, when $s = 1$, inserting (II.17) into the inductive hypothesis (II.15) yields

$$\sup_{|\gamma|_X \le \mathrm{R}_\varepsilon} \left| \mathcal{I}_\varepsilon^{*_\varepsilon q_1} * \delta\mathcal{I}_\varepsilon * \cdots * \mathcal{I}_\varepsilon^{*_\varepsilon q_r} * \delta\mathcal{I}_\varepsilon * \mathcal{I}_\varepsilon^{*_\varepsilon q_{r+1}}(\alpha,\gamma) \right| \le (3C)^{r+1} \prod_{\ell=1}^{r} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{q_\ell^{5/2}\sqrt{\varepsilon}}\right\} \left\{ e^{-\kappa \min\{p\varepsilon,1\}\|\alpha\|} + e^{-\kappa p_0(\varepsilon)^2} \right\} \tag{II.18}$$

Applying (II.18) to (II.14), it follows that

$$\begin{aligned}
&\left| \int \left[(\mathcal{I}_\varepsilon + \delta\mathcal{I}_\varepsilon)^{*_\varepsilon p} - \mathcal{I}_\varepsilon^{*_\varepsilon p} \right](\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha) \right| \\
&\quad \le \sum_{r=1}^{p} \sum_{\substack{q_1,\cdots,q_{r+1}\ge 0 \\ q_1+\cdots+q_{r+1}=p-r}} (3C)^{r+1} \prod_{\ell=1}^{r} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{q_\ell^{5/2}\sqrt{\varepsilon}}\right\} \\
&\qquad\qquad \int d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)\ \left\{ e^{-\kappa \min\{p\varepsilon,1\}\|\alpha\|} + e^{-\kappa p_0(\varepsilon)^2} \right\} \qquad \text{(II.19)} \\
&\quad \le \mathrm{const} \sum_{r=1}^{\infty} \sum_{q_1,\cdots,q_r\ge 0} (3C)^{r+1} \prod_{\ell=1}^{r} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{q_\ell^{5/2}\sqrt{\varepsilon}}\right\} \\
&\quad \le \mathrm{const} \sum_{r=1}^{\infty} \left[3C \sum_{q\ge 0} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\right\} \right]^r
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{q\ge 0} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\right\} &\le \sum_{0\le q\le \frac{1}{\varepsilon^{2/5}}} \sqrt{\varepsilon} + \sum_{q\ge \frac{1}{\varepsilon^{2/5}}} \tfrac{1}{q^{5/2}\sqrt{\varepsilon}} \le \mathrm{const}\left(\tfrac{\sqrt{\varepsilon}}{\varepsilon^{2/5}} + \tfrac{1}{\sqrt{\varepsilon}}\left(\tfrac{1}{\varepsilon^{2/5}}\right)^{-3/2} \right) \\
&\le \mathrm{const}\, \varepsilon^{\frac{1}{10}}
\end{aligned} \tag{II.20}$$

we get that

$$\left| \int \left[(\mathcal{I}_\varepsilon + \delta\mathcal{I}_\varepsilon)^{*_\varepsilon p} - \mathcal{I}_\varepsilon^{*_\varepsilon p} \right](\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha) \right| = O\big(\varepsilon^{\frac{1}{10}}\big)$$

and the Theorem follows from

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{p\to\infty} \int \mathcal{I}_\varepsilon^{*_\varepsilon p}(\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)$$

which was proven in [I, Theorem III.1]. ■

III. Correlation Functions

By definition, an n–point correlation function at inverse temperature β is an expression of the form

$$\frac{\text{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)}{\text{Tr}\ e^{-\beta K}}$$

Here $\psi^{(\dagger)}$ refers to either ψ or $\psi^\dagger$ and

$$\psi^{(\dagger)}(\tau,\mathbf{x}) = e^{K\tau}\psi^{(\dagger)}(\mathbf{x})e^{-K\tau}$$

The time–ordering operator $\mathbb{T}$ orders the product $\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)$ with smaller times to the right. In the case of equal times, $\psi^\dagger$'s are placed to the right of ψ's. We already have functional integral representations for the denominator, which is just the partition function. In this section, we outline the analogous construction of functional integral representations for the numerator.

Recall that a partition P of the interval $[0,\beta]$ is a finite set of points τ_ℓ, $0\le\ell\le p$, that obeys

$$0=\tau_0\le\tau_1\le\cdots\le\tau_{p-1}\le\tau_p=\beta \tag{III.1}$$

We shall only consider partitions all of whose subintervals $\tau_\ell-\tau_{\ell-1}$ are of roughly the same size. We denote by $p=p(P)$ the number of intervals in the partition P and set $\varepsilon=\varepsilon(P)=\frac{\beta}{p}$. For the rest of this section we fix $\beta>0$, $n\in\mathbb{N}$ and $0=\beta_0\le\beta_1\le\beta_2\le\cdots\le\beta_{n+1}=\beta$. Then

$$\begin{aligned}\frac{\text{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)}{\text{Tr}\ e^{-\beta K}}\\ &= \frac{\text{Tr}\ e^{-(\beta-\beta_n)K}\psi^{(\dagger)}(\mathbf{x}_n)e^{-(\beta_n-\beta_{n-1})K}\psi^{(\dagger)}(\mathbf{x}_{n-1})\cdots e^{-(\beta_2-\beta_1)K}\psi^{(\dagger)}(\mathbf{x}_1)e^{-\beta_1 K}}{\text{Tr}\ e^{-\beta K}}\end{aligned}$$

Definition III.1

(a) A $(\beta_0,\cdots,\beta_{n+1})$–partition is a partition $P=\left\{\ \tau_\ell\ \middle|\ 0\le\ell\le p\ \right\}$ of the interval $[\beta_0,\beta_{n+1}]$

- (i) that contains the points $\beta_1,\ \beta_2,\ \cdots,\ \beta_n$ and for which
- (ii) $\frac{1}{2}\varepsilon(P)\le\tau_\ell-\tau_{\ell-1}\le 2\varepsilon(P)$ for all $1\le\ell\le p$.

(b) We denote by $\mathcal{P}=\mathcal{P}(\beta_0,\cdots,\beta_{n+1})$ the set of all $(\beta_0,\cdots,\beta_{n+1})$–partitions. When we say that

$$\lim_{p\to\infty} f(P)=F$$

we mean that for every $\eta>0$ there is an $N\in\mathbb{N}$ such that $|F-f(P)|<\eta$ for all $P\in\mathcal{P}(\beta_0,\cdots,\beta_{n+1})$ with $p(P)\ge N$.

The analog of [I, Theorem III.1] is

Theorem III.2 *Let* $\mathrm{R}(P,\ell)>0$*, for each* $P\in\mathcal{P}$ *and* $1\le\ell\le p(P)$*, and assume that*

$$\lim_{p\to\infty}\sum_{\ell=1}^{p(P)} e^{-\frac{1}{2}\mathrm{R}(P,\ell)^2} \;=\; 0 \tag{III.2}$$

Then,

$$\begin{aligned}
&\mathrm{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^n \psi^{(\dagger)}(\beta_j,\mathbf{x}_j)\\
&=\lim_{p\to\infty}\int\prod_{\ell=1}^{p(P)}\Big[d\mu_{\mathrm{R}(P,\ell)}(\phi^*_{\tau_\ell},\phi_{\tau_\ell})\, e^{-\int d\mathbf{y}\ |\phi_{\tau_\ell}(\mathbf{y})|^2}\Big\langle\, \phi_{\tau_{\ell-1}}\,\Big|\, e^{-(\tau_\ell-\tau_{\ell-1})\,K}\,\Big|\,\phi_{\tau_\ell}\Big\rangle\Big]\prod_{j=1}^n\phi_{\beta_j}(\mathbf{x}_j)^{(*)}
\end{aligned}$$

with the convention that $\phi_0=\phi_\beta$ *.*

Example III.3 Let $C>\sqrt{2}$. Any $\mathrm{R}(P,\ell)$'s that obey $\mathrm{R}(P,\ell)\ge C\sqrt{\ln p(P)}$ satisfy the hypothesis of Theorem III.2, because

$$\sum_{\ell=1}^{p(P)} e^{-\frac{1}{2}\mathrm{R}(P,\ell)^2}\le\sum_{\ell=1}^{p(P)} p(P)^{-\frac{1}{2}C^2}\le p(P)^{1-\frac{1}{2}C^2}$$

