A CLASS OF FERMI LIQUIDS

J. Feldman, H. Knörrer, D. Lehmann and E. Trubowitz

I Introduction

In these lectures, we consider a many-body system that is somewhat unusual in that the Fermi surface survives the turning on of all sufficiently weak short range interactions. The system consists of a gas of fermions with prescribed, strictly positive, density, together with a crystal lattice of magnetic ions. The fermions interact with each other through a two-body potential. The lattice provides periodic scalar and vector background potentials. As well, the ions oscillate, generating phonons and then the fermions interact with the phonons. At the present time our result is restricted to $d = 2$ space dimensions. But we believe that the difficulties preventing the extension to $d = 3$ are technical rather than physical and are working to overcome them.

To start, turn off the fermion-fermion and fermion-phonon interactions. Then we have a gas of independent fermions, each with Hamiltonian

$$H_0 = \tfrac{1}{2m} \left(i\nabla + \mathbf{a}(\mathbf{x})\right)^2 + U(\mathbf{x})$$

We assume that the vector and scalar potentials $\mathbf{a}$, U are periodic with respect to some lattice Γ in $\mathbb{R}^2$. By convention, bold face characters are two component vectors. Because the Hamiltonian commutes with lattice translations it is possible to simultaneously diagonalize the Hamiltonian and the generators of lattice translations. Call the eigenvalues and eigenvectors $\varepsilon_\nu(\mathbf{k})$ and $\phi_{\nu,\mathbf{k}}(\mathbf{x})$ respectively. They obey

$$\begin{aligned} H_0\phi_{\nu,\mathbf{k}}(\mathbf{x}) &= \varepsilon_\nu(\mathbf{k})\phi_{\nu,\mathbf{k}}(\mathbf{x}) \\ \phi_{\nu,\mathbf{k}}(\mathbf{x}+\gamma) &= e^{i<\mathbf{k},\gamma>}\phi_{\nu,\mathbf{k}}(\mathbf{x}) \qquad \forall\ \gamma\in\Gamma \end{aligned} \tag{I.1}$$

The crystal momentum $\mathbf{k}$ runs over $\mathbb{R}^2/\Gamma^\#$ where

$$\Gamma^\# = \left\{\ b\in\mathbb{R}^2\ \middle|\ <b,\gamma>\in 2\pi\mathbb{Z}\ \forall\ \gamma\in\Gamma\ \right\}$$

is the dual lattice to Γ. The band index $\nu\in\mathbb{N}$ just labels the eigenvalues for boundary condition $\mathbf{k}$ in increasing order.

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In the grand canonical ensemble, the Hamiltonian H is replaced by $H - \mu N$ where N is the number operator and the chemical potential μ is used to control the density of the gas. At very low temperature, which is the physically interesting domain, only those pairs $\nu, \mathbf{k}$ for which $\varepsilon_\nu(\mathbf{k}) \approx \mu$ are important. To keep things as simple as possible, we assume that $\varepsilon_\nu(\mathbf{k}) \approx \mu$ only for one value ν_0 of ν and we put on a fixed ultraviolet cutoff so that we consider only those crystal momenta for which $|\varepsilon_{\nu_0}(\mathbf{k}) - \mu|$ is smaller than some fixed small constant.

Precisely, we denote $e(\mathbf{k}) = \varepsilon_{\nu_0}(\mathbf{k}) - \mu$ and make the following assumptions.

Hypothesis I. *The dispersion relation* $e(\mathbf{k})$ *is a real-valued, real analytic function on a compact subset* B *of* $\mathbb{R}^d$. *For all points* $\mathbf{p} \in \mathrm{B}$,

$$\nabla e(\mathbf{p}) \neq 0$$

Hypothesis II. *The Fermi curve*

$$\mathrm{F} = \left\{ \mathbf{p} \in \mathrm{B} \;\middle|\; e(\mathbf{p}) = 0 \right\}$$

for e *is a simple closed curve, whose curvature is bounded away from zero.*

Hypothesis III. *For all* $\mathbf{q} \in \mathbb{R}^d$,

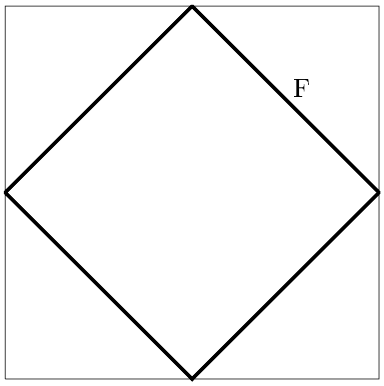
$$-\mathrm{F} + \mathbf{q} \neq \mathrm{F}$$

By definition,

$$-\mathrm{F} + \mathbf{q} = \left\{ \mathbf{p} \in \mathbb{R}^2 \;\middle|\; -\mathbf{p} + \mathbf{q} \in \mathrm{B} \text{ and } e(-\mathbf{p} + \mathbf{q}) = 0 \right\}$$

It is Hypothesis III that makes this class of models somewhat unusual and permits the system to remain a Fermi liquid when the interaction is turned on. If $\mathbf{a} = 0$ then, taking the complex conjugate of (I.1), we see that $\varepsilon_\nu(-\mathbf{k}) = \varepsilon_\nu(\mathbf{k})$ so that Hypothesis III is violated for $\mathbf{q} = 0$. Hence the presence of a nonzero vector potential is essential.

In order to make the hypotheses as simple looking as possible, we have made them much stronger than necessary. One model that violates these hypotheses, not only for technical reasons but because it exhibits different physics, is the Hubbard model at half filling. Its Fermi surface looks like



This Fermi curve is not smooth, violating Hypothesis I, has zero curvature almost everywhere, violating Hypothesis II and is reflection invariant so that $\mathrm{F} = -\mathrm{F}$, violating Hypothesis III with $\mathbf{q} = 0$.

The interacting models are formally characterized by the Euclidean Green's functions

$$\left\langle \prod_{i=1}^{n} \psi_{p_i} \bar\psi_{q_i} \right\rangle = \frac{\int \left(\prod_{i=1}^{n} \psi_{p_i}\bar\psi_{q_i}\right) e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)} \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}} \tag{I.2a}$$

The action

$$\mathcal{A}(\psi,\bar\psi) \;=\; -\int đk\; \big(ik_0 e(\mathbf{k})\big)\bar\psi_k\psi_k \;-\; \int đk\, \mathcal{E}(\lambda,\mathbf{k})\bar\psi_k\psi_k \;-\; \mathcal{V}(\psi,\bar\psi) \tag{I.2b}$$

We now take some time to explain this formula. The fermion fields are vectors

$$\psi_k = \begin{pmatrix} \psi_{k,\uparrow} \\ \psi_{k,\downarrow} \end{pmatrix} \qquad \bar\psi_k = (\,\bar\psi_{k,\uparrow} \quad \bar\psi_{k,\downarrow}\,)$$

whose components $\psi_{k,\sigma}, \bar\psi_{k,\sigma}$, $k = (k_0,\mathbf{k}) \in \mathcal{B} = (-1,1)\times \mathrm{B}$, $\sigma \in \{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other.

$$\overset{(-)}{\psi}_{k,\sigma}\overset{(-)}{\psi}_{p,\tau} = -\overset{(-)}{\psi}_{p,\tau}\overset{(-)}{\psi}_{k,\sigma}$$

We have deliberately chosen $\bar\psi$ to be a row vector and ψ to be a column vector so that

$$\bar\psi_k\psi_p = \bar\psi_{k,\uparrow}\psi_{p,\uparrow} + \bar\psi_{k,\downarrow}\psi_{p,\downarrow} \qquad \psi_k\bar\psi_p = \begin{pmatrix} \psi_{k,\uparrow}\bar\psi_{p,\uparrow} & \psi_{k,\uparrow}\bar\psi_{p,\downarrow} \\ \psi_{k,\downarrow}\bar\psi_{p,\uparrow} & \psi_{k,\downarrow}\bar\psi_{p,\downarrow} \end{pmatrix}$$

In the argument $k = (k_0,\mathbf{k})$, the last d components $\mathbf{k}$ are to be thought of as a crystal momentum and the first component k_0 as the dual variable to an imaginary time. Hence the $\sqrt{-1}$ in $ik_0 - e(\mathbf{k})$. For convenience only, we have put an ultraviolet cutoff on k_0 as well as on $\mathbf{k}$. In the full model k_0 runs over $\mathbb{R}$ and $\mathbf{k}$ is replaced by $(\nu,\mathbf{k})$ with ν summed over $\mathbb{N}$ and $\mathbf{k}$ integrated over $\mathbb{R}^d/\Gamma^\#$. The relationship between the position space field $\psi(\xi)$, with $\xi = (t,\mathbf{x})$ running over (imaginary)time$\times$space, and the momentum space field ψ_k is given, in our single band approximation, by

$$\begin{aligned} \psi_k \;&=\; \int d\xi\; e^{-ik_0 t}\phi_{\nu_0,\mathbf{k}}(\mathbf{x})\psi(\xi) \\ \psi(\xi) \;&=\; \int đk\; e^{ik_0 t}\overline{\phi_{\nu_0,\mathbf{k}}(\mathbf{x})}\psi_k \end{aligned} \tag{I.3}$$

where

$$đk \;=\; \frac{dk_0}{2\pi}\; đ\mathbf{k} \;=\; \frac{d^{d+1}k}{(2\pi)^{d+1}}$$

The general spin independent form of the interaction is

$$\mathcal{V}(\psi,\bar\psi) \;=\; \tfrac{\lambda}{2}\int \prod_{i=1}^{4} đk_i\; {\scriptstyle (2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)}\; \bar\psi_{k_1}\psi_{k_3}\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\, \bar\psi_{k_2}\psi_{k_4} \tag{I.4}$$

Spin independence is imposed purely for notational convenience. It plays no role. The delta function δ is that for $\mathbb{R}^d/\Gamma^\#$ and imposes the appropriate conservation of crystal momentum for the present setting. The function $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$ implements the fermion-fermion and fermion-phonon interaction. Its precise value does not concern us. We just assume

Hypothesis IV. *The interaction is short range. That is* $\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle \in C^\infty$.

