Reviews in Mathematical Physics
Vol. 15, No. 9 (2003) 1121–1169

SINGLE SCALE ANALYSIS OF MANY FERMION SYSTEMS PART 4: SECTOR COUNTING

JOEL FELDMAN*

Department of Mathematics, University of British Columbia
Vancouver, B.C., Canada V6T 1Z2
feldman@math.ubc.ca
http://www.math.ubc.ca/~feldman/

HORST KNÖRRER† and EUGENE TRUBOWITZ‡

Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland
†*knoerrer@math.ethz.ch*
‡*trub@math.ethz.ch*
†*http://www.math.ethz.ch/~knoerrer/*

For a two-dimensional, weakly coupled system of fermions at temperature zero, one principal ingredient used to control the composition of the associated renormalization group maps is the careful counting of the number of quartets of sectors that are consistent with conservation of momentum. A similar counting argument is made to show that particle–particle ladders are irrelevant in the case of an asymmetric Fermi curve.

Keywords: Fermi liquid; renormalization; fermionic functional integral; Fermi surface.

Contents

*Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Forschungsinstitut für Mathematik, ETH Zürich.

XVIII. Introduction to Part 4

In the application of the results of Parts 1 through 3 to many fermion systems ([1–3]) the effective potential and all the quantities derived from it will conserve particle number. Particle number conservation implies that sectorized functions $\varphi((\cdot, s_1), \ldots, (\cdot, s_n)) \in \mathcal{F}_0(n; \Sigma)$, where Σ is a sectorization, vanish unless the configuration $s_1, \ldots, s_n$ of sectors is consistent with conservation of momentum (for a more precise statement see Definition XX.1 and Remark XX.2). We shall count the number of configurations $s_1, \ldots, s_n$ of sectors consistent with conservation of momentum that satisfy certain constraints. The results are used to compare different norms for four-point functions (Proposition XIX.1), and to compare norms associated to different sectorizations at different scales (Proposition XIX.4). The latter is crucial for a multi scale analysis of many fermion systems ([1–3]). Notation tables are provided at the end of the paper.

We retain the assumptions that the dispersion relation $e(\mathbf{k})$ is $r + d + 1$ times differentiable, with $r \geq 2$ and $d = 2$, and that its gradient does not vanish on the Fermi curve $F = \{\mathbf{k} \in \mathbb{R}^d | e(\mathbf{k}) = 0\}$. All the above results hold under additional geometric assumptions on the geometry of the Fermi curve F. First of all, we assume throughout the rest of the paper that the Fermi curve F is **strictly convex**, with curvature bounded away from zero. If the dispersion relation $e(\mathbf{k})$ is that of a background electric field alone then $e(\mathbf{k}) = e(-\mathbf{k})$ and the Fermi curve F is symmetric about the origin. That is, $\mathbf{k} \in F$ if and only if $-\mathbf{k}$ in F.

Definition XVIII.1. (i) Since F is strictly convex, for each point $\mathbf{k} \in F$ there is a unique point $a(\mathbf{k}) \in F$ different from $\mathbf{k}$ such that the tangent lines to F at $\mathbf{k}$ and $a(\mathbf{k})$ are parallel. $a(\mathbf{k})$ is called the antipode of $\mathbf{k}$.

(ii) We say that F is symmetric about a point $\mathbf{p} \in \mathbb{R}^2$ if $F = \{2\mathbf{p} - \mathbf{k} | \mathbf{k} \in F\}$.

Example XVIII.2. If F is symmetric about a point $\mathbf{p}$ then $a(\mathbf{k}) = 2\mathbf{p} - \mathbf{k}$ for all $\mathbf{k} \in F$.

Symmetry of the Fermi curve about a point allows for the formation of Cooper pairs and the phase transition to a superconducting state. In [1–3] we show that this is the only instability in a broad class of short range many fermion models. We now make a precise asymmetry assumption on the geometry of the Fermi surface.

Definition XVIII.3. Choose an orientation for F.

(i) Let $\mathbf{k} \in F$, $\vec{\mathrm{t}}$ the oriented unit tangent vector to F at $\mathbf{k}$ and $\vec{\mathrm{n}}$ the inward pointing unit normal vector to F at $\mathbf{k}$. Then there is a function $\varphi_{\mathbf{k}}(s)$, defined on a neighborhood of 0 in $\mathbb{R}$, such that $s \mapsto \mathbf{k} + s\vec{\mathrm{t}} + \varphi_{\mathbf{k}}(s)\vec{\mathrm{n}}$ is an oriented parametrization of F near $\mathbf{k}$.