Remark III.4 In fact Theorem III.2 does not use condition (ii) of Definition III.1.a. It suffices to require (III.2). For example, any $\mathrm{R}(P,\ell)$'s that obey

$$\mathrm{R}(P,\ell)\ge C\sqrt{\ln\tfrac{1}{\tau_\ell-\tau_{\ell-1}}}\qquad\text{where } P=\{0=\tau_0,\tau_1,\cdots,\tau_{p-1}\le\tau_p=\beta\}\text{ and } C>\sqrt{2}$$

work, as long as the mesh $\|P\|=\max\limits_{1\le\ell\le p(P)}(\tau_\ell-\tau_{\ell-1})$ tends to zero, because

$$\sum_{\ell=1}^{p(P)} e^{-\frac{1}{2}\mathrm{R}(P,\ell)^2}\le\sum_{\ell=1}^{p(P)}(\tau_\ell-\tau_{\ell-1})^{\frac{1}{2}C^2}\le\|P\|^{\frac{1}{2}C^2-1}\beta$$

Proof of Theorem III.2: We may assume, without loss of generality, that the number $\psi^\dagger$'s is the same as the number of ψ's so that the operator $\prod_{j=1}^n\big(\psi^{(\dagger)}(\mathbf{x}_j)e^{-(\beta_{j+1}-\beta_j)K}\big)$ commutes with the number operator. Otherwise, both sides are zero. (To see that the right hand side vanishes, use invariance under $\phi_{\tau_\ell}\to\phi_{\tau_\ell}e^{i\theta}$.) So, by the definition of I_r (given in the statement of [I, Theorem II.26]), [I, Proposition II.20] and [I, Proposition II.28] (we'll prove boundedness of the appropriate operator shortly), the integral on the right hand side can be written

$$\mathrm{Tr}\prod_{\ell=1}^{p(P)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi^-_\ell\,\mathrm{I}_{\mathrm{R}(P,\ell)}\Psi^+_\ell$$

where the product is ordered with smaller indices on the right,

$$\Psi_\ell^- = \prod_{j=1}^{n} \begin{cases} \psi(\mathbf{x}_j) & \text{if } \beta_j = \tau_\ell \text{ and } \psi^{(\dagger)}(\mathbf{x}_j) = \psi(\mathbf{x}_j) \\ \mathbb{1} & \text{otherwise} \end{cases}$$

and

$$\Psi_\ell^+ = \prod_{j=1}^{n} \begin{cases} \psi^\dagger(\mathbf{x}_j) & \text{if } \beta_j = \tau_\ell \text{ and } \psi^{(\dagger)}(\mathbf{x}_j) = \psi^\dagger(\mathbf{x}_j) \\ \mathbb{1} & \text{otherwise} \end{cases}$$

Replacing all the $\mathrm{I}_{\mathrm{R}(P,\ell)}$'s by $\mathbb{1}$ gives the trace on the left hand side.

Recall that $P^{(m)}$ is the orthogonal projection on the m particle space $\mathcal{B}_m(X)$ and that P_m is the orthogonal projection on $\bigoplus_{\ell\le m} \mathcal{B}_\ell(X)$. Since K and I_r preserve particle number, $\psi^\dagger$ increases it by one and ψ decreases it by one, there are, for each $m \in \mathbb{N}\cup\{0\}$, integers $m-n \le m_\ell \le m+n$, $1 \le \ell \le p(P)$, such that

$$P^{(m)} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi_\ell^- \mathrm{I}_{\mathrm{R}(P,\ell)}\Psi_\ell^+ = \prod_{\ell=1}^{p(P)} P^{(m_\ell)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi_\ell^- \mathrm{I}_{\mathrm{R}(P,\ell)}\Psi_\ell^+$$

Recall from (II.2) that, if $\frac{m}{|X|} \ge 2\nu$ (the constant ν was defined just after (II.2)) and $\tau \ge 0$, then

$$\big\|P^{(m)}e^{-\tau K}\big\| \le e^{-\frac{\lambda_0}{4|X|}m^2\tau} \tag{III.3}$$

By [I, Lemma II.13], the local density operator $\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$, when restricted to the m particle space $\mathcal{B}_m$, has eigenvalues ℓ running over the integers from 0 to m. As a consequence

$$\big\|\psi(\mathbf{x})P^{(m)}\big\| \le \sqrt{m} \qquad \text{and} \qquad \big\|P^{(m)}\psi^\dagger(\mathbf{x})\big\| \le \sqrt{m} \tag{III.4}$$

Hence if each J_ℓ, $1 \le \ell \le p(P)$, is either $\mathrm{I}_{\mathrm{R}(P,\ell)}$ or $\mathbb{1}$, we have

$$\Big\|P^{(m)} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi_\ell^- J_\ell \Psi_\ell^+\Big\| \le e^{-\frac{\lambda_0}{4|X|}(m-n)^2\beta}(m+n)^{\frac{n}{2}}$$

assuming that $\frac{m-n}{|X|} \ge 2\nu$. Pick any $\gamma > 0$ with $2\gamma < \frac{\lambda_0\beta}{4|X|}$. Then there is a constant (depending only on γ, $\frac{\lambda_0\beta}{4|X|}$ and n) such that

$$\Big\|P^{(m)} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi_\ell^- J_\ell \Psi_\ell^+\Big\| \le \mathrm{const}\, e^{-2\gamma m^2}$$

This supplies the boundedness required for the application of [I, Proposition II.28] referred to earlier. As in [I, (III.6)], this also implies that

$$\Big|\mathrm{Tr}\,(\mathbb{1}-P_{\tilde m}) \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell-\tau_{\ell-1})\,K}\Psi_\ell^- J_\ell \Psi_\ell^+\Big| \le Ce^{-\gamma\tilde m^2} \tag{III.5}$$

for all sufficiently large $\tilde{m}$.

If one J_ℓ with $1 \le \ell \le p(P)$, say $\ell = \ell_0$, is $\mathrm{I}_{\mathrm{R}(P,\ell)} - \mathbb{1}$ and each of the others is either $\mathrm{I}_{\mathrm{R}(P,\ell)}$ or $\mathbb{1}$, then, by part (d) of [I, Theorem II.26] and the fact that $K \ge -K_0$ (where, by (II.2), $K_0 = \frac{\lambda_0}{8}\nu^2|X|$)

$$\Big\| P^{(m)} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell - \tau_{\ell-1})\,K} \Psi_\ell^- J_\ell \Psi_\ell^+ \Big\| \le e^{K_0\beta}(m+n)^{\frac{n}{2}}|X| 2^{m+n+1} e^{-\mathrm{R}(P,\ell_0)^2/2}$$

and

$$\Big| \mathrm{Tr}\, P_{\tilde{m}} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell - \tau_{\ell-1})\,K} \Psi_\ell^- J_\ell \Psi_\ell^+ \Big| \le C_{\tilde{m}} e^{-\mathrm{R}(P,\ell_0)^2/2} \tag{III.6}$$