The net coefficient $e(\mathbf{k}) - \mathcal{E}(\lambda, \mathbf{k})$ of $\bar\psi_k\psi_k$ in $\mathcal{A}$ has been deliberately split into two parts, with $\mathcal{E}(\lambda, \mathbf{k})$ chosen to satisfy an explicit renormalization condition. This is called renormalization of the dispersion relation. It is done to ensure that $\langle \prod_{i=1}^n \psi_{p_i}\bar\psi_{q_i}\rangle$ is C^∞ in λ at $\lambda = 0$. Define the proper self energy $\Sigma(p)$ for the action $\mathcal{A}$ by the equation

$$\Big(ip_0 - e(\mathbf{p}) - \Sigma(p)\Big)^{-1} (2\pi)^{d+1}\delta(p-q) \;=\; \frac{\int \psi_p\bar\psi_q \;\; e^{\mathcal{A}(\psi)} \;\prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}{\int \; e^{\mathcal{A}(\psi)} \;\prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}$$

The counterterm $\mathcal{E}(\lambda, \mathbf{k})$ is chosen so that

$$\Sigma\big(0, \mathbf{p}\big)\Big|_{\mathbf{p}\in\mathbf{F}} \;=\; 0$$

To give a rigorous definition of (I.2) one must introduce cutoffs and then take the limit in which the cutoffs are removed. To impose an infrared cutoff in the spatial directions one may put the system in a finite periodic box $\mathbb{R}^d/L\Gamma$. To impose an infrared cutoff in the zero direction one may make the inverse temperature $\beta < \infty$. Then momenta $k = (k_0, \mathbf{k})$ are restricted to lie on the lattice

$$\begin{aligned} k_0 &\in \tfrac{\pi}{\beta}\big(2\mathbb{Z}+1\big) \\ \mathbf{k} &\in \tfrac{1}{L}\Gamma^\# \end{aligned}$$

The ultraviolet cutoffs further restrict $|k_0| \le 1$, $|e(\mathbf{k})| \le 1$. Then the Grassmann algebra becomes finite dimensional and (I.2b) with the integral symbol reinterpreted as

$$\int đk \; f(k) \;=\; \tfrac{1}{\beta} \sum_{\substack{k_0 \in \frac{\pi}{\beta}\left(2\mathbb{Z}+1\right) \\ |k_0|\le 1}} \; \tfrac{1}{L^2} \sum_{\substack{\mathbf{k}\in \frac{1}{L}\Gamma^\# \\ |e(\mathbf{k})|\le 1}} \; f(k)$$

is a well-defined element of that algebra.

Theorem. *Let* $d = 2$ *and Hypotheses I-IV be satisfied. There is an* $r > 0$ *and a dispersion relation counterterm* $\mathcal{E}(\lambda, \mathbf{k})$, *such that the limits*

$$\lim_{\beta, L\to\infty} \frac{\int \prod_{i=1}^n \psi_{p_i}\bar\psi_{q_i} \;\; e^{\mathcal{A}(\psi,\bar\psi)} \;\prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}{\int \; e^{\mathcal{A}(\psi,\bar\psi)} \;\prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}$$

exist in the sense of distributions and are independent of the order in which the limits are taken. The counterterm and the limit are both analytic functions of the coupling constant λ *for* $|\lambda| < r$. *Furthermore, there is a jump in the average occupation number* $n_{\mathbf{k}}$ *at the Fermi curve. Precisely, if*

$$n_{\mathbf{k}} \;=\; \lim_{x_0 \searrow 0} \int đk_0 \, e^{\imath k_0 x_0} \Big(ik_0 - e(\mathbf{k}) - \Sigma(k_0, \mathbf{k})\Big)^{-1}$$

then

$$\lim_{\varepsilon \searrow 0} n_{\mathbf{p}-\varepsilon\nu_{\mathbf{p}}} - n_{\mathbf{p}+\varepsilon\nu_{\mathbf{p}}} = \Big(1 + i\tfrac{\partial}{\partial k_0}\Sigma(0,\mathbf{p})\Big)^{-1} \geq 1 - O(\lambda)$$

for all $\mathbf{p}$ *on the Fermi curve* $\mathbf{F}$. *Here,* $\nu_{\mathbf{p}}$ *is the outward pointing unit normal to* $\mathbf{F}$ *at* $\mathbf{p}$. *In other words, the infinite volume system is a Fermi liquid.*

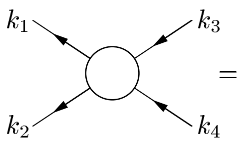
Our main goal here is to explain why this Theorem is true, though the complete proof [FKLT] is too long to include. There are two main aspects to that proof: the control of four legged Feynman diagrams and the control of high orders of perturbation theory. The first aspect is discussed in §II while the second is discussed in §III.

II Four Legged Diagrams

Spin plays no role in this section. So we supress it. We also relax the condition $d = 2$ allowing all $d \geq 2$. Feynman diagrams in this model have lines

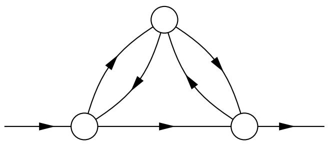
$$= \frac{1}{ik_0 - e(\mathbf{k})}$$

and vertices

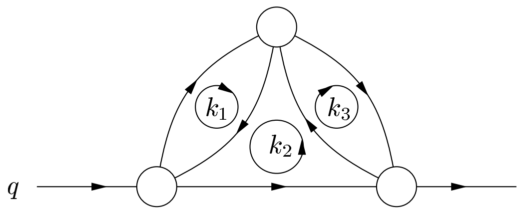


$$= (2\pi)^{d+1}\delta(k_1 + k_2 - k_3 - k_4)\lambda\,\langle k_1, k_2|\mathrm{V}|k_3, k_4\rangle$$

For example



is one graph contributing to the proper self energy. This is a three loop graph. Choosing the loops as in



we see that the value of this graph is

$$\int đk_1\, đk_2\, đk_3\ \frac{1}{i(k_1)_0 - e(\mathbf{k}_1)}\ \frac{1}{i(k_1+k_2)_0 - e(\mathbf{k}_1+\mathbf{k}_2)}$$
$$\frac{1}{i(k_2+k_3)_0 - e(\mathbf{k}_2+\mathbf{k}_3)}\ \frac{1}{i(k_3)_0 - e(\mathbf{k}_3)}\ \frac{1}{i(k_2+q)_0 - e(\mathbf{k}_2+\mathbf{q})}$$
$$\langle k_1+k_2, k_3|\mathrm{V}|k_1, k_2+k_3\rangle\,\langle k_1, k_2+q|\mathrm{V}|q, k_1+k_2\rangle\,\langle k_2+k_3, q|\mathrm{V}|k_2+q, k_3\rangle$$

It is not clear that this integral converges. The domain of integration is compact, because of the ultraviolet cutoff, but the integrand is singular.

To check for convergence one first does "naive power counting" bounds. In field theory propagator singularities occur at points. Then power counting just comes down to some simple dimensional analysis. Here there are singularities on curves, like $(k_1)_0 = 0$, $\mathbf{k}_1 \in F$. We have to have a simple yet precise way of measuring whether the integrand is large a lot. To do so we decompose the propagator

$$\begin{aligned} C(k) &= \frac{1}{ik_0 - e(\mathbf{k})} \\ &= \sum_{j=-\infty}^{0} C^{(j)} \end{aligned}$$

where

$$C^{(j)}(k) = \frac{1}{ik_0 - e(\mathbf{k})}\chi\big(2^j \le |ik_0 - e(\mathbf{k})| < 2^{j+1}\big)$$

Note, the perhaps bizarre, convention that j is negative. As j tends to *minus* infinity, 2^j approaches zero and, on the support of $C^{(j)}$, $|ik_0 - e(\mathbf{k})|$ approaches zero. Naive power counting just uses

Lemma II.1. *Let d be arbitrary and Hypothesis I be satisfied. Then*

a)
$$\|C^{(j)}\|_\infty = \sup_k |C^{(j)}(k)| \le 2^{-j}$$

b)
$$\|C^{(j)}\|_1 = \int đk\ |C^{(j)}(k)| \le \mathrm{const}\, 2^j$$

Proof. Part a) is obvious because, by construction, $|ik_0 - e(\mathbf{k})| \ge 2^j$ on the support of $C^{(j)}(k)$.

For part b) observe that

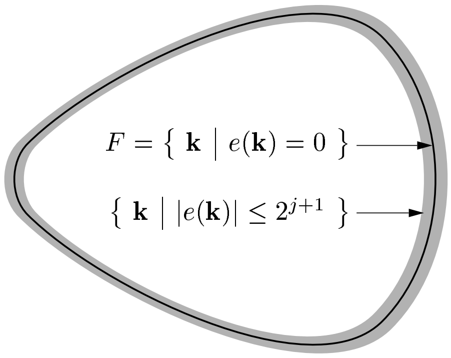
$$\begin{aligned} \mathrm{vol}\,\{\ k = (k_0, \mathbf{k})\ \big|\ C^{(j)}(k) \ne 0\ \} & \\ &\le \mathrm{vol}\,\{\ k_0\ \big|\ |k_0| \le 2^{j+1}\ \}\mathrm{vol}\,\{\ \mathbf{k} \in B\ \big|\ |e(\mathbf{k})| \le 2^{j+1}\ \} \\ &\le 2^{j+2}\mathrm{vol}\,\{\ \mathbf{k} \in B\ \big|\ |e(\mathbf{k})| \le 2^{j+1}\ \} \end{aligned}$$

The set $\{\ \mathbf{k}\in B\ \big|\ |e(\mathbf{k})|\le 2^{j+1}\ \}$ consists of a shell of thickness $O(2^j)$ around F and hence has volume bounded by $\mathrm{const}\,2^j$ so that

$$\text{(II.1)}\qquad \mathrm{vol}\{\ k=(k_0,\mathbf{k})\ \big|\ C^{(j)}(k)\ne 0\ \}\le \mathrm{const}\,2^{2j}$$

and

$$\begin{aligned}\|C^{(j)}\|_1 &= \int đk\ |C^{(j)}(k)| \le \sup_k |C^{(j)}(k)|\mathrm{vol}\{\ k=(k_0,\mathbf{k})\ \big|\ C^{(j)}(k)\ne 0\ \}\\ &\le \mathrm{const}\,2^j\end{aligned}$$



■

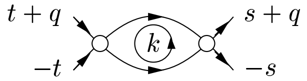
We remark that the smoothness condition $\nabla e(\mathbf{k})\ne 0$ of Hypothesis I was used to get the volume bound (II.1). The corresponding volume for the Hubbard model at half filling is $|j|2^{2j}$ which leads to $\|C^{(j)}\|_1\le \mathrm{const}\,|j|2^j$.