(ii) We say that F is strongly asymmetric if there is $n_0 \in \mathbb{N}$, with $n_0 \le r$, such that for each $\mathbf{k} \in F$ there exists an $n \le n_0$ such that

$$\varphi_{\mathbf{k}}^{(n)}(0) \neq \varphi_{a(\mathbf{k})}^{(n)}(0)\,.$$

Remark XVIII.4. (i) By construction, $\varphi_{\mathbf{k}}(0) = \dot{\varphi}_{\mathbf{k}}(0) = 0$ and $\ddot{\varphi}_{\mathbf{k}}(0)$ is the curvature of F at $\mathbf{k}$.

(ii) If F is symmetric under inversion in some point $\mathbf{p} \in \mathbb{R}^2$, then $\varphi_{\mathbf{k}} = \varphi_{a(\mathbf{k})}$ for all $\mathbf{k} \in F$.

(iii) In [4] we show that independent electrons in a suitably chosen periodic electromagnetic background field have a dispersion relation whose associated Fermi curve, for suitably chosen chemical potential, is smooth, strictly convex, strongly asymmetric and has nonzero curvature everywhere.

(iv) In [1–3] we show that a many fermion system with a strongly asymmetric Fermi surface and weak, short range interaction is a Fermi liquid.

Throughout the rest of the paper we assume, unless otherwise stated, that the Fermi surface is strictly convex and either symmetric about a point or strictly asymmetric in the sense of Definition XVIII.3. In Sec. XXII, we derive a sector counting result that holds only for strongly asymmetric Fermi curves and use it to get an estimate on particle–particle bubbles that is better than the logarithmic divergence that, in the case of a symmetric Fermi surface, is responsible for the Cooper instability.

We emphasize that for the sector counting arguments of Sec. XX, the fact that the model is in two space dimensions is crucial. Propositions XX.10 and XX.11 would not hold in a three dimensional situation. See [1, Sec. II, Subsec. 8].

XIX. Comparison of Norms

Theorem XV.3 indicates that ladders give the dominant contributions to $w_{0,4}$. The $|\cdot|_{3,\Sigma}$ norm of ladders will be estimated in Sec. XXII and [5]. To control the $N(w;\cdots)$ norms of w, we develop a bound on the $|\cdot|_{1,\Sigma}$ norm of a ladder in terms of its $|\cdot|_{3,\Sigma}$ norm.

Proposition XIX.1. *Let Σ be a sectorization of length $\frac{1}{M^{2j/3}} \le \mathfrak{l} \le \frac{1}{M^{j/2}}$ at scale $j \ge 4$. Furthermore let $\varphi \in \mathcal{F}_0(4,\Sigma)$ and $f \in \check{\mathcal{F}}_{4;\Sigma}$ be particle number conserving functions. Then*

$$|\varphi|_{1,\Sigma} \le {\rm const}\,\frac{1}{\mathfrak{l}}|\varphi|_{3,\Sigma} \quad \textit{and} \quad |f|^{\sim}_{1,\Sigma} \le {\rm const}\,\frac{1}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma}$$

with a constant const *that is independent of M, j, Σ.*

This proposition is proven after Lemma XXI.1. In the renormalization group analysis, we go from scale to scale. After integrating out scale j, we shall have an effective potential $\mathcal{W}$ with a representative w, sectorized at scale j; and we will have an estimate on the norm of w. To apply Theorem XV.3 at scale $j+1$ we then need a representative for $\mathcal{W}$ that is sectorized at scale $j+1$ and estimates on it. This change of sectorizations is implemented by

Definition XIX.2. Let $j, i \geq 2$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively. If $i \neq j$, define, for functions φ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma')^n$ and f on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma')^n$,

$$\begin{aligned} &\varphi_\Sigma(\eta_1, \ldots, \eta_m; (\xi_1, s_1), \ldots, (\xi_n, s_n)) \\ &\quad = \sum_{s'_1, \ldots, s'_n \in \Sigma'} \int d\xi'_1 \cdots d\xi'_n \varphi(\eta_1, \ldots, \eta_m; (\xi'_1, s'_1), \ldots, (\xi'_n, s'_n)) \prod_{\ell=1}^{n} \hat{\chi}_{s_\ell}(\xi'_\ell, \xi_\ell) \\ &f_\Sigma(\check{\eta}_1, \ldots, \check{\eta}_m; (\xi_1, s_1), \ldots, (\xi_n, s_n)) \\ &\quad = \sum_{s'_1, \ldots, s'_n \in \Sigma'} \int d\xi'_1 \cdots d\xi'_n f(\check{\eta}_1, \ldots, \check{\eta}_m; (\xi'_1, s'_1), \ldots, (\xi'_n, s'_n)) \prod_{\ell=1}^{n} \hat{\chi}_{s_\ell}(\xi'_\ell, \xi_\ell) \end{aligned}$$

where χ_s, $s \in \Sigma$ is the partition of unity of Lemma XII.3 and (XIII.2). If φ is translation invariant and antisymmetric under permutation of its η arguments, then $\varphi_\Sigma \in \mathcal{F}_m(n; \Sigma)$. For $i = j$ and $\Sigma' = \Sigma$, define $\varphi_\Sigma = \varphi$ and $f_\Sigma = f$.