with the constant $C_{\tilde{m}}$ depending on $K_0\beta$, n and $|X|$ as well as $\tilde{m}$. Using the usual telescoping decomposition of a difference of products and applying the bounds (III.5) and (III.6) now gives

$$\begin{aligned} &\Big| \mathrm{Tr}\ e^{-\beta K} \mathbb{T} \prod_{j=1}^{n} \psi^{(\dagger)}(\beta_j, \mathbf{x}_j) - \mathrm{Tr} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell - \tau_{\ell-1})\,K} \Psi_\ell^- \mathrm{I}_{\mathrm{R}(P,\ell)} \Psi_\ell^+ \Big| \\ &\qquad = \Big| \mathrm{Tr} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell - \tau_{\ell-1})\,K} \Psi_\ell^- \mathbb{1} \Psi_\ell^+ - \mathrm{Tr} \prod_{\ell=1}^{p(P)} e^{-(\tau_\ell - \tau_{\ell-1})\,K} \Psi_\ell^- \mathrm{I}_{\mathrm{R}(P,\ell)} \Psi_\ell^+ \Big| \\ &\qquad \le 2Ce^{-\gamma \tilde{m}^2} + \sum_{\ell_0=1}^{p(P)} C_{\tilde{m}} e^{-\mathrm{R}(P,\ell_0)^2/2} \end{aligned}$$

for all sufficiently large $\tilde{m}$. The claim follows by choosing, for each $\varepsilon > 0$, $\tilde{m}$ large enough that $2Ce^{-\gamma\tilde{m}^2} < \frac{\varepsilon}{2}$ and then choosing p large enough that the remaining sum is smaller than $\frac{\varepsilon}{2}$. ■

The analog of [I, Theorem III.13] is

Theorem III.5 *Let*

$$\max_{1 \le \ell \le p(P)} \mathrm{R}(P,\ell) \le \left(\tfrac{1}{\varepsilon(P)}\right)^{\frac{1}{6(n+3)(|X|+1)}}$$

for each $P \in \mathcal{P}$ and assume that

$$\lim_{p\to\infty} \sum_{\ell=1}^{p(P)} e^{-\frac{1}{2}\mathrm{R}(P,\ell)^2} \;=\; 0$$

Then,

$$\begin{aligned}&\mathrm{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)\\ &\quad=\lim_{p\to\infty}\int\prod_{\ell=1}^{p(P)}\Big[d\mu_{\mathrm{R}(P,\ell)}(\phi^*_{\tau_\ell},\phi_{\tau_\ell})\ e^{-\int d\mathbf{y}\ [\phi^*_{\tau_\ell}(\mathbf{y})-\phi^*_{\tau_{\ell-1}}(\mathbf{y})]\phi_{\tau_\ell}(\mathbf{y})}\ e^{-(\tau_\ell-\tau_{\ell-1})K(\phi^*_{\tau_{\ell-1}},\phi_{\tau_\ell})}\Big]\\ &\qquad\qquad\prod_{j=1}^{n}\phi_{\beta_j}(\mathbf{x}_j)^{(*)}\end{aligned}$$

with the convention that $\phi_0=\phi_\beta$ *. Recall that* $K(\alpha^*,\phi)$ *was defined in [I, Corollary III.7].*

Proof: We give the proof for the case that $0=\beta_0<\beta_1<\beta_2<\cdots<\beta_n<\beta=\beta_{n+1}$. The proofs for the other cases require only very minor changes. Let, as in [I, Examples III.15 and III.17],

$$\begin{aligned}\mathcal{I}_\varepsilon(\alpha,\phi)&=e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}e^{F(\varepsilon,\alpha^*,\phi)} &&=e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\left\langle\alpha\,\middle|\,e^{-\varepsilon K}\,\middle|\,\phi\right\rangle\\ \tilde{\mathcal{I}}_\varepsilon(\alpha,\phi)&=e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}e^{F(\varepsilon,\alpha^*,\phi)-\mathcal{F}_0(\varepsilon,\alpha^*,\phi)} &&=e^{-\frac12\|\alpha\|^2-\frac12\|\phi\|^2}\left\langle\alpha\,\middle|\,e^{-\varepsilon K}\,\middle|\,\phi\right\rangle e^{-\mathcal{F}_0(\varepsilon,\alpha^*,\phi)}\end{aligned}$$

where $\mathcal{F}_0$ was defined in [I, Corollary III.7]. Recall that

$$\tilde{\mathcal{I}}_\varepsilon(\alpha,\phi)=\exp\Big\{-\tfrac12\|\alpha\|^2-\tfrac12\|\phi\|^2+\int d\mathbf{x}\ \alpha^*(\mathbf{x})\phi(\mathbf{x})-\varepsilon K(\alpha^*,\phi)\Big\}$$

For any partition $P=\{0=\tau_0<\tau_1<\cdots<\tau_p=\beta\}\in\mathcal{P}$, set, for $1\le\ell\le m\le p(P)$,

$$\begin{aligned}\mathfrak{I}_{\ell,m}(\phi,\phi')&=\big(\mathcal{I}_{\varepsilon_\ell}*_{\mathrm{R}(P,\ell)}\mathcal{I}_{\varepsilon_{\ell+1}}*_{\mathrm{R}(P,\ell+1)}\cdots*_{\mathrm{R}(P,m-1)}\mathcal{I}_{\varepsilon_m}\big)(\phi,\phi')\\ \tilde{\mathfrak{I}}_{\ell,m}(\phi,\phi')&=\big(\tilde{\mathcal{I}}_{\varepsilon_\ell}*_{\mathrm{R}(P,\ell)}\tilde{\mathcal{I}}_{\varepsilon_{\ell+1}}*_{\mathrm{R}(P,\ell+1)}\cdots*_{\mathrm{R}(P,m-1)}\tilde{\mathcal{I}}_{\varepsilon_m}\big)(\phi,\phi')\end{aligned}$$

where $\varepsilon_\ell=\tau_\ell-\tau_{\ell-1}$. The convolution $*_r$ was introduced in [I, Definition III.14]. By Theorem III.2, if $\beta_j=\tau_{\ell_j}$ for $1\le j\le n$,

$$\begin{aligned}&\mathrm{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j)\\ &\quad=\lim_{p\to\infty}\int\prod_{j=0}^{n}d\mu_{\mathrm{R}(P,\ell_j)}(\phi^*_{\beta_j},\phi_{\beta_j})\ \prod_{j=0}^{n}\mathfrak{I}_{\ell_j+1,\ell_{j+1}}(\phi_{\beta_j},\phi_{\beta_{j+1}})\ \prod_{j=1}^{n}\phi_{\beta_j}(\mathbf{x}_j)^{(*)}\end{aligned}\tag{III.7}$$

where $\ell_0=\beta_0=0$, $\ell_{n+1}=p(P)$, $\phi_{\beta_{n+1}}=\phi_0$ and $\mathrm{R}(P,0)=\mathrm{R}(P,p(P))$. On the other hand, the right hand side of the claim of the current Theorem is

$$\lim_{p\to\infty}\int\prod_{j=0}^{n}d\mu_{\mathrm{R}(P,\ell_j)}(\phi^*_{\beta_j},\phi_{\beta_j})\ \prod_{j=0}^{n}\tilde{\mathfrak{I}}_{\ell_j+1,\ell_{j+1}}(\phi_{\beta_j},\phi_{\beta_{j+1}})\ \prod_{j=1}^{n}\phi_{\beta_j}(\mathbf{x}_j)^{(*)}\tag{III.8}$$