In the infrared Φ_4^4 model $C^{(j)}=\frac{1}{k^2}\chi\big(2^j\le|k|<2^{j+1}\big)$ so that the analog of Lemma II.1 for Φ_4^4 model is $\|C^{(j)}\|_\infty\le 2^{-2j}$, $\|C^{(j)}\|_1\le \mathrm{const}\,2^{-2j}2^{4j}\ \le\ \mathrm{const}\,2^{2j}$. The replacement $j\to 2j$ can be viewed simply as a change of units. Then it is not too surprising that Lemma II.1 implies [FT2, FMRT1] that models satisfying Hypotheses I and IV obey bounds typical of strictly renormalizable models in the infrared regime. Two legged are linearly divergent and must be renormalized. Four legged subdiagrams are marginal and all other subdiagrams are convergent. As is normal for infrared models, the two legged counterterm is finite and the marginality of four legged subdiagrams does not require a counterterm. The four legged subdiagrams are divergent only for certain exceptional momenta and then only logarithmically divergent. These logarithmic singularities are integrable and hence do not prevent diagrams from being well defined. But they can cause the values of diagrams containing many four legged subdiagrams to be anomalously large through

$$\int đk\,\ln^n|k|\sim n!$$

Normally, under these circumstances one of two possibilities occur. The renormalization group flow of the four point function is either asymptotically free or is to a nontrivial fixed point and is acompanied by some interesting physics, like mass generation or symmetry breaking. We shall now see that under Hypotheses I-IV, the bounds which give marginality of four legged subdiagrams are not saturated. Four legged subdiagrams are in fact convergent. The models behave more like superrenormalizable models than strictly renormalizable ones.

The Particle-Particle Bubble. As a concrete example, we'll first impose only Hypotheses I and IV and do the naive power counting bound explicitly on one simple, but very important, graph – the particle-particle bubble



If the total momentum entering from the left is q, the value of this graph is

$$B(s,t,q) = \int \mathchar'26\mkern-12mu d k\ C(-k+q)C(k)\,\langle -k+q,k|\mathrm{V}|t+q,-t\rangle\,\langle s+q,-s|\mathrm{V}|-k+q,k\rangle$$

Decomposing the two propagators into scales and then bounding the integral by the supremum of the integrand times the volume of the support of the integrand, we have

(II.2)

$$\begin{aligned}
|B(s,t,q)| &= \left|\sum_{j_1,j_2\le 0}\int_{\mathcal{B}} \mathchar'26\mkern-12mu d k\ C_{j_1}C_{j_2}\,\langle -k+q,k|\mathrm{V}|t+q,-t\rangle\,\langle s+q,-s|\mathrm{V}|-k+q,k\rangle\right| \\
&\le \sum_{j_1,j_2\le 0}\|V\|_\infty^2 2^{-j_1-j_2}\mathrm{vol}\big\{\ k\in\mathcal{B}\ \big|\ |ik_0-e(\mathbf{k})|\le 2^{j_1+1}, \\
&\qquad\qquad |i(-k+q)_0-e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\} \\
&\le \sum_{j_1,j_2\le 0}\|V\|_\infty^2 2^{-j_1-j_2}2^{\min\{j_1,j_2\}}\mathrm{vol}\big\{\ \mathbf{k}\in\mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}, |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\}
\end{aligned}$$

since $|k_0|\le 2^{j_1+1}$ and $|k_0-q_0|\le 2^{j_2+1}$. Even without using Hypotheses II and III we can bound the volume

(II.3)

$$\begin{aligned}
&\mathrm{vol}\big\{\ \mathbf{k}\in\mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}\ ,\ |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\} \\
&\le \min\big\{\mathrm{vol}\big\{\ \mathbf{k}\in\mathrm{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}\ \big\}, \mathrm{vol}\big\{\ \mathbf{k}\in\mathrm{B}\ \big|\ |e(-\mathbf{k}+\mathbf{q})|\le 2^{j_1+1}\ \big\}\big\} \\
&\le \mathrm{const}\ \min\big\{2^{j_2},2^{j_2}\big\} = \mathrm{const}\,2^{\min\{j_1,j_2\}}
\end{aligned}$$

This gives

$$\begin{aligned}\sup_{s,t,q} |B(s,t,q)| &\le \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{-j_1-j_2} 2^{2\min\{j_1,j_2\}} \\ &= \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{-|j_1-j_2|} \\ &= \sum_{j_1\le 0} \mathrm{const}\, \|V\|_\infty^2\end{aligned} \tag{II.4}$$

Recall that 2^j, and not j, has the units of energy. This bound allows the supremum of $|B(s,t,q)|$ to diverge logarithmically.

In the event that $e(\mathbf{k}) = e(-\mathbf{k})$, violating Hypothesis III, and $\mathbf{q} = 0$ we have

$$\begin{aligned}&\mathrm{vol}\{\ \mathbf{k} \in \mathrm{B}\ \big|\ |e(\mathbf{k})| \le 2^{j_1+1}\ ,\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \} \\ &= \mathrm{vol}\{\ \mathbf{k} \in \mathrm{B}\ \big|\ |e(\mathbf{k})| \le 2^{\min\{j_1,j_2\}+1}\ \} \\ &= O\Big(2^{\min\{j_1,j_2\}+1}\Big)\end{aligned}$$

and (II.3) is saturated. In this case $q = 0$ really is an exceptional momentum for $B(q)$ which really does have a singularity at $q = 0$. We can see that the singularity is integrable by bounding

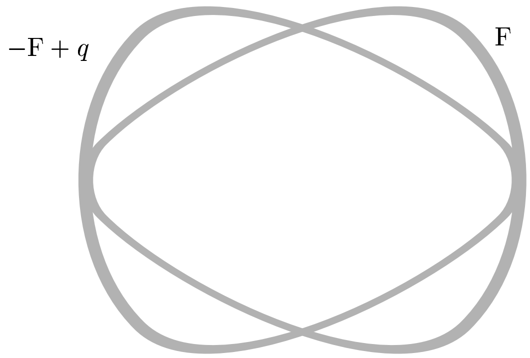
$$\begin{aligned}\sup_{s,t} \int đq |B(s,t,q)| \ &\le\ \|V\|_\infty^2 \int đq\, đk |C(-k+q)C(k)| \\ &=\ \|V\|_\infty^2 \int đq\, đk |C(q)C(k)| \\ &=\ \sum_{j_1,j_2\le 0} \|V\|_\infty^2 \int đq\, đk |C_{j_1}(q)C_{j_2}(k)| \\ &=\ \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 \|C_{j_1}\|_1 \|C_{j_2}\|_1 \\ &\le\ \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{j_1} 2^{j_2} \\ &\le\ \mathrm{const}\, \|V\|_\infty^2\end{aligned}$$

We now show that, when Hypotheses II,III are turned on, (II.4) is not saturated and that four legged subgraphs are actually convergent so that the model acts superrenormalizable. By Hypotheses III and analyticity (or even with just Hypothesis II if $\mathbf{q} \ne 0$) the Fermi curve F can only meet the reflected translated Fermi curve $-F + \mathbf{q}$ transversely or with a tangency of some finite order. Hence there is an $\epsilon > 0$ such that

(II.3a)

$$\mathrm{vol}\{\ \mathbf{k} \in \mathrm{B}\ \big|\ |e(\mathbf{k})| \le 2^{j_1+1}\ ,\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \} \le \mathrm{const}\, 2^{\min\{j_1,j_2\}} 2^{\epsilon \max\{j_1,j_2\}}$$

Here $\mathrm{const}\, 2^{\min\{j_1,j_2\}}$ is the thickness of each component of the intersection of the two shells and $\mathrm{const}\, 2^{\epsilon\max\{j_1,j_2\}}$ is a bound on the length of each component. Even though this bound is intuitively obvious, we give a complete proof in Lemma II.2, below.



Substituting (II.3a) into (II.2) gives

$$
\begin{aligned}
\sup_{s,t,q} |B(s,t,q)| &\le \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{-j_1-j_2} 2^{2\min\{j_1,j_2\}} 2^{\epsilon \max\{j_1,j_2\}} \\
&= \sum_{j_1,j_2\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{-|j_1-j_2|} 2^{\epsilon \max\{j_1,j_2\}} \\
&= \sum_{j\le 0} \mathrm{const}\, \|V\|_\infty^2 2^{\epsilon j} \\
&< \infty
\end{aligned}
$$

We conclude that, when Hypothesis III is turned, on the particle-particle bubble becomes uniformly bounded.

Lemma II.2. *There is a constant* const *and an* $\epsilon > 0$ *such that for all* $j_1, j_2 < 0$

$$\sup_{\mathbf{q}} \mathrm{vol}\big\{\ \mathbf{k}\ \big|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \big\} \le \mathrm{const}\, 2^{\min\{j_1,j_2\}} 2^{\epsilon \max\{j_1,j_2\}}$$

Proof. We may assume without loss of generality that $j_1 \le j_2$. Otherwise make the change of variables $\mathbf{p} = -\mathbf{k} + \mathbf{q}$. Put

$$\mathcal{M} = \big\{\ (\mathbf{q},\mathbf{k}) \in \mathbb{R}^d \times \mathbb{R}^d\ \big|\ |e(\mathbf{k})| \le 2,\ |e(\mathbf{q}-\mathbf{k})| \le 2\ \big\}$$

Since $\mathcal{M}$ is compact, it suffices to show

Claim. *For each* $(\mathbf{q}^{(0)}, \mathbf{k}^{(0)}) \in \mathcal{M}$ *there are constants* const > 0, $n \in \mathbb{N}$ *and there are neighbourhoods* U *of* $\mathbf{q}^{(0)}$ *and* V *of* $\mathbf{k}^{(0)}$ *in* $\mathbb{R}^d$ *such that for all* $j_1 \le j_2 < 0$ *and all* $\mathbf{q} \in U$

$$\mathrm{vol}\big\{\ \mathbf{k} \in V\ \big|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \big\} \le \mathrm{const}\, 2^{j_1} 2^{j_2/n}$$

Proof that the Lemma follows from the claim. Note first that the const and the n in the statement of the claim may depend on $\mathbf{q}^{(0)}$ and $\mathbf{k}^{(0)}$. Since $\mathcal{M}$ is compact, it can be covered

$$\mathcal{M} \subset \bigcup_{i=1}^{m} U_i \times V_i$$

with U_i and V_i being the neghbourhoods of the Claim for some choice $(\mathbf{q}_i^{(0)}, \mathbf{k}_i^{(0)})$ of $(\mathbf{q}^{(0)}, \mathbf{k}^{(0)})$. Then the const of the Lemma is bounded by $\sum_{i=1}^{m} \mathrm{const}_i$ and the ϵ of the Lemma is bounded by $\min_{1\le i\le m} 1/n_i$.