Remark XIX.3. (i) If $u \in \mathcal{F}_0(2; \Sigma')$ is an antisymmetric, spin independent and particle number conserving function then

$$\check{u}_\Sigma(k) = \check{u}(k)\, (\tilde{\nu}^{(\geq j)}(k))^2\,.$$

(ii) For a function φ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma')^n$ one has $(\varphi_\Sigma)^\sim = (\varphi^\sim)_\Sigma$.

(iii) Let $j, i_1, i_2 \geq 2$ with $i_2 > i_1$. Let Σ, Σ_1 and Σ_2 be sectorizations at scales j, i_1 and i_2 respectively. Then, for each function φ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma)^n$ and each function f on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^n$

$$(\varphi_{\Sigma_1})_{\Sigma_2} = \varphi_{\Sigma_2} \quad \text{and} \quad (f_{\Sigma_1})_{\Sigma_2} = f_{\Sigma_2}\,.$$

Proposition XIX.4. *Let $j > i \geq 2$, $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$ and $\frac{1}{M^{i-3/2}} \leq \mathfrak{l}' \leq \frac{1}{M^{(i-1)/2}}$ with $4\mathfrak{l} < \mathfrak{l}'$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively. Let $\varphi \in \mathcal{F}_m(n; \Sigma'$ and $f \in \check{\mathcal{F}}_m(n; \Sigma')$ be particle number conserving functions.*

(i) *If $m \neq 0$*

$$|\varphi_\Sigma|_{1,\Sigma} \leq \mathit{const}^n\ \mathfrak{c}_{j-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^n |\varphi|_{1,\Sigma'}\,.$$

(ii) *If f is antisymmetric in its (ξ, s) arguments, then for all p*

$$|f_\Sigma|^\sim_{p,\Sigma} \leq \mathit{const}^n\ \mathfrak{c}_{j-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{n+m-p-1} |f|^\sim_{p,\Sigma'}\,.$$

Moreover, if $\mathfrak{l} \geq \frac{1}{M^{2/3(j-1)}}$, $\mathfrak{l}' \leq \frac{1}{6}\sqrt{\mathfrak{l}}$ and $n \geq 3$

$$|f_\Sigma|^\sim_{1,\Sigma} \leq \mathit{const}^n\ \mathfrak{c}_{j-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{n+m-3} \left(|f|^\sim_{1,\Sigma'} + \frac{1}{\mathfrak{l}'}|f|^\sim_{3,\Sigma'}\right).$$

(iii) *If f is antisymmetric in its (ξ, s) arguments, then for all p*

$$|f_{\Sigma'}|^{\sim}_{p,\Sigma'} \le {\rm const}^n\ \mathfrak{c}_{i-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{p-m} |f|^{\sim}_{p,\Sigma}\,.$$

Here const is a constant that is independent of M, j, Σ.

This proposition is proved after Lemma XXI.4.

Remark XIX.5. Since for $m = 0$ the norms $|\varphi|_{p,\Sigma}$ and $|\varphi|^{\sim}_{p,\Sigma}$ agree, Proposition XIX.4(ii) implies that, in the case that $\mathfrak{l} \ge \frac{1}{M^{2/3(j-1)}}$ and $4\mathfrak{l} < \mathfrak{l}' < \frac{1}{6}\sqrt{\mathfrak{l}}$, for antisymmetric $\varphi \in \mathcal{F}_0(n; \Sigma')$

$$\begin{aligned}
&|\varphi_\Sigma|_{1,\Sigma} \le {\rm const}\ \mathfrak{c}_{j-1}|\varphi|_{1,\Sigma'} && \text{if } n = 2\\
&|\varphi_\Sigma|_{3,\Sigma} \le {\rm const}\ \mathfrak{c}_{j-1}|\varphi|_{3,\Sigma'} && \text{if } n = 4\\
&|\varphi_\Sigma|_{1,\Sigma} \le {\rm const}^n\ \mathfrak{c}_{j-1}\left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{n-3}\left(|\varphi|_{1,\Sigma'} + \frac{1}{\mathfrak{l}'}|\varphi|_{3,\Sigma'}\right) && \text{if } n \ge 4\,.
\end{aligned}$$