We apply Proposition III.6.b, below with $\mathcal{I}_\ell$ replaced by $\mathcal{I}_{\varepsilon_\ell}$, $\tilde{\mathcal{I}}_\ell$ replaced by $\tilde{\mathcal{I}}_{\varepsilon_\ell}$, $\zeta_\ell = \varepsilon_\ell^{3/2}$, $r_\ell = \mathrm{R}(P,\ell)$, $\kappa = \frac{1}{12}$, $p = p(P)$ and $C_\beta = \beta$. If $p(P)$ is sufficiently large, the hypotheses of the Proposition are satisfied because then

$$C_\beta \left(\pi \max_\ell r_\ell^2\right)^{(n+3)(|X|+1)} \zeta_\ell^{1-\kappa} \le \beta\pi^{(n+3)(|X|+1)} \left(\tfrac{1}{\varepsilon(P)}\right)^{\frac{1}{3}} \varepsilon_\ell^{\frac{3}{2}(1-\frac{1}{12})} \le 2\beta\pi^{(n+3)(|X|+1)}\varepsilon_\ell^{\frac{25}{24}} \le \varepsilon_\ell$$

since $\frac{1}{\varepsilon(P)} \le \frac{2}{\varepsilon_\ell}$, and, by [I, Example III.17] with $r = \left(\frac{2}{\varepsilon_\ell}\right)^{\frac{1}{24|X|}} \ge r_{\ell-1}, r_\ell$,

$$\|\mathcal{I}_{\varepsilon_\ell} - \tilde{\mathcal{I}}_{\varepsilon_\ell}\|_{r_{\ell-1},r_\ell} \le e^{\varepsilon_\ell K_0}\ \mathrm{const}\,\varepsilon_\ell^2 \left(\tfrac{2}{\varepsilon_\ell}\right)^{\frac{1}{4|X|}} |X|\ e^{\mathrm{const}\,|X|\varepsilon_\ell^2 (2/\varepsilon_\ell)^{\frac{1}{4|X|}}} \le \varepsilon_\ell^{3/2} = \zeta_\ell$$

and, as [I, Example III.15] (just replace the $q-1$ appearances of I_r by $\mathrm{I}_{r_\ell}, \cdots, \mathrm{I}_{r_{m-1}}$ and the q appearances of $e^{-\varepsilon K}$ by $e^{-\varepsilon_\ell K}, \cdots, e^{-\varepsilon_m K}$),

$$\|\mathcal{I}_{\varepsilon_\ell} *_{r_\ell} \mathcal{I}_{\varepsilon_{\ell+1}} *_{r_{\ell+1}} \cdots *_{r_{m-1}} \mathcal{I}_{\varepsilon_m}\|_{r_{\ell-1},r_m} \le e^{(\varepsilon_\ell+\cdots+\varepsilon_m)K_0}$$

The Theorem now follows by Proposition III.6.b, (III.7) and (III.8). ■

Proposition III.6 *Let $K_0 > 0$, $0 < \kappa < 1$, $C_\beta \ge 1$ and $p \in \mathbb{N}$. Let $r_0, \cdots, r_p \ge 1$, $\varepsilon_1, \cdots, \varepsilon_p > 0$ and $\zeta_1, \cdots, \zeta_p > 0$ and assume that $\varepsilon_1 + \cdots + \varepsilon_p \le C_\beta$. For each $1 \le \ell \le p$, let $\mathcal{I}_\ell, \tilde{\mathcal{I}}_\ell : \mathbb{C}^{2|X|} \to \mathbb{C}$. Define, for each $1 \le \ell \le m \le p$,*

$$\mathfrak{I}_{\ell,m} = \mathcal{I}_\ell *_{r_\ell} \mathcal{I}_{\ell+1} *_{r_{\ell+1}} \cdots *_{r_{m-1}} \mathcal{I}_m \qquad \tilde{\mathfrak{I}}_{\ell,m} = \tilde{\mathcal{I}}_\ell *_{r_\ell} \tilde{\mathcal{I}}_{\ell+1} *_{r_{\ell+1}} \cdots *_{r_{m-1}} \tilde{\mathcal{I}}_m$$

and assume that

$$\|\mathcal{I}_\ell - \tilde{\mathcal{I}}_\ell\|_{r_{\ell-1},r_\ell} \le \zeta_\ell \qquad \|\mathfrak{I}_{\ell,m}\|_{r_{\ell-1},r_m} \le e^{(\varepsilon_\ell+\cdots+\varepsilon_m)K_0}$$

where

$$\|\mathcal{I}\|_{r,r'} = \sup_{\substack{\phi,\phi'\in\mathbb{C}^X \\ |\phi|_X \le r,\ |\phi'|_X \le r'}} \left|\mathcal{I}(\phi,\phi')\right|$$

(a) If

$$C_\beta \left(\pi r_{\ell-1}^2\right)^{|X|} \zeta_\ell^{1-\kappa} \left(\pi r_\ell^2\right)^{|X|} \le \varepsilon_\ell \text{ for } \ell = 1, \cdots, p$$

then, setting $\zeta = \max_{1\le\ell\le p} \zeta_\ell$,

$$\begin{aligned} \|\tilde{\mathfrak{I}}_{1,p}\|_{r_0,r_p} &\le e^{(\varepsilon_1+\cdots+\varepsilon_p)(K_0+\zeta^\kappa)} \\ \|\tilde{\mathfrak{I}}_{1,p} - \mathfrak{I}_{1,p}\|_{r_0,r_p} &\le \zeta^\kappa e^{(\varepsilon_1+\cdots+\varepsilon_p)(K_0+\zeta^\kappa)} \end{aligned}$$

(b) Let $n \in \mathbb{N}$ and $0 \le \ell_1 \le \cdots \le \ell_n \le p$. If $r_1, \cdots, r_p \le r$ with

$$C_\beta \left(\pi r^2\right)^{(n+3)(|X|+1)} \zeta_\ell^{1-\kappa} \le \varepsilon_\ell \text{ for } \ell = 1, \cdots, p$$

then

$$
\begin{aligned}
\Big| \int \prod_{\ell=0}^{p-1} d\mu_{r_\ell}(\phi_\ell^*,\phi_\ell)\ \prod_{\ell=1}^{p} \mathcal{I}_\ell(\phi_{\ell-1},\phi_\ell)\ \prod_{j=1}^{n} \phi_{\ell_j}(\mathbf{x}_j)^{(*)} \\
- \int \prod_{\ell=0}^{p-1} d\mu_{r_\ell}(\phi_\ell^*,\phi_\ell)\ \prod_{\ell=1}^{p} \tilde{\mathcal{I}}_\ell(\phi_{\ell-1},\phi_\ell)\ \prod_{j=1}^{n} \phi_{\ell_j}(\mathbf{x}_j)^{(*)} \Big| \\
= \Big| \int \prod_{j=0}^{n} d\mu_{r_{\ell_j}}(\phi_{\ell_j}^*,\phi_{\ell_j})\ \prod_{j=0}^{n} \mathfrak{I}_{\ell_j+1,\ell_{j+1}}(\phi_{\ell_j},\phi_{\ell_{j+1}})\ \prod_{j=1}^{n} \phi_{\ell_j}(\mathbf{x}_j)^{(*)} \\
- \int \prod_{j=0}^{n} d\mu_{r_{\ell_j}}(\phi_{\ell_j}^*,\phi_{\ell_j})\ \prod_{j=0}^{n} \tilde{\mathfrak{I}}_{\ell_j+1,\ell_{j+1}}(\phi_{\ell_j},\phi_{\ell_{j+1}})\ \prod_{j=1}^{n} \phi_{\ell_j}(\mathbf{x}_j)^{(*)} \Big| \\
\le \zeta^\kappa e^{(\varepsilon_1+\cdots+\varepsilon_p)(K_0+\zeta^\kappa)}
\end{aligned}
$$

where $\ell_0 = 0$, $\ell_{n+1} = p$ *and, as usual,* $\phi_p = \phi_0$.