Proof of the claim. For maximum clarity, we first give the proof for $d = 2$. The claim is trivial if $e(\mathbf{k}^{(0)}) \neq 0$ or $e(\mathbf{q}^{(0)} - \mathbf{k}^{(0)}) \neq 0$. If $e(\mathbf{k}^{(0)}) \neq 0$, we can choose V so that $\left\{\ \mathbf{k} \in V \ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \right\}$ is empty except for finitely many values of j_1 and j_2. If $e(\mathbf{k}^{(0)}) = 0$ but $e(\mathbf{q}^{(0)} - \mathbf{k}^{(0)}) \neq 0$, we can choose U, V so that $\left\{\ \mathbf{k} \in V \ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \right\}$ is empty except for finitely many values of j_2 and we can bound

$$\mathrm{vol}\left\{\ \mathbf{k} \in V \ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1}\ \right\} \le \mathrm{const}\, 2^{j_1}$$

So we assume

$$e(\mathbf{k}^{(0)}) = e(\mathbf{q}^{(0)} - \mathbf{k}^{(0)}) = 0$$

Since $\nabla e(\mathbf{k})\big|_{\mathbf{k}=\mathbf{k}^{(0)}} \neq 0$ there are neighbourhoods V' of $\mathbf{k}^{(0)}$ and $X \times Y$ of $(0,0)$ in $\mathbb{R}^2$ and a diffeomorphism

$$\pi : X \times Y \longrightarrow V'$$

such that

$$\begin{aligned} e(\pi(x,y)) &= y \\ \pi(0,0) &= \mathbf{k}^{(0)} \end{aligned}$$

and $\frac{\partial\, \mathbf{k}}{\partial(x,y)}$ is a nowhere vanishing bounded function. Define

$$E(\mathbf{q}, x) = e\big(\mathbf{q} - \pi(x,0)\big)$$

Since, for all $\mathbf{q}$ in a neighbourhood of $\mathbf{q}^{(0)}$ and all $(x,y) \in X \times Y$,

$$\begin{aligned} |E(\mathbf{q}, x)| &= \left|e\big(\mathbf{q} - \pi(x,0)\big) - e\big(\mathbf{q} - \pi(x,y)\big) + e\big(\mathbf{q} - \pi(x,y)\big)\right| \\ &\le \mathrm{const}\, |y| + \left|e\big(\mathbf{q} - \pi(x,y)\big)\right| \end{aligned}$$

We have, for all $\mathbf{q}$ in a neighbourhood of $\mathbf{q}^{(0)}$

$$\begin{aligned} &\mathrm{vol}\left\{\ \mathbf{k} \in V' \ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(-\mathbf{k}+\mathbf{q})| \le 2^{j_2+1}\ \right\} \\ &\le \mathrm{const}\,\mathrm{vol}\left\{\ (x,y) \in X \times Y \ \middle|\ |y| \le 2^{j_1+1},\ \left|e\big(\mathbf{q} - \pi(x,y)\big)\right| \le 2^{j_2+1}\ \right\} \\ &\le \mathrm{const}\,\mathrm{vol}\left\{\ (x,y) \in X \times Y \ \middle|\ |y| \le 2^{j_1+1},\ |E(\mathbf{q},x)| \le \mathrm{const}\, 2^{j_2+1}\ \right\} \\ &\le \mathrm{const}\, 2^{j_1}\mathrm{vol}\left\{\ x \in X \ \middle|\ |E(\mathbf{q},x)| \le \mathrm{const}\, 2^{j_2+1}\ \right\} \end{aligned}$$

Since $e(\mathbf{q}^{(0)} - \mathbf{k}^{(0)}) = 0$, we have that $E(\mathbf{q}^{(0)}, 0) = 0$. By analyticity, if the order of the zero of $E(\mathbf{q}^{(0)}, x)$ at $x = 0$ is not finite, Hypothesis III is violated. So, if X has been chosen small enough, there is a nowhere vanishing smooth function $\delta(x)$ and a natural number n such that

$$E(\mathbf{q}^{(0)}, x) = \delta(x) x^n$$

So $E(\mathbf{q}^{(0)}, x)$ takes the normal form z^n under the diffeomorphic change of coordinates $z = \sqrt[n]{\delta(x)}\, x$. As we now move $\mathbf{q}$ away from $\mathbf{q}^{(0)}$, the $n^{\rm th}$ order zero for $E(\mathbf{q}^{(0)}, x)$ probably splits up into a number of distinct zeros of $E(\mathbf{q}, x)$. So we cannot retain the normal form z^n for $E(\mathbf{q}, x)$. But we can have a normal form which is a polynomial of degree n.

Put, for $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$

$$P(z; a) = z^n + a_1 z^{n-1} + \cdots + a_n$$

By the theory of "universal unfoldings" there are neighbourhoods $X' \subset X$, Z of 0 in $\mathbb{R}$ and U' of $\mathbf{q}^{(0)}$ in $\mathbb{R}^2$, functions $a_r(\mathbf{q})$, $0 \le r \le m$ on U' with

$$a_r(\mathbf{q}^{(0)}) = 0 \qquad\qquad |a_r(\mathbf{q})| \le 1$$

and a diffeomorphism

$$\begin{aligned} \Phi : X' \times U' &\longrightarrow Z \times U' \\ (x, \mathbf{q}) &\longmapsto (\phi_{\mathbf{q}}(x), \mathbf{q}) \end{aligned}$$

such that

(i) $\phi_{\mathbf{q}}(0) = 0$ for all $\mathbf{q} \in U'$

(ii) $E(\mathbf{q}, x) = P\Big(\phi_{\mathbf{q}}(x); \big(a_1(\mathbf{q}), \cdots, a_m(\mathbf{q})\big)\Big)$

(iii) $(\phi_{\mathbf{q}}^{-1})^* dx = \rho(z, \mathbf{q}) dz$ with $|\rho(z, \mathbf{q})| \le \text{const}$

Therefore, for $d = 2$, Lemma II.2 follows from Lemma II.3 below.

For dimensions $d > 2$ the function $E(\mathbf{q}, x)$ is replaced by $E(\mathbf{q}, x_1, \cdots, x_{d-1})$. We can always arrange (by using a rotation in $\mathbb{R}^{d-1}$), that when $\mathbf{q} = \mathbf{q}^{(0)}$ and $x_2 = \cdots = x_{d-1} = 0$, we have $E(\mathbf{q}^{(0)}, x) = \delta(x_1) x_1^n$ with $\delta(x_1)$ bounded away from zero. By the same unfolding of singularities argument as above, for each fixed $\mathbf{q}, x_2, \cdots, x_{d-1}$, the set of allowed x_1 has volume at most const $2^{j_2/n}$. ■

Lemma II.3. *There is a constant $K(m)$ such that for all $a_1, \cdots a_m \in \mathbb{R}$ and $0 \le \delta \le 1$*

$$\mathrm{vol}\{\ z \in \mathbb{R}\ \big|\ |z^m + a_1 z^{m-1} + \cdots + a_m| \le \delta\ \} \le K(m)\ \sqrt[m]{\delta}$$

Proof. We use induction on m. The case $m = 1$ is trivial. Consider $m = 2$. We can always translate z to make $a_1 = 0$. If $a_2 \ge -\delta$

$$\{\ z \in \mathbb{R}\ \big|\ |z^2 + a_2| \le \delta\ \} \subset \{\ z \in \mathbb{R}\ \big|\ |z^2| \le 2\delta\ \}$$

which trivially has volume $2\sqrt{2\delta}$. If $a_2 = -a^2 < -\delta$ then

$$\{\ z \in \mathbb{R}\ \big|\ |z^2 + a_2| \le \delta\ \} = \Big[\sqrt{a^2 - \delta}, \sqrt{a^2 + \delta}\Big] \cup \Big[-\sqrt{a^2 + \delta}, -\sqrt{a^2 - \delta}\Big]$$

which has volume

$$2\left(\sqrt{a^2+\delta}-\sqrt{a^2-\delta}\right)=2\frac{(a^2+\delta)-(a^2-\delta)}{\sqrt{a^2+\delta}+\sqrt{a^2-\delta}}\le 2\frac{2\delta}{\sqrt{2\delta}}$$

Now suppose $m>2$. Write

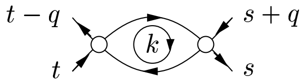
$$z^m+a_1z^{m-1}+\cdots+a_m=g(z)h(z)$$

where $g(z),h(z)$ are monic polynomials of degree $m-2$ and 2 respectively. Clearly

$$\begin{aligned}&\left\{\ z\in\mathbb{R}\ \middle|\ |z^m+a_1z^{m-1}+\cdots+a_m|\le\delta\ \right\}\\&\qquad\subset\left\{\ z\in\mathbb{R}\ \middle|\ |g(z)|\le\delta^{\frac{m-2}{m}}\ \right\}\cup\left\{\ z\in\mathbb{R}\ \middle|\ |h(z)|\le\delta^{\frac{2}{m}}\ \right\}\end{aligned}$$

By the induction hypothesis and the case $m=2$ respectively, the volumes of the two sets is bounded by $K(m-2)\delta^{1/m}$ and $K(2)\delta^{1/m}$. ■

The Particle-Hole Bubble. Of course the particle-particle bubble is just one graph. As a second example we consider the second most important graph in our class of models – the particle hole bubble



The value of this bubble is

$$B_2(s,t,q)=\int đk\ C(k+q)C(k)\left\langle t-q,k|\mathrm{V}|k,t\right\rangle\left\langle s,k|\mathrm{V}|k+q,s+q\right\rangle$$

When Hypothesis II is satisfied and when q is bounded away from zero, we can apply the same argument as in the particle-particle bubble, now using the fact that shells around F and $F+q$ have small intersections.

$$\begin{aligned}|B_2(s,t,q)|&\le\sum_{j_1,j_2\le 0}\|V\|_\infty^2\int_{\mathcal{B}}đk\ \left|C_{j_1}(k+q)C_{j_2}(k)\right|\\&\le\sum_{j_1,j_2\le 0}\|V\|_\infty^2 2^{-j_1-j_2}\mathrm{vol}\Big\{\ k\in\mathcal{B}\ \Big|\ |ik_0-e(\mathbf{k})|\le 2^{j_1+1},\\&\qquad\qquad |i(k+q)_0-e(\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \Big\}\end{aligned}$$