The resectorization of functions on $\mathfrak{X}^n_\Sigma = (\check{\mathcal{B}} \mathbin{\dot\cup} (\mathcal{B} \times \Sigma))^n$ is defined just as in Definition XIX.2. To be precise, recall from Remark XVI.3 and Definition XVI.2(iii) that

$$\mathfrak{X}^n_\Sigma = \dot{\bigcup_{i_1,\dots,i_n \in \{0,1\}}} \mathfrak{X}_{i_1}(\Sigma) \times \cdots \times \mathfrak{X}_{i_n}(\Sigma)$$

where $\mathfrak{X}_0(\Sigma) = \check{\mathcal{B}}$ and $\mathfrak{X}_1(\Sigma) = \mathcal{B} \times \Sigma$. Furthermore, for each $\vec{\imath} = (i_1, \dots, i_n) \in \{0,1\}^n$, the map Ord gives a bijection between functions on $\mathfrak{X}_{i_1}(\Sigma) \times \cdots \times \mathfrak{X}_{i_n}(\Sigma)$ and functions on $\check{\mathcal{B}}^{m(\vec{\imath})} \times (\mathcal{B} \times \Sigma)^{n-m(\vec{\imath})}$, where $m(\vec{\imath}) = n - i_1 - \cdots - i_n$.

Definition XIX.6. Let $j, i \ge 2$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively.

(i) Let $\vec{\imath} = (i_1, \dots, i_n) \in \{0,1\}^n$ and f a function on $\mathfrak{X}_{i_1}(\Sigma') \times \cdots \times \mathfrak{X}_{i_n}(\Sigma')$. Then f_Σ is the function on $\mathfrak{X}_{i_1}(\Sigma) \times \cdots \times \mathfrak{X}_{i_n}(\Sigma)$ determined by $\mathrm{Ord}(f_\Sigma) = (\mathrm{Ord}\, f)_\Sigma$.
(ii) If f is a function on $\mathfrak{X}^n_{\Sigma'}$, its resectorization f_Σ is the function on $\mathfrak{X}^n_\Sigma$ determined by

$$f_\Sigma|_{\vec{\imath}} = (f|_{\vec{\imath}})_\Sigma \quad \text{for all } \vec{\imath} \in \{0,1\}^n\,.$$

From Proposition XIX.4, we have

Corollary XIX.7. *Let $j > i \ge 2$, $\frac{1}{M^{j-3/2}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$ and $\frac{1}{M^{i-3/2}} \le \mathfrak{l}' \le \frac{1}{M^{(i-1)/2}}$ with $4\mathfrak{l} < \mathfrak{l}'$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively. Let $f \in \check{\mathcal{F}}_{n;\Sigma'}$ be an antisymmetric particle number conserving function. Then for all p*

$$|f_\Sigma|^{\sim}_{p,\Sigma} \le {\rm const}^n\ \mathfrak{c}_{j-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{n-p-1} |f|^{\sim}_{p,\Sigma'}\,.$$

Moreover, if $\mathfrak{l} \geq \frac{1}{M^{2/3(j-1)}}$, $\mathfrak{l}' \leq \frac{1}{6}\sqrt{\mathfrak{l}}$ *and* $n \geq 4$

$$|f_\Sigma|\tilde{}_{1,\Sigma} \leq {\scriptstyle const}^n\ \mathfrak{c}_{j-1} \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right]^{n-3} \left(|f|\tilde{}_{1,\Sigma'} + \frac{1}{\mathfrak{l}'}|f|\tilde{}_{3,\Sigma'}\right).$$

In the renormalization group analysis of [1–3], the numbers $\rho_{0;n}$ used as weights in the norms N_j of Definition XV.1 do not depend on the scale j. As pointed out in Remark XV.2, boundedness in j of the norms N_j implies that the coefficient of $t^{\mathbf{0}}$ in $|w_{0,2}|_{1,\Sigma}$ has positive power counting (that is, tends to zero as a power of $\frac{1}{M^j}$) and the coefficient of $t^{\mathbf{0}}$ in $|w_{0,4}|_{3,\Sigma}$ has neutral power counting. The other contributions $w_{m,n}$ behave well with respect to resectorization.

Corollary XIX.8. *Fix* $\frac{1}{2} < \aleph < \frac{2}{3}$ *and let* $j \geq \frac{3}{2-3\aleph}$. *Let* Σ_{j+1} *and* Σ_j *be sectorizations of length* $\mathfrak{l}_{j+1} = \frac{1}{M^{\aleph(j+1)}}$ *at scale* $j+1$ *and* $\mathfrak{l}_j = \frac{