Proof: The proof is very similar to the proof of [I, Proposition III.16]. ∎

The analog of Theorem II.2 is

Theorem III.7 *Let* R_ε *and* $p_0(\varepsilon)$ *obey Hypothesis II.1 and* $j(\varepsilon)$ *obey (II.1). Let* $\beta > 0$. *Set* $\mathrm{R}(P,\ell) = \mathrm{R}_{\varepsilon(P)}$, *for each partition* $P = \big\{0 = \tau_0 < \tau_1 < \cdots < \tau_p = \beta\big\} \in \mathcal{P}$ *and* $1 \le \ell \le p(P)$. *Then, with the conventions that* $\varepsilon = \varepsilon(P) = \frac{\beta}{p(P)}$, $p = p(P)$, $\varepsilon_\ell = \tau_\ell - \tau_{\ell-1}$ *and* $\phi_0 = \phi_\beta$,

$$
\begin{aligned}
\mathrm{Tr}\ e^{-\beta K}\mathbb{T} \prod_{j=1}^{n} \psi^{(\dagger)}(\beta_j,\mathbf{x}_j) \\
= \lim_{p\to\infty} \int \prod_{\ell=1}^{p} \Big[d\mu_{\mathrm{R}_\varepsilon}(\phi_{\tau_\ell}^*,\phi_{\tau_\ell})\ \zeta_\varepsilon(\phi_{\tau_{\ell-1}},\phi_{\tau_\ell})\ e^{\mathcal{A}(\varepsilon_\ell,\phi_{\tau_{\ell-1}}^*,\phi_{\tau_\ell})} \Big]\ \prod_{j=1}^{n} \phi_{\beta_j}(\mathbf{x}_j)^{(*)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}(\varepsilon_\ell,\alpha^*,\phi) = -\tfrac{1}{2}\|\alpha\|^2 + \iint_{X^2} d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^* j(\varepsilon_\ell;\mathbf{x},\mathbf{y})\phi(\mathbf{y}) - \tfrac{1}{2}\|\phi\|^2 \\
- \tfrac{\varepsilon_\ell}{2} \iint_{X^2} d\mathbf{x}d\mathbf{y}\ \alpha(\mathbf{x})^*\alpha(\mathbf{y})^*\, v(\mathbf{x},\mathbf{y})\, \phi(\mathbf{x})\phi(\mathbf{y})
\end{aligned}
$$

and $\zeta_\varepsilon(\alpha,\phi)$ *is the characteristic function of* $|\alpha - \phi|_X \le p_0(\varepsilon)$.

We prove this theorem following the proof of Lemma III.10, below. Until we start the proof of Theorem III.7, we fix a partition $P = \big\{0 = \tau_0 < \tau_1 < \cdots < \tau_p = \beta\big\} \in \mathcal{P}$ and set $\varepsilon = \frac{\beta}{p}$, $\varepsilon_\ell = \tau_\ell - \tau_{\ell-1}$ and $r = \mathrm{R}_\varepsilon$. We let $\mathcal{I}_\varepsilon$ and $\delta\mathcal{I}_\varepsilon$ be defined as in Example II.3 and write $\mathcal{I}_\ell$ for $\mathcal{I}_{\varepsilon_\ell}$ and $\hat{\mathcal{I}}_\ell$ for $\mathcal{I}_{\varepsilon_\ell} + \delta\mathcal{I}_{\varepsilon_\ell}$. Thus

$$\begin{aligned}
\mathcal{I}_\ell(\alpha,\phi) &= e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2} e^{F(\varepsilon_\ell,\alpha^*,\phi)} = e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2} \left\langle \alpha \,\middle|\, e^{-\varepsilon_\ell K} \,\middle|\, \phi \right\rangle \\
\hat{\mathcal{I}}_\ell(\alpha,\phi) &= e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2} e^{F(\varepsilon_\ell,\alpha^*,\phi)-\mathcal{F}_1(\varepsilon_\ell,\alpha^*,\phi)}\zeta_\varepsilon(\alpha,\phi) \\
\delta\mathcal{I}_\ell(\alpha,\phi) &= e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2} e^{F(\varepsilon_\ell,\alpha^*,\phi)}\big\{e^{-\mathcal{F}_1(\varepsilon_\ell,\alpha^*,\phi)}\zeta_\varepsilon(\alpha,\phi) - 1\big\}
\end{aligned}$$

where $\mathcal{F}_1$ was defined in [I, Proposition III.6]. Recall that

$$-\tfrac{1}{2}\|\alpha\|^2 - \tfrac{1}{2}\|\phi\|^2 + F(\varepsilon_\ell,\alpha^*,\phi) - \mathcal{F}_1(\varepsilon_\ell,\alpha^*,\phi) = \mathcal{A}(\varepsilon_\ell,\alpha^*,\phi)$$

We also introduce analogs of $\mathcal{I}_\ell$ and $\hat{\mathcal{I}}_\ell$ that contain the appropriate correlation fields from $\prod_{j=1}^n \phi_{\beta_j}(\mathbf{x}_j)^{(*)}$.

$$\mathcal{C}_\ell(\alpha,\phi) = \mathcal{I}_\ell(\alpha,\phi)\Phi_\ell(\phi) \qquad \hat{\mathcal{C}}_\ell(\alpha,\phi) = \hat{\mathcal{I}}_\ell(\alpha,\phi)\Phi_\ell(\phi) \qquad \delta\mathcal{C}_\ell(\alpha,\phi) = \delta\mathcal{I}_\ell(\alpha,\phi)\Phi_\ell(\phi) \tag{III.9}$$

where

$$\Phi_\ell(\phi) = \prod_{j=1}^n \begin{cases} \phi_{\beta_j}(\mathbf{x}_j) & \text{if } \beta_j = \tau_\ell \text{ and } \psi^{(\dagger)}(\mathbf{x}_j) = \psi(\mathbf{x}_j) \\ \phi_{\beta_j}(\mathbf{x}_j)^* & \text{if } \beta_j = \tau_\ell \text{ and } \psi^{(\dagger)}(\mathbf{x}_j) = \psi^\dagger(\mathbf{x}_j) \\ 1 & \text{otherwise} \end{cases}$$

The various convolutions are

$$\begin{aligned}
\mathfrak{I}_{\ell,m}(\phi,\phi') &= \big(\mathcal{I}_\ell *_r \mathcal{I}_{\ell+1} *_r \cdots *_r \mathcal{I}_m\big)(\phi,\phi') \\
\mathfrak{C}_{\ell,m}(\phi,\phi') &= \big(\mathcal{C}_\ell *_r \mathcal{C}_{\ell+1} *_r \cdots *_r \mathcal{C}_m\big)(\phi,\phi') \\
\hat{\mathfrak{C}}_{\ell,m}(\phi,\phi') &= \big(\hat{\mathcal{C}}_\ell *_r \hat{\mathcal{C}}_{\ell+1} *_r \cdots *_r \hat{\mathcal{C}}_m\big)(\phi,\phi')
\end{aligned} \tag{III.10}$$

We have proven, in Theorem III.2, that the left hand side in Theorem III.7 is

$$\lim_{p\to\infty} \int \mathfrak{C}_{1,p}(\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)$$

On the other hand, the right hand side in Theorem III.7 is

$$\lim_{p\to\infty} \int \hat{\mathfrak{C}}_{1,p}(\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)$$

Lemma III.8 *Let $1 \le \ell \le m \le p$ and write $\bar\varepsilon = \varepsilon_\ell + \cdots + \varepsilon_m$. Then*