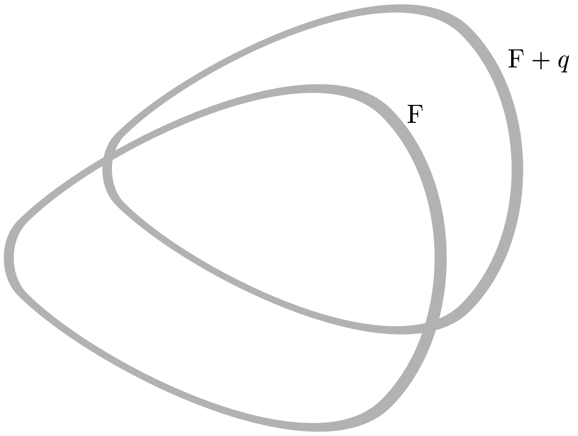
In the event that $|q_0|\ge$ const >0, it is only possible to satisfy the conditions $|k_0|\le 2^{j_1+1}$ and $|(k+q)_0|\le 2^{j_2+1}$ simultaneously if $\max\{j_1,j_2\}\ge$ const . In this case

$$\begin{aligned}
&|B_2(s,t,q)| \\
&\le \sum_{\substack{j_1,j_2\le 0\\ \max\{j_1,j_2\}\ge \mathrm{const}}} \|V\|_\infty^2 2^{-j_1-j_2}2^{\min\{j_1,j_2\}}\mathrm{vol}\big\{k\in\mathcal{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}, |e(\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\big\} \\
&\le \sum_{\substack{j_1,j_2\le 0\\ \max\{j_1,j_2\}\ge \mathrm{const}}} \mathrm{const}\,\|V\|_\infty^2 2^{-j_1-j_2}2^{2\min\{j_1,j_2\}} \\
&= \sum_{\substack{j_1,j_2\le 0\\ \max\{j_1,j_2\}\ge \mathrm{const}}} \mathrm{const}\,\|V\|_\infty^2 2^{-|j_1-j_2|} \\
&= \sum_{0\ge j\ge \mathrm{const}} \mathrm{const}\,\|V\|_\infty^2 \\
&< \infty
\end{aligned}$$

On the other hand, if $|\mathbf{q}|\ge$ const

$$\begin{aligned}
&|B_2(s,t,q)| \\
&\le \sum_{j_1,j_2\le 0} \|V\|_\infty^2 2^{-j_1-j_2}2^{\min\{j_1,j_2\}}\mathrm{vol}\big\{\ k\in\mathcal{B}\ \big|\ |e(\mathbf{k})|\le 2^{j_1+1}, |e(\mathbf{k}+\mathbf{q})|\le 2^{j_2+1}\ \big\} \\
&\le \sum_{j_1,j_2\le 0} \mathrm{const}\,\|V\|_\infty^2 2^{-j_1-j_2}2^{2\min\{j_1,j_2\}}2^{\epsilon\max\{j_1,j_2\}} < \infty
\end{aligned}$$

Once again const $2^{\min\{j_1,j_2\}}$ is the thickness of each component of the intersection of the two shells and const $2^{\epsilon\max\{j_1,j_2\}}$ is a bound on the length of each component.



So $B_2(s,t,q)$ is uniformly bounded if q is kept away from zero. As an illustration of what happens when q is small, consider $q=0$. If we had analyticity in k_0 we could observe that

$$\int d\!\!\!{}^{-}\mathbf{k}\int d\!\!\!{}^{-}k_0\ \frac{f(\mathbf{k})}{[ik_0-e(\mathbf{k})]^2}\,\langle t,k|\mathrm{V}|k,t\rangle\,\langle s,k|\mathrm{V}|k,s\rangle = 0$$

simply by closing the k_0 contour in the half-plane not containing the pole $k_0 = -ie(\mathbf{k})$. However our ultraviolet cutoff destroys that analyticity so we have to work a bit harder. Changing variables to

$$\begin{aligned} x &= k_0 \\ y &= e(\mathbf{k}) \end{aligned}$$

and some angular variable(s) and performing the integral over the angular variable(s), we have

$$\begin{aligned} B_2(s,t,0) &= \int_{\mathcal{B}} đk \ \frac{1}{[ik_0 - e(\mathbf{k})]^2} \langle t,k|\mathrm{V}|k,t\rangle \langle s,k|\mathrm{V}|k,s\rangle \\ &= \int dxdy \ \frac{1}{[ix-y]^2} I(x,y) \end{aligned}$$

with $I(x,y)$ being some function that is C^∞ at $x=y=0$. Making the further change of variables to polar coordinates

$$\begin{aligned} B_2(s,t,0) &= \int drd\theta \ \frac{r}{i[re^{i\theta}]^2} I(r\cos\theta, r\sin\theta) \\ &= \int drd\theta \ \frac{r}{i[re^{i\theta}]^2} [I(0,0) + O(r)] \end{aligned}$$

The potentially logarithmically divergent term

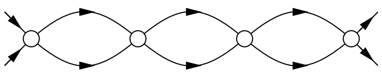
$$\int drd\theta \ \frac{1}{ire^{2i\theta}} I(0,0)$$

vanishes because

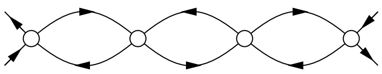
$$\int_0^{2\pi} d\theta \ e^{in\theta} = 0$$

for all nonzero integers n. Hence $B_2(s,t,0)$ is bounded. By working harder still we can bound $B_2(s,t,q)$ for all small q.

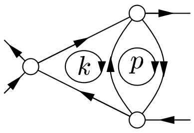
Higher Order Diagrams. We can always Wick order the interaction. Then no tadpoles appear in Feynman diagrams and every higher order graph falls into one of two categories. There are strings of bubbles, like



and



that can be treated as above. And there are graphs which have overlapping loops, like



In the example, the k-loop and the p-loop share a line and hence, by definition, overlap. Consequently, after the propagators have been decomposed into scales, the integrand of the value of this diagram contains all three factors $C^{(j_1)}(k)C^{(j_2)}(p)C^{(j_3)}(p-k)$. The support properties of these propagators constrains the domain of integration to

$$\left\{\ (\mathbf{k},\mathbf{p}) \in \mathrm{B}\times\mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(\mathbf{p})| \le 2^{j_2+1},\ |e(\mathbf{p}-\mathbf{k})| \le 2^{j_3+1}\ \right\}$$

The naive bound

$$\begin{aligned}
&\mathrm{vol}\left\{\ (\mathbf{k},\mathbf{p}) \in \mathrm{B}\times\mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(\mathbf{p})| \le 2^{j_2+1},\ |e(\mathbf{p}-\mathbf{k})| \le 2^{j_3+1}\ \right\}\\
&\le \mathrm{vol}\left\{\ (\mathbf{k},\mathbf{p}) \in \mathrm{B}\times\mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(\mathbf{p})| \le 2^{j_2+1}\ \right\}\\
&= \mathrm{vol}\left\{\ \mathbf{k} \in \mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1}\ \right\}\mathrm{vol}\left\{\ \mathbf{p} \in \mathrm{B}\ \middle|\ |e(\mathbf{p})| \le 2^{j_2+1}\ \right\}\\
&\le \mathrm{const}\, 2^{j_1} 2^{j_2}
\end{aligned}$$

allows the value of the diagram to have logarithmic singularities.

Fortunately, the third condition gives some "volume improvement" over naive power counting. To see this, make the change of variables

$$\begin{aligned}
\mathbf{k} &= \mathbf{k}(x,\theta) & x &= e(\mathbf{k})\\
\mathbf{p} &= \mathbf{p}(y,\phi) & y &= e(\mathbf{p})
\end{aligned}$$

with θ and ϕ each being a set of $d-1$ angular variables. This change of variables has bounded Jacobean so that

$$\begin{aligned}
&\mathrm{vol}\left\{\ (\mathbf{k},\mathbf{p}) \in \mathrm{B}\times\mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(\mathbf{p})| \le 2^{j_2+1},\ |e(\mathbf{p}-\mathbf{k})| \le 2^{j_3+1}\ \right\}\\
&\le \mathrm{const}\,\mathrm{vol}\left\{\ (x,\theta,y,\phi)\ \middle|\ |x| \le 2^{j_1+1}, |y| \le 2^{j_2+1}, \left|e\big(\mathbf{p}(y,\phi)-\mathbf{k}(x,\theta)\big)\right| \le 2^{j_3+1}\ \right\}
\end{aligned}$$

By the Mean Value Theorem (applied twice)

$$\left|e\big(\mathbf{p}(y,\phi)-\mathbf{k}(x,\theta)\big) - e\big(\mathbf{p}(0,\phi)-\mathbf{k}(0,\theta)\big)\right| \le \mathrm{const}\, 2^{j_1} + \mathrm{const}\, 2^{j_2}$$

so that

$$\begin{aligned}
&\mathrm{vol}\left\{\ (\mathbf{k},\mathbf{p}) \in \mathrm{B}\times\mathrm{B}\ \middle|\ |e(\mathbf{k})| \le 2^{j_1+1},\ |e(\mathbf{p})| \le 2^{j_2+1},\ |e(\mathbf{p}-\mathbf{k})| \le 2^{j_3+1}\ \right\}\\
&\le \mathrm{const}\,\mathrm{vol}\Big\{(x,\theta,y,\phi)\ \Big|\ |x| \le 2^{j_1+1},\ |y| \le 2^{j_2+1},\\
&\qquad\qquad\qquad \left|e\big(\mathbf{p}(0,\phi)-\mathbf{k}(0,\theta)\big)\right| \le \mathrm{const}\, 2^{\max\{j_1,j_2,j_3\}}\Big\}\\
&\le \mathrm{const}\, 2^{j_1}2^{j_2}\mathrm{vol}\left\{\ (\theta,\phi)\ \middle|\ \left|e\big(\mathbf{p}(0,\phi)-\mathbf{k}(0,\theta)\big)\right| \le \mathrm{const}\, 2^{\max\{j_1,j_2,j_3\}}\ \right\}\\
&\le \mathrm{const}\, 2^{j_1}2^{j_2}2^{\epsilon\max\{j_1,j_2,j_3\}}
\end{aligned}$$

for some $\epsilon > 0$. To prove the "volume improvement" bound of the last inequality, first use a compactness argument like that at the beginning of the proof of Lemma

II.1, to reduce consideration to a small ball in (ϕ, θ) space. Then recopy the last half of the proof of the Claim in Lemma II.1, in which the bound

$$\mathrm{vol}\{\ x \in X \ \big|\ |E(\mathbf{q}, x)| \le \mathrm{const}\, 2^{j_2+1}\ \} \le \mathrm{const}\, 2^{j_2/n}$$

is proven. Replace 2^{j_2} by $2^{\max\{j_1,j_2,j_3\}}$, $\mathbf{q}$ by $\mathbf{p}(0,\phi)$ and x by θ. You now have a proof. Note, in particular, that for each fixed ϕ, the function $e\big(\mathbf{p}(0,\phi) - \mathbf{k}(0,\theta)\big)$ cannot be identically zero in θ by Hypothesis III. This "volume improvement" ensures that all four legged subdiagrams have convergent, rather than marginal, power counting. See [FKLT, FST].