(a) For any $\gamma > 0$,

$$\big|\mathfrak{I}_{\ell,m}(\alpha,\phi)\big| \le c_1 e^{-\frac{1}{2}\min\{1,\bar\varepsilon\lambda_0\gamma\}t}$$

where

$$t = \tfrac{1}{2}\big(\|\alpha\|^2 + \|\phi\|^2\big) \qquad c_1 = e^{\bar\varepsilon\lambda_0(\nu+\gamma)^2|X|}$$

(b) We have

$$\big|\mathfrak{I}_{\ell,m}(\alpha,\phi)\big| \le c_2\big(\tfrac{1}{\bar\varepsilon\lambda_0}e^{-c_3\bar\varepsilon t^2}+e^{-\frac{t}{8}}\big)$$

where

$$t=\tfrac{1}{2}\big(\|\alpha\|^2+\|\phi\|^2\big)\quad c_2=65e^{(1+\bar\varepsilon\lambda_0\nu^2)|X|}\quad c_3=\tfrac{\lambda_0}{40|X|}$$

(c) Let $\beta>0$ and assume that $0<\varepsilon_\ell+\cdots+\varepsilon_m\le\beta$. If r is large enough (depending only on ν and $|X|$), then there is a constant const *(depending only on $|X|$, β, λ_0 and ν) such that*

$$\Big|\mathfrak{I}_{\ell,m}(\alpha,\phi)-e^{-\frac{1}{2}\|\alpha\|^2-\frac{1}{2}\|\phi\|^2}\left\langle\,\alpha\,\middle|\,e^{-\bar\varepsilon K}\,\middle|\,\phi\,\right\rangle\Big|\le\text{const}\Big\{(m-\ell)e^{-\frac{1}{4}r^2}+\tfrac{1}{\sqrt{\bar\varepsilon}}e^{-\frac{\lambda_0}{54|X|}r^4\bar\varepsilon}\Big\}$$

The proof of this lemma is virtually the same as the proof of its analog, Lemma II.4.

For the rest of this section, except where otherwise specified, all constants may depend on $|X|$, v, $\|\mathrm{h}\|_{1,\infty}$, c_j, β, μ and n. They may not depend on the partition P and, in particular, on $\varepsilon=\varepsilon(P)$ or $p=p(P)$.

Lemma III.9 *Let $\mathfrak{I}_{\ell,m}(\alpha,\phi)$ be as in (III.10). There are constants a_1, a_2 and a_3 such that*

$$\big|\mathfrak{I}_{\ell,m}(\alpha,\phi)\big|\le a_1\big(e^{-a_2\|\alpha-\phi\|^2}+e^{-a_3p_0(\varepsilon)^2}\big)$$

for all $1\le\ell\le m\le p$.

The proof of this lemma is virtually identical to that of its analog, Lemma II.7.

Lemma III.10 *Under the notation and hypotheses of Theorem III.7 there are constants C and $\kappa>0$ such that the following holds. Let $0<\varepsilon<1$ and $1\le\ell\le m\le p$ and set $q=m-\ell+1$. Write $|\gamma|_+=\max\{1,|\gamma|_X\}$.*

(a) Denote by n' the total number of $\phi_{\beta_j}(\mathbf{x}_j)^{(*)}$'s in $\mathfrak{C}_{\ell,m}$, as defined in (III.9) and (III.10). Then

$$\sup_\phi\big|\mathfrak{C}_{\ell,m}(\alpha,\phi)\big|\le C\;|\alpha|_+^{n'}\;\Big\{e^{-\kappa\min\{1,q\varepsilon\}\|\alpha\|}+e^{-\kappa p_0(\varepsilon)^2}\Big\}$$

(b) Denote by n'' is the total number of $\phi_{\beta_j}(\mathbf{x}_j)^{(*)}$'s in $\mathfrak{C}_{\ell,m}\,\delta\mathcal{C}_{m+1}$, as defined in (III.9) and (III.10). Then

$$\begin{aligned}\int\big|\mathfrak{C}_{\ell,m}(\alpha,\phi)\,\delta\mathcal{C}_{m+1}(\phi,\gamma)\big|\;|\gamma|_+^{\tilde n}\;e^{-\kappa Q\varepsilon\|\gamma\|}\;d\mu_r(\phi^*,\phi)d\mu_r(\gamma^*,\gamma)&\\ \le C\;|\alpha|_+^{\tilde n+n''}\;\Big\{e^{-\kappa\min\{1,(Q+q+1)\varepsilon\}\|\alpha\|}+e^{-\kappa p_0(\varepsilon)^2}\Big\}\min\Big\{\sqrt{\varepsilon},\tfrac{1}{q^{5/2}\sqrt{\varepsilon}}\Big\}&\end{aligned}$$

for all $\tilde n+n''\le n$ and $Q\ge0$.

Proof: We start by observing, just as in (II.10), that

$$\begin{aligned} \big|\mathfrak{I}_{\ell,m}(\alpha,\phi)\big| &\le \tilde{C} e^{-2\kappa\bar{\varepsilon}\|\alpha\|}\big(e^{-3\kappa\|\alpha-\phi\|}+e^{-3\kappa p_0(\varepsilon)^2}\big)\min\Big\{1,\tfrac{1}{\bar\varepsilon}e^{-\kappa\bar\varepsilon\|\phi\|_{\ell^4}^4}+e^{-\kappa\|\phi\|}\Big\} \\ \big|\delta\mathcal{I}_{m+1}(\phi,\gamma)\big| &\le \tilde{C} e^{-\kappa\varepsilon\|\phi\|}\big(e^{-3\kappa\|\phi-\gamma\|}+e^{-3\kappa p_0(\varepsilon)^2}\big)\,\varepsilon^2\big(1+|\phi|_X^6 e^{-\kappa\varepsilon\|\phi\|_{\ell^4}^4}+|\gamma-\phi|_X^6 e^{-\kappa\|\phi-\gamma\|}\big) \end{aligned} \tag{III.11}$$

(a) From the definitions (III.9) and (III.10), we have

$$\begin{aligned} \big|\mathfrak{C}_{\ell,m}(\alpha,\phi)\big| \le \int \prod_{j=1}^{k} d\mu_r(\phi_j^*,\phi_j)\ \big|\mathfrak{I}_{\ell,\ell_1}(\phi_0,\phi_1)\big|\,|\phi_1|_X\,\big|\mathfrak{I}_{\ell_1+1,\ell_2}(\phi_1,\phi_2)\big|\,|\phi_2|_X\ \cdots \\ \cdots\big|\mathfrak{I}_{\ell_k+1,m}(\phi_k,\phi_{k+1})\big|\,|\Phi_m(\phi_{k+1})|_X \end{aligned} \tag{III.12}$$

Here $\phi_0=\alpha$ and $\phi_{k+1}=\phi$. If $k=0$, then $\ell_k=\ell-1$. The n' of the statement of the Lemma is

$$n' = \begin{cases} k & \text{if } \Phi_m(\phi_{k+1})=1 \\ k+1 & \text{if } \Phi_m(\phi_{k+1})\ne 1 \end{cases}$$