III A Single Slice Fermionic Cluster Expansion

In this section, we concentrate on the problem of summing high orders of perturbation theory. To do so we shall consider an artificial model which retains the essential difficulties that now interest us, but which is restricted to a single slice. The toy world consists of

- $d+1$ dimensional Euclidean space-time
- four types of fermions, denoted $\psi_\uparrow$, $\psi_\downarrow$, $\bar\psi_\uparrow$ and $\bar\psi_\downarrow$, that play the roles of spin up and spin down electrons and positrons/holes
- momenta "morally" in the range $2^j \le |p| \le 2^{j+1}$.

This is typical of one slice of a relativistic quantum field theory. In a many body model we would have $2^j \le |p_0| + \big||\mathbf{p}| - k_F\big| \le 2^{j+1}$. We will discuss the implications of this set of momenta later. In a realistic model we would have to sum over j using the renormalization group, which provides a machinized version of the techniques of the previous section.

I say "morally" because momentum is never actually going to appear in the toy world. Instead we are going to mimic the assumed momentum range by two space-time properties of the model. First, because the momentum space of our toy world has volume $2^{j(d+1)}$ the Pauli exclusion principle says that there can be at most one $\psi_\uparrow$, for example, in any region of volume $2^{-j(d+1)}$ in position space. Thus we define the fields of our model to be

$$\big\{\psi_\uparrow(x), \psi_\downarrow(x), \bar\psi_\uparrow(x), \bar\psi_\downarrow(x)\ \big|\ x \in W := 2^{-j}\mathbb{Z}^{d+1}\big\}$$

They are the generators of a Grassmann algebra, meaning

$$\overset{(-)}{\psi}_\alpha(x)\overset{(-)}{\psi}_\beta(y) = -\overset{(-)}{\psi}_\beta(y)\overset{(-)}{\psi}_\alpha(x)$$

and in particular

$$\Big(\overset{(-)}{\psi}_\alpha(x)\Big)^2 = 0$$

The second concerns the propagator. That is, the free two point Euclidean Green's function. The interacting two point Euclidean Green's function is

$$S_2(x, x') = \frac{\int \psi_\uparrow(x)\bar\psi_\uparrow(x')e^{-\lambda V}\, d\mu_C}{\int e^{-\lambda V}\, d\mu_C}$$

where the interaction

$$V = \frac{1}{2} \sum_{y \in W} 2^{-j(d+1)} \bar\psi_\uparrow(y) \bar\psi_\downarrow(y) \psi_\downarrow(y) \psi_\uparrow(y)$$

and

$$d\mu_C = \exp\Big\{ \sum_{\substack{z,z' \in W \\ \sigma \in \{\uparrow,\downarrow\}}} \bar\psi_\sigma(z') C^{-1}(z',z) \psi_\sigma(z) \Big\} \prod_{\substack{z \in W \\ \sigma \in \{\uparrow,\downarrow\}}} d\psi_\sigma(z) d\bar\psi_\sigma(z)$$

is the Grassmann Gaussian measure with covariance C, to be specified shortly.

Here are all of their properties of Grassmann Gaussian measures that we are going to use. The symbol $\int \cdot\, d\mu_C$ is a linear functional that assigns a complex number to every polynomial in the fields and that obeys

$$\begin{aligned}
1. \qquad \int \psi_\sigma(x) \bar\psi_{\sigma'}(y) d\mu_C &= \delta_{\sigma,\sigma'} C(x,y) \\
\int \prod_{i=1}^n \psi_{\sigma_i}(x_i) \bar\psi_{\sigma'_i}(y_i) d\mu_C &= \det\Big[\delta_{\sigma_i,\sigma'_{i'}} C(x_i, y_{i'})\Big]_{\substack{1\le i\le n \\ 1 \le i' \le n}} \\
2. \qquad \int \psi_\sigma(x) F(\psi,\bar\psi) d\mu_C &= \sum_{y \in W} C(x,y) \int \frac{\delta}{\delta\bar\psi_\sigma(y)} F(\psi,\bar\psi) d\mu_C \\
\int \bar\psi_\sigma(y) F(\psi,\bar\psi) d\mu_C &= -\sum_{x \in W} C(x,y) \int \frac{\delta}{\delta\psi_\sigma(x)} F(\psi,\bar\psi) d\mu_C
\end{aligned}$$

Except for signs, the Grassmann derivatives behave like normal ones. They are defined by

$$\frac{\delta}{\delta \overset{(-)}{\psi}_\alpha(x)} \prod_{j=1}^n \overset{(-)}{\psi}_{\sigma_j}(y_j) = \sum_{k=1}^n (-1)^{k-1} \prod_{\substack{j=1 \\ j \ne k}}^n \overset{(-)}{\psi}_{\sigma_j}(y_j) \begin{cases} 1 & \text{if } \overset{(-)}{\psi}_{\sigma_k}(y_k) = \overset{(-)}{\psi}_\alpha(x) \\ 0 & \text{if } \overset{(-)}{\psi}_{\sigma_k}(y_k) \ne \overset{(-)}{\psi}_\alpha(x) \end{cases}$$

The $(-1)^{k-1}$ in the definition of the derivative is the sign of the permutation that moves $\overset{(-)}{\psi}_{\sigma_k}(y_k)$ to the left hand end of the product.

We assume, as the second characteristic of our momentum range, that the covariance C decays at a rate typical of a smooth function whose Fourier transform has support in a neighbourhood of $|p| = 2^j$. Precisely,

$$|C(x,y)| \le \kappa 2^{(d+1)j/2} e^{-2^j |x-y|}$$

The coefficient $2^{(d+1)j/2}$ is chosen to give power counting typical of a strictly renormalizeable field theory. The position space behaviour of the many-Fermion propagator is somewhat more complicated than this. More about this later.

Theorem III.1. *Let $S_{2,n}(x,x')$ be the coefficient of λ^n in the formal power series expansion of $S_2(x,x')$. That is, $S_2(x,x') = \sum_{n=0}^\infty S_{2,n}(x,x')\lambda^n$. There exists a constant R, independent of j, x, x', such that*

$$\sup_x \sum_{x'} |S_{2,n}(x,x')| \le K_j R^n$$

In other words S_2 is analytic in $|\lambda| < \frac{1}{R}$. In other words, the sum of all connected Feynman diagrams converges for all $|\lambda| < \frac{1}{R}$. Similar bounds apply to the other Euclidean Green's functions.

Logic of the Proof. We first describe the logic of the proof. Denote by $S_2(x, x'; \Lambda)$ the two point function of the model gotten by restricting the world to a finite subset Λ of W. As a preliminary step, we will show, by Hadamard's inequality, that both the numerator and denominator of $S_2(x, x'; \Lambda)$ are entire functions of λ. The denominator $\int e^{-\lambda V} d\mu_C$ can have many Λ dependent zeros. But when $\lambda = 0$, the denominator is one so that $S_2(x, x'; \Lambda)$ is meromorphic on all of $\mathbb{C}$ and analytic at zero. We shall develop a formal power series expansion for $S_2(x, x'; \Lambda)$ with the property that for every N

$$S_2(x, x'; \Lambda) = \sum_{n=0}^{N} S_{2,n}(x, x'; \Lambda)\lambda^n + O\left(\lambda^{N+1}\right) \tag{III.1a}$$

A priori we do not claim that the tail $O\left(\lambda^{N+1}\right)$ is uniform in Λ. Nevertheless, since $S_2(x, x'; \Lambda)$ is analytic at zero we must have

$$S_2(x, x'; \Lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{d^n}{d\lambda^n} S_2(x, x'; \Lambda)\Big|_{\lambda=0}$$

with

$$\frac{1}{n!} \frac{d^n}{d\lambda^n} S_2(x, x'; \Lambda)\Big|_{\lambda=0} = S_{2,n}(x, x'; \Lambda)$$

by (III.1a). Hence

$$S_2(x, x'; \Lambda) = \sum_{n=0}^{\infty} S_{2,n}(x, x'; \Lambda)\lambda^n \tag{III.1b}$$

for all λ smaller than the (possibly Λ dependent) radius convergence of the right hand side. We remark in passing that $S_{2,n}(x, x'; \Lambda)$ must be the sum of all connected Feynman diagrams of order n with 2 external legs, since we have an asymptotic expansion.

The heart of the proof is to show that there exists a constant R, independent of Λ, j and a constant K_j independent of Λ such that

$$\sup_x \sum_{x'} |S_{2,n}(x, x'; \Lambda)| \le K_j R^n \tag{III.2}$$

As a consequence, equation (III.1b) applies for all $|\lambda| < R^{-1}$. Any zeroes of the denominator that appear in this disk must be cancelled by zeroes of the numerator. It shall also be clear from the proof of (III.2) that the limits $S_{2,n}(x, x') = \lim\limits_{\Lambda \to W} S_{2,n}(x, x'; \Lambda)$ exist. This will prove, by the Lebesgue dominated convergence theorem, that

$$S_2(x, x') = \lim_{\Lambda \to W} S_2(x, x'; \Lambda) = \sum_{n=0}^{\infty} S_{2,n}(x, x')\lambda^n$$

for all $|\lambda| < R^{-1}$ and that the coefficients $S_{2,n}(x,x')$ obey the bound (III.2).