Insert the first bound of (III.11) into (III.12). Set $q_1=\ell_1-\ell+1$, $q_2=\ell_2-\ell_1$, $\cdots$, $q_k=\ell_k-\ell_{k-1}$ and $q_{k+1}=m-\ell_k$. Also set $\bar\varepsilon_1=\tau_{\ell_1}-\tau_{\ell-1}$, $\bar\varepsilon_2=\tau_{\ell_2}-\tau_{\ell_1}$, $\cdots$, $\bar\varepsilon_{k+1}=\tau_m-\tau_{\ell_k}$. By the second condition in part (a) of Definition III.1, each $\bar\varepsilon_i\ge\frac12 q_i\varepsilon$. Also $q_1+\cdots+q_{k+1}=q$. When inserting the first bound of (III.11) into (III.12), discard all factors $\min\Big\{1,\frac{1}{\bar\varepsilon_j}e^{-\kappa\bar\varepsilon_j\|\phi_j\|_{\ell^4}^4}+e^{-\kappa\|\phi_j\|}\Big\}$. To this point, the right hand side of (III.12) is bounded by a constant times

$$\int\prod_{i=1}^{k+1}\Big\{e^{-\kappa q_i\varepsilon\|\phi_{i-1}\|}\big(e^{-3\kappa\|\phi_{i-1}-\phi_i\|}+e^{-3\kappa p_0(\varepsilon)^2}\big)\Big\}\ \prod_{i=1}^{k}|\phi_i|_X\,|\Phi_m(\phi_{k+1})|_X\ \prod_{j=1}^{k}d\mu_r(\phi_j^*,\phi_j)$$

We now deal with the factors $\prod_{i=1}^{k}|\phi_i|_X$ and $|\Phi_m(\phi_{k+1})|_X$. Use that, for any field $|\phi|_X\le r$, $a>0$ and $0<b<1$

$$\begin{aligned} \big(e^{-a\|\alpha-\phi\|}+e^{-ap_0(\varepsilon)^2}\big)|\phi|_+ &\le e^{-a\|\alpha-\phi\|}|\alpha|_+ + e^{-a\|\alpha-\phi\|}|\alpha-\phi|_X + e^{-ap_0(\varepsilon)^2}|\phi|_+ \\ &\le e^{-a\|\alpha-\phi\|}|\alpha|_+ + \|\alpha-\phi\|e^{-a\|\alpha-\phi\|} + re^{-ap_0(\varepsilon)^2} \\ &\le C_{a,b}\big(e^{-ab\|\alpha-\phi\|}+e^{-abp_0(\varepsilon)^2}\big)|\alpha|_+ \end{aligned} \tag{III.13}$$

to "move" each of the $n'\le n$ fields in $\prod_{i=1}^{k}|\phi_i|_X\,|\Phi_m(\phi_{k+1})|_X$ to $|\phi_0|_+=|\alpha|_+$. We may choose b so that $3\kappa b^n\ge 2\kappa$. Consequently, to this point, the right hand side of (III.12) is bounded by an (n–dependent) constant times

$$\begin{aligned} |\alpha|_+^{n'}&\int\prod_{i=1}^{k+1}\Big\{e^{-\kappa q_i\varepsilon\|\phi_{i-1}\|}\big(e^{-2\kappa\|\phi_{i-1}-\phi_i\|}+e^{-2\kappa p_0(\varepsilon)^2}\big)\Big\}\prod_{j=1}^{k}d\mu_r(\phi_j^*,\phi_j) \\ &\le |\alpha|_+^{n'}\int\Big[2^{k+1}e^{-2\kappa p_0(\varepsilon)^2}+\prod_{i=1}^{k+1}\Big(e^{-\kappa q_i\varepsilon\|\phi_{i-1}\|}e^{-2\kappa\|\phi_{i-1}-\phi_i\|}\Big)\Big]\prod_{j=1}^{k}d\mu_r(\phi_j^*,\phi_j) \end{aligned}$$

Now use that, for $Q\varepsilon \le 1$ (if $Q\varepsilon > 1$, replace $Q\varepsilon$ by 1),

$$\begin{aligned}\kappa q_i\varepsilon\|\phi\| + \kappa\|\phi-\gamma\| + \kappa Q\varepsilon\|\gamma\| &\ge \kappa q_i\varepsilon\|\phi\| + \kappa Q\varepsilon\|\phi-\gamma\| + \kappa Q\varepsilon\|\gamma\| \\ &\ge \kappa(Q+q_i)\varepsilon\|\phi\|\end{aligned}$$

to prove that

$$e^{-\kappa q_i\varepsilon\|\phi\|}e^{-2\kappa\|\phi-\gamma\|}e^{-\kappa Q\varepsilon\|\gamma\|} \le e^{-\kappa\min\{1,(Q+q_i)\varepsilon\}\|\phi\|}e^{-\kappa\|\phi-\gamma\|} \tag{III.14}$$

Applying this k times we have that the right hand side of (III.12) is bounded by a constant times

$$\begin{aligned}|\alpha|_+^{n'}\int\Big[2^{k+1}e^{-2\kappa p_0(\varepsilon)^2} + e^{-\kappa\min\{1,\Sigma_i q_i\varepsilon\}\|\alpha\|}\prod_{i=1}^{k+1}e^{-\kappa\|\phi_{i-1}-\phi_i\|}\Big]\prod_{j=1}^{k}d\mu_r(\phi_j^*,\phi_j) \\ \le |\alpha|_+^{n'}\Big[2^{k+1}(\pi r^2)^{k|X|}e^{-2\kappa p_0(\varepsilon)^2} + e^{-\kappa\min\{1,\Sigma_i q_i\varepsilon\}\|\alpha\|}D^k\Big]\end{aligned}$$

with

$$D = \int e^{-\kappa\|\gamma\|}\ d\mu(\gamma^*,\gamma)$$

As $\sum_{i=1}^{k+1} q_i = q$, the bound follows.

(b) The proof is similar to that of Lemma II.8. ∎

Proof of Theorem III.7: We need to show that, in the notation of (III.10), the integral $\int\big[\hat{\mathfrak{C}}_{1,p} - \mathfrak{C}_{1,p}\big](\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)$ converges to zero as $p = \frac{\beta}{\varepsilon} \to \infty$. Recall that $\hat{\mathcal{C}}_\ell = \mathcal{C}_\ell + \delta\mathcal{C}_\ell$ and expand

$$\begin{aligned}\hat{\mathfrak{C}}_{1,p} - \mathfrak{C}_{1,p} = \sum_{\rho=1}^{p}\ \sum_{1\le q_1<q_2<\cdots<q_\rho\le p}\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \mathfrak{C}_{q_1+1,q_2-1} *_r \delta\mathcal{C}_{q_2} *_r \cdots \\ \cdots *_r \mathcal{C}_{q_{\rho-1}+1,q_\rho-1} *_r \delta\mathcal{C}_{q_\rho} *_r \mathfrak{C}_{q_\rho+1,p}\end{aligned}$$

Hence

$$\begin{aligned}&\Big|\int\big[\hat{\mathfrak{C}}_{1,p} - \mathfrak{C}_{1,p}\big](\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)\Big| \\ &\quad\le \sum_{\rho=1}^{p}\ \sum_{1\le q_1<\cdots<q_\rho\le p}\int d\mu_r(\alpha^*,\alpha)\ \sup_{|\gamma|_X\le r}\big|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho} *_r \mathfrak{C}_{q_\rho+1,p}(\alpha,\gamma)\big|\end{aligned} \tag{III.15}$$

We now prove by backwards induction that, for each $\rho \ge \sigma \ge 0$,

$$\begin{aligned}&\sup_{|\gamma|_X\le r}\big|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho} *_r \mathfrak{C}_{q_\rho+1,p}(\alpha,\gamma)\big| \\ &\quad\le (3C)^{\rho-\sigma+1}\prod_{\ell=\sigma+1}^{\rho}\min\Big\{\sqrt{\varepsilon}, \tfrac{1}{(q_\ell-q_{\ell-1}-1)^{5/2}\sqrt{\varepsilon}}\Big\} \\ &\qquad\int d\mu_r(\gamma^*,\gamma)\ \big|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\sigma}(\alpha,\gamma)\big|\,|\gamma|_+^{n_\sigma}\Big\{e^{-\kappa\min\{(p-q_\sigma)\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2}\Big\}\end{aligned} \tag{III.16}$$

where n_σ is the number of $\phi_{\beta_j}(\mathbf{x}_j)^{(*)}$'s with $\beta_j > \tau_{q_\sigma}$ and C is the constant of Lemma III.10. For the final case, $\sigma = 0$, the factor $\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\sigma}(\alpha,\gamma)$ in the integrand is to be replaced by the kernel of the identity operator.