Analyticity in Finite Volume. Fix a finite subset $\Lambda \subset W$. We now show that when the interaction V is restricted to Λ, the denominator

$$Z(\Lambda) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\left(-\tfrac{\lambda}{2}\right)^n 2^{-nj(d+1)} \sum_{y_1,\cdots,y_n\in\Lambda} \int \prod_{i=1}^{n} \bar\psi_\uparrow(y_i)\bar\psi_\downarrow(y_i)\psi_\downarrow(y_i)\psi_\uparrow(y_i)\ d\mu_C$$

in the definition of $S_2(x,x';\Lambda)$ is entire in λ. The proof that the numerator is also entire is similar.

The value of the integral is

$$\begin{aligned}\int \prod_{i=1}^{n} &\bar\psi_\uparrow(y_i)\bar\psi_\downarrow(y_i)\psi_\downarrow(y_i)\psi_\uparrow(y_i)\ d\mu_C \\ &= \int \prod_{i=1}^{n} \bar\psi_\uparrow(y_i)\psi_\uparrow(y_i)\ d\mu_C \int \prod_{i=1}^{n} \bar\psi_\downarrow(y_i)\psi_\downarrow(y_i)\ d\mu_C \\ &= \det\left[C(y_i,y_{i'})\right]_{1\le i,i'\le n} \det\left[C(y_i,y_{i'})\right]_{1\le i,i'\le n}\end{aligned}$$

For these determinants to be nonzero, all of the $y_{i'}$'s must be distinct. For, otherwise, the determinants have two identical columns. So, by Hadamard's inequality,

$$\begin{aligned}\left|\det\left[C(y_i,y_{i'})\right]_{1\le i,i'\le n}\right| &\le \prod_{i=1}^{n}\left[\sum_{i'=1}^{n} |C(y_i,y_i')|^2\right]^{1/2} \\ &\le \prod_{i=1}^{n}\left[\sum_{i'=1}^{n} |\kappa^2 2^{(d+1)j} e^{-22^j|y_i-y_{i'}|}\right]^{1/2} \\ &\le \prod_{i=1}^{n}\left[\sum_{y\in\Lambda} |\kappa^2 2^{(d+1)j} e^{-22^j|y_i-y|}\right]^{1/2} \\ &\le \tilde\kappa_j^n\end{aligned}$$

and the coefficient of λ^n in the Taylor series expansion of $Z(\Lambda)$ is bounded by

$$\tfrac{1}{n!}\left(\tfrac{|\lambda|}{2}\right)^n 2^{-nj(d+1)}|\Lambda|^n \tilde\kappa_j^{2n}$$

which implies that $Z(\Lambda)$ is entire in λ. The bound, however, blows up badly with $|\Lambda|$. The bulk of the effort in this proof goes into deriving a Λ independent bound.

The Expansion. We now describe the expansion used. To emphasize that everything is uniform in Λ, we supress Λ. The first step is to use integration by parts (that is property 2.) to turn the $\psi_\uparrow(x)$ of the two point function into a covariance:

$$\begin{aligned}S_2(x,x') &= \frac{\int \psi_\uparrow(x)\bar\psi_\uparrow(x')e^{-\lambda V}d\mu_C}{\int e^{-\lambda V}d\mu_C} \\ &= C(x,x') + \frac{\sum_y \lambda C(x,y)\int \bar\psi_\uparrow(x')\frac{\delta\ \ V}{\delta\bar\psi_\uparrow(y)}e^{-\lambda V}d\mu_C}{\int e^{-\lambda V}d\mu_C}\end{aligned}$$

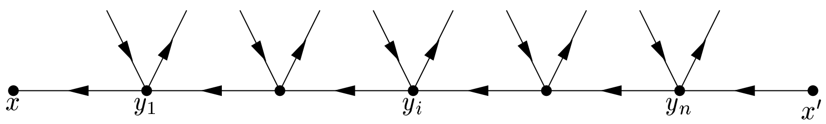
The first term is the trivial Feynman diagram giving the free value of S_2. For the second, apply integration by parts again to turn the $\bar\psi_\uparrow(x')$ into another propagator.

$$S_2(x,x') = C(x,x') - \frac{\sum_{y,y'} \lambda C(x,y)C(y',x') \int \left[\frac{\delta}{\delta\psi_\uparrow(y')}\frac{\delta}{\delta\bar\psi_\uparrow(y)} V\right] e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C}$$
$$- \frac{\sum_{y,y'} \lambda^2 C(x,y)C(y',x') \int \frac{\delta\ V}{\delta\bar\psi_\uparrow(y)}\frac{\delta\ V}{\delta\psi_\uparrow(y')} e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C}$$

In each step select any $\overset{(-)}{\psi}$ downstairs and use integration by parts to turn it into one end of a propagator. When a term has no fields downstairs, the $\int e^{-\lambda V} d\mu_C$ in the numerator exactly cancels that in the denominator, leaving a Feynman diagram. This was how the trivial diagram C arose. Leave such terms alone. Upon completion of the expansion, we have $S_2(x,x')$ expressed as the sum of all connected two point Feynman diagrams.

To illustrate the principal difficulty in bounding S_2 consider the following n^{th} order term that arises in the midst of the expansion:

(III.3)



$$= \frac{\lambda^n}{2^n} 2^{-j(d+1)n} \sum_{y_1,\cdots,y_n\in W} \Big(\prod_{i=1}^{n+1} C(y_{i-1},y_i)\Big) \frac{\int \prod_{m=1}^n \bar\psi_\downarrow(y_m)\psi_\downarrow(y_m) e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C}$$

where $y_0 = x$ and $y_{n+1} = x'$. The functional integral

$$\begin{aligned}
&\int \prod_{m=1}^n \bar\psi_\downarrow(y_m)\psi_\downarrow(y_m) e^{-\lambda V} d\mu_C \\
&= -\sum_{z\in W} C(z,y_1) \int \frac{\delta}{\delta\psi_\downarrow(z)} \left[\psi_\downarrow(y_1) \prod_{m=2}^n \bar\psi_\downarrow(y_m)\psi_\downarrow(y_m) e^{-\lambda V}\right] d\mu_C \\
&= -\sum_{i=1}^n C(y_i,y_1) \int \psi_\downarrow(y_1)\cdots \cancel{\psi_\downarrow(y_i)} \cdots \bar\psi_\downarrow(y_n)\psi_\downarrow(y_n) e^{-\lambda V} d\mu_C \\
&\qquad\qquad - \tfrac{\lambda}{2} 2^{-j(d+1)} \sum_{y_{n+1}\in W} C(y_{n+1},y_1) A_{n+1}
\end{aligned}$$

where

$$A_{n+1} = \int \psi_\downarrow(y_1)\cdots\psi_\downarrow(y_n)\bar\psi_\uparrow(y_{n+1})\bar\psi_\downarrow(y_{n+1})\psi_\uparrow(y_{n+1}) e^{-\lambda V} d\mu_C$$

We did a single integration by parts to get rid of $\bar\psi_\downarrow(y_1)$ and ended up with n terms of order λ^n. The A_{n+1} terms are of order λ^{n+1}. If we perform $n-1$ further integrations by parts to get rid of $\bar\psi_\downarrow(y_2),\cdots,\bar\psi_\downarrow(y_n)$ we will generate $n!$ diagrams of order λ^n. Naive bounds on these $n!$ terms will fail to produce an acceptable bound on $S_{2,n}$.

Fortunately, the Pauli exclusion priciple saves us. Note first that, if $y_m = y_{m'}$ for any $m \neq m'$, then $\psi_\downarrow(y_m)\psi_\downarrow(y'_m) = -\psi_\downarrow(y'_m)\psi_\downarrow(y_m)$ so that $\psi_\downarrow(y_m)\psi_\downarrow(y'_m) = 0$ and hence $\psi_\downarrow(y_1)\prod_{m=2}^{n}\bar\psi_\downarrow(y_m)\psi_\downarrow(y_m) = 0$. Let

$$A_i = \int \psi_\downarrow(y_1)\cdots\cancel{\psi_\downarrow(y_i)}\cdots\bar\psi_\downarrow(y_n)\psi_\downarrow(y_n)e^{-\lambda V}d\mu_C$$

Then we may bound

$$\begin{aligned}\sum_{i=1}^{n}|C(y_i,y_1)A_i| &\le \max_{1\le i\le n}\left|e^{2^j|y_1-y_i|/2}C(y_i,y_1)A_i\right|\sum_{i=1}^{n}e^{-2^j|y_1-y_i|/2}\\
&\le \max_{1\le i\le n}\left|\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}A_i\right|\sum_{y\in 2^{-j}\mathbb{Z}^{d+1}}e^{-2^j|y_1-y|/2}\\
&= \max_{1\le i\le n}\left|\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}A_i\right|\sum_{x\in\mathbb{Z}^{d+1}}e^{-|x|/2}\\
&= \mathcal{E}\max_{1\le i\le n}\left|\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}A_i\right| \qquad \text{(III.4)}\end{aligned}$$

where $\mathcal{E} = \sum_{x\in\mathbb{Z}^{d+1}}^{n}e^{-|x|/2} < \infty$. The crucial consequence of the Pauli exclusion principle, that the y_i's all are different, was used in going from line one to line two. Think of $2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}$ as a propagator (replacing $C(y_1,y_i)$) for a line in a graph. This propagator joins a vertex at y_1 to a vertex at y_i. The fields $\overset{(-)}{\psi}$ downstairs in the functional integral A_i are external legs for the graph.

Incorporating the A_{n+1} terms just gives

$$\begin{aligned}&\left|\int\prod_{m=1}^{n}\bar\psi_\downarrow(y_m)\psi_\downarrow(y_m)e^{-\lambda V}d\mu_C\right|\\
&\le \mathcal{E}\max_{1\le i\le n}\left|\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}A_i\right| + \tfrac{|\lambda|}{2}2^{-j(d+1)}\sum_{y_{n+1}\in W}C(y_{n+1},y_1)\,|A_{n+1}|\\
&\le \max\Big\{\max_{1\le i\le n}\left|(\mathcal{E}+1)\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_i|/2}A_i\right|,\\
&\qquad \tfrac{|\lambda|}{2}2^{-j(d+1)}\sum_{y_{n+1}\in W}(\mathcal{E}+1)\kappa 2^{j(d+1)/2}e^{-2^j|y_1-y_{n+1}|/2}\,|A_{n+1}|\Big\} \qquad \text{(III.5)}\end{aligned}$$

Proof of the Main Bound. We now develop the full bound, proceeding by induction. In each step of the induction we integrate by parts once and apply the above bounding procedure. At the end of step s of the induction procedure we will have the bound

$$|S_2(x,x')| \le \max_{G\in\mathcal{G}_s}B(G) \qquad \text{(III.6)}$$

where $\mathcal{G}_s$ is the set of all "incomplete" Feynman diagrams, like (III.3), that are formed by s or fewer integration by parts. Each $G \in \mathcal{G}_s$ has

- two one-legged vertices, labelled x and x'
- $\omega(G)$ four-legged vertices labelled $y_1, \cdots, y_{\omega(G)}$
- at most s propagators with each propagator joining a pair of legs selected from the $4\omega(G)+2$ legs of the $\omega(G)+2$ vertices. The positions of the two vertices at the ends of line ℓ are denoted y_{i_ℓ} and y_{f_ℓ}. The set of legs of G that are not paired to form propagators is denoted $F(G)$ and consists of those fields that are downstairs in the functional integral and have not yet been either an initiator or a target of an integration by parts. If $F(G)$ is not empty the number of propagators is exactly s, because exactly s integration by parts have been performed.

The bound on G that appears in (III.6) is

$$B(G) = \frac{\lambda^{\omega(G)}}{2^{\omega(G)}} \sum_{y_1, \cdots, y_{\omega(G)}} 2^{-j(d+1)\omega(G)} \prod_{\ell \in G} \left[(\mathcal{E}+1)\kappa 2^{j(d+1)/2} e^{-2^j |y_{i_\ell} - y_{f_\ell}|/2} \right] \left| \frac{\int \prod_{f \in F(G)} \overset{(-)}{\psi}_{\sigma_f}(y_f) e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C} \right|$$

At the very beginning of the induction $s = 0$ and $\mathcal{G}_0$ consists of precisely one graph G_0, which has two one-legged vertices, no propagators and

$$B(G_0) = \left| \frac{\int \psi_\uparrow(x) \bar{\psi}_\uparrow(x') e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C} \right|$$

The verification of the inductive hypothesis is virtually identical to the proof of (III.5). It suffices to replace $\prod_{m=1}^n \bar{\psi}_\downarrow(y_m)\psi_\downarrow(y_m)$ by $\prod_{f \in F(G)} \overset{(-)}{\psi}_{\sigma_f}(y_f)$.

In order to isolate the n^{th} order contribution to S_2, it suffices to insert the projection

$$P_{\le n} = \frac{d^n}{d\lambda^n} \Big|_{\lambda=0}$$

onto n^{th} order everywhere. Then

$$|S_{2,n}(x,x')| \le \max_{G \in \mathcal{G}_s} B_n(G) \tag{III.6a}$$

with

$$B_n(G) = \frac{1}{2^{\omega(G)}} \sum_{y_1, \cdots, y_{\omega(G)}} 2^{-j(d+1)\omega(G)} \prod_{\ell \in G} \left[(\mathcal{E}+1)\kappa 2^{j(d+1)/2} e^{-2^j |y_{i_\ell} - y_{f_\ell}|/2} \right] \left| P_{\le n-\omega(G)} \frac{\int \prod_{f \in F(G)} \overset{(-)}{\psi}_{\sigma_f}(y_f) e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C} \right|$$

A graph having s propagators must be of order at least $(s-1)/2$. So (III.6a) becomes independent of s for $s \ge 2n+1$ and we have

$$|S_{2,n}(x,x')| \le \frac{1}{2^n} \max_G \sum_{y_1, \cdots, y_n} 2^{-j(d+1)n} \prod_{\ell \in G} \left[(\mathcal{E}+1)\kappa 2^{j(d+1)/2} e^{-2^j |y_{i_\ell} - y_{f_\ell}|/2} \right]$$

The maximum is over all connected Feynman digrams with two one-legged vertices, labelled x, x' and n four-legged vertices labelled $y_1, \cdots, y_n$. In preparation for bounding the graph G, select a spanning tree T for G. A spanning tree is a subgraph $T \subset G$ which has no loops and contains all the vertices of G. Bound all factors $e^{-2^j|y_{i_\ell}-y_{f_\ell}|/2}$ that are associated with lines $\ell \in G \setminus T$ by one. Then apply

$$\sum_{y\in W} e^{-2^j|y'-y|/2} \le \mathcal{E}$$

to each vertex of G starting with those farthest from x in the partial ordering of T. The result is

$$\begin{aligned}
&\sum_{x'} |S_{2,n}(x,x')| \\
&\qquad \le \frac{1}{2^n} \max_G \sum_{y_1,\cdots,y_n,x'} 2^{-j(d+1)n} \prod_{\ell\in G} \Big[(\mathcal{E}+1)\kappa 2^{j(d+1)/2} e^{-2^j|y_{i_\ell}-y_{f_\ell}|/2}\Big] \\
&\qquad \le \frac{1}{2^n} 2^{-j(d+1)n} \max_G (\kappa\mathcal{E}+\kappa)^{|G|} 2^{|G|j(d+1)/2} \sum_{y_1,\cdots,y_n,x'} \prod_{\ell\in T} \Big[e^{-2^j|y_{i_\ell}-y_{f_\ell}|/2}\Big] \\
&\qquad \le \frac{1}{2^n} 2^{-j(d+1)n} \max_G (\kappa\mathcal{E}+\kappa)^{|G|} 2^{|G|j(d+1)/2} \mathcal{E}^{n+1}
\end{aligned}$$

As we are currently considering an n^{th} order diagram contributing to the two point function

$$|G| = \frac{2+4n}{2} = 2n+1$$

and the final bound is

$$\begin{aligned}
\sum_{x'} |S_{2,n}(x,x')| &\le \frac{1}{2^n} 2^{-j(d+1)n} (\kappa\mathcal{E}+\kappa)^{2n+1} 2^{(2n+1)j(d+1)/2} \mathcal{E}^{n+1} \\
&\le \frac{(\mathcal{E}+1)^{3n+2}\kappa^{2n+1}}{2^n} 2^{j(d+1)/2} ,
\end{aligned}$$

which proves the Theorem with $R = \frac{1}{2}\kappa^2(\mathcal{E}+1)^3$ and $K_j = \kappa(\mathcal{E}+1)^2 2^{j(d+1)/2}$.

More generally, for a p point function, $|G| = \frac{p+4n}{2} = 2n + p/2$, the number of sums controlled by the tree decay is $n+p-1$ and

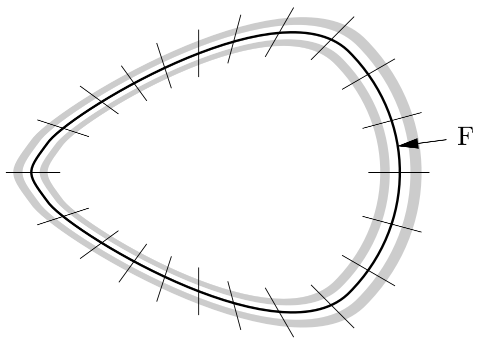
$$\begin{aligned}
\sum_{x_2,\cdots,x_p} |S_{p,n}(x_1,\cdots,x_p)| &\le \frac{1}{2^n} 2^{-j(d+1)n} (\kappa\mathcal{E}+\kappa)^{2n+p/2} 2^{(2n+p/2)j(d+1)/2} \mathcal{E}^{n+p-1} \\
&\le \frac{\kappa^{2n+p/2}(\mathcal{E}+1)^{3n+3p/2-1}}{2^n} 2^{pj(d+1)/4}
\end{aligned}$$

■

As we mentioned earlier, the hypothesis

$$|C(x,y)| \le \kappa 2^{(d+1)j/2} e^{-2^j|x-y|}$$

while typical of a strictly renormalizeable field theory is not satisfied by the many-Fermion propagator. The problem is that the momentum space shell $\{\ k \ \big|\ 2^j \le |ik_0 - e(\mathbf{k})| < 2^{j+1}\ \}$ has two very different characteristic lengths: a macroscopic diameter of order 1 and a microscopic thickness of order 2^j. By decomposing the Fermi surface into a union of $2^{-(d-1)j}$ "rectangles" of side 2^j, one can write the



many-Fermion field at scale j as a sum $\psi^{(j)} = \sum_\alpha \psi^{(j,\alpha)}$ of $2^{-(d-1)j}$ independent fields with each "coloured" field having a covariance that obeys $|C(x,y)| \le \kappa 2^{dj} e^{-2^j|x-y|}$ One can then apply the methods of Theorem III.1. Of course one still has to control all the colour sums. We have succeeded [FMRT1] in doing so for $d = 2$ and are working to extend the control to $d = 3$. This is the only place in the proof of the Theorem of section I that we have to restrict to $d = 2$.

References

[FKLT] J. Feldman, H. Knörrer, D. Lehmann and E. Trubowitz, *in preparation.*

[FMRT1] J. Feldman, J. Magnen, V. Rivassseau and E. Trubowitz, *An Infinite Volume Expansion for Many Fermion Green's Functions*, Helvetica Physica Acta **65**, 679–721.

[FMRT2] ______, *Fermionic Many-Body Models*, Mathematical Quantum Theory I: Field Theory and Many-Body Theory (J. Feldman, R. Froese and L. Rosen, eds.), CRM Proceedings & Lecture Notes.

[FMRT3] ______, *Two Dimensional Many Fermion Systems as Vector Models*, Europhysics Letters **24**, 521–526.

[FMRT4] ______, *An Intrinsic 1/N Expansion for Many Fermion Systems*, Europhysics Letters **24**, 437–442.

[FST] J. Feldman, M. Salmhofer and E. Trubowitz, *Renormalization Theory for Many Fermion Systems: One-Band Models with Non-Nested Fermi Surfaces, in preparation.*

[FT1] J. Feldman and E. Trubowitz, *Perturbation Theory for Many Fermion Systems*, Helvetica Physica Acta **63**, 156–260.

[FT2] ______, *The Flow of an Electron-Phonon System to the Superconducting State*, Helvetica Physica Acta **64**, 214–357.

Dept of Mathematics, University of British Columbia, Vancouver, B.C. CANADA V6T 1Z2

E-mail: feldman@math.ubc.ca

Mathematik, ETH-Zentrum, CH-8092 Zürich, SWITZERLAND
E-mail: knoerrer@math.ethz.ch

Mathematik, ETH-Zentrum, CH-8092 Zürich, SWITZERLAND
E-mail: lehmann@math.ethz.ch

Mathematik, ETH-Zentrum, CH-8092 Zürich, SWITZERLAND
E-mail: trub@math.ethz.ch