Consider the initial case, $\sigma = \rho$. By Lemma III.10.a,

$$\begin{aligned}
&\sup_{|\gamma|_X \le r} \left|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho} *_r \mathfrak{C}_{q_\rho+1,p}(\alpha,\gamma)\right| \\
&\le \int d\mu_r(\gamma'^*,\gamma') \sup_{|\gamma|_X \le r} \left|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho}(\alpha,\gamma')\ \mathfrak{C}_{q_\rho+1,p}(\gamma',\gamma)\right| \\
&\le C\int d\mu_r(\gamma'^*,\gamma') \left|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho}(\alpha,\gamma')\right| |\gamma'|_+^{n_\rho} \left\{e^{-\kappa\min\{1,(p-q_\rho)\varepsilon\}\|\gamma'\|} + e^{-\kappa p_0(\varepsilon)^2}\right\}
\end{aligned}$$

which provides the induction hypothesis for $\sigma = \rho$. Now assume that the induction hypothesis holds for σ. Observe that

$$\begin{aligned}
&\int d\mu_r(\gamma^*,\gamma)\ \left|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\sigma}(\alpha,\gamma)\right|\ |\gamma|_+^{n_\sigma}\ \left\{e^{-\kappa\min\{(p-q_\sigma)\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2}\right\} \\
&\qquad \le \int d\mu_r(\alpha'^*,\alpha') d\mu_r(\phi^*,\phi) d\mu_r(\gamma^*,\gamma)\ |\gamma|_+^{n_\sigma}\ \left\{e^{-\kappa\min\{(p-q_\sigma)\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2}\right\} \\
&\qquad\qquad \left|\big(\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_{\sigma-1}}\big)(\alpha,\alpha')\ \mathfrak{C}_{q_{\sigma-1}+1,q_\sigma-1}(\alpha',\phi)\ \delta\mathcal{C}_{q_\sigma}(\phi,\gamma)\right|
\end{aligned} \tag{III.17}$$

By Lemma III.10.b, with $\ell = q_{\sigma-1}+1$ and $m = q_\sigma - 1$ and $q = m - \ell + 1 = q_\sigma - q_{\sigma-1} - 1$,

$$\begin{aligned}
&\int d\mu_r(\phi^*,\phi) d\mu_r(\gamma^*,\gamma)\ |\gamma|_+^{n_\sigma} \left\{e^{-\kappa\min\{(p-q_\sigma)\varepsilon,1\}\|\gamma\|} + e^{-\kappa p_0(\varepsilon)^2}\right\} \left|\mathfrak{C}_{q_{\sigma-1}+1,q_\sigma-1}(\alpha',\phi)\ \delta\mathcal{C}_{q_\sigma}(\phi,\gamma)\right| \\
&\le C\ |\alpha'|_+^{n_{\sigma-1}}\ \Big\{\Big[e^{-\kappa\min\{(p-q_{\sigma-1})\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2}\Big] \\
&\qquad + e^{-\kappa p_0(\varepsilon)^2}\Big[e^{-\kappa\min\{(q_\sigma-q_{\sigma-1})\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2}\Big]\Big\} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{(q_\sigma-q_{\sigma-1}-1)^{5/2}\sqrt{\varepsilon}}\right\} \\
&\le 3C\ |\alpha'|_+^{n_{\sigma-1}}\ \left\{e^{-\kappa\min\{(p-q_{\sigma-1})\varepsilon,1\}\|\alpha'\|} + e^{-\kappa p_0(\varepsilon)^2}\right\} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{(q_\sigma-q_{\sigma-1}-1)^{5/2}\sqrt{\varepsilon}}\right\}
\end{aligned} \tag{III.18}$$

Here we used Lemma III.10.b with $Q\varepsilon = \min\{(p-q_\sigma)\varepsilon, 1\}$ for the first term in the curly bracket and with $Q = 0$ for the second term in the curly bracket. Inserting this result into (III.17) and then applying the inductive hypothesis (III.16) yields (III.16) with σ replaced by $\sigma - 1$ and γ replaced by α'. In particular, when $\sigma = 1$ and $q_0 = 0$, inserting (III.18) into the inductive hypothesis (III.16) yields

$$\begin{aligned}
&\sup_{|\gamma|_X \le r} \left|\mathfrak{C}_{1,q_1-1} *_r \delta\mathcal{C}_{q_1} *_r \cdots *_r \delta\mathcal{C}_{q_\rho} *_r \mathfrak{C}_{q_\rho+1,p}(\alpha,\gamma)\right| \\
&\qquad \le (3C)^{\rho+1}\ |\alpha|_+^{n}\ \prod_{\ell=1}^{\rho} \min\left\{\sqrt{\varepsilon}, \tfrac{1}{(q_\ell-q_{\ell-1}-1)^{5/2}\sqrt{\varepsilon}}\right\} \left\{e^{-\kappa\min\{p\varepsilon,1\}\|\alpha\|} + e^{-\kappa p_0(\varepsilon)^2}\right\}
\end{aligned} \tag{III.19}$$

Applying (III.19) to (III.15), it follows that

$$\Big|\int \big[\hat{\mathfrak{C}}_{1,p} - \mathfrak{C}_{1,p}\big](\alpha,\alpha)\ d\mu_r(\alpha^*,\alpha)\Big|$$

$$\begin{aligned}
&\le \sum_{\rho=1}^{p} \sum_{1\le q_1<\cdots<q_\rho\le p} (3C)^{\rho+1} \prod_{\ell=1}^{\rho} \min\Big\{\sqrt{\varepsilon}, \tfrac{1}{(q_\ell-q_{\ell-1}-1)^{5/2}\sqrt{\varepsilon}}\Big\} \\
&\qquad\qquad\qquad \int d\mu_r(\alpha^*,\alpha)\ |\alpha|_+^n\ \Big\{e^{-\kappa\min\{\beta,1\}\|\alpha\|} + e^{-\kappa p_0(\varepsilon)^2}\Big\} \\
&\le \text{const} \sum_{\rho=1}^{p} \sum_{1\le q_1<\cdots<q_\rho\le p} (3C)^{\rho+1} \prod_{\ell=1}^{\rho} \min\Big\{\sqrt{\varepsilon}, \tfrac{1}{(q_\ell-q_{\ell-1}-1)^{5/2}\sqrt{\varepsilon}}\Big\} \\
&\le \text{const} \sum_{\rho=1}^{\infty} \sum_{\tilde q_1,\cdots,\tilde q_\rho\ge 0} (3C)^{\rho+1} \prod_{\ell=1}^{\rho} \min\Big\{\sqrt{\varepsilon}, \tfrac{1}{\tilde q_\ell^{5/2}\sqrt{\varepsilon}}\Big\} \qquad \text{with } \tilde q_\ell = q_\ell - q_{\ell-1} - 1 \\
&\le \text{const}\, \varepsilon^{\frac{1}{10}}
\end{aligned}$$

by (II.20). The Theorem follows from

$$\mathrm{Tr}\ e^{-\beta K}\mathbb{T}\prod_{j=1}^{n}\psi^{(\dagger)}(\beta_j,\mathbf{x}_j) \ = \ \lim_{p\to\infty}\int \mathfrak{C}_{1,p}(\alpha,\alpha)\ d\mu_{\mathrm{R}_\varepsilon}(\alpha^*,\alpha)$$

which was proven in Theorem III.2. ■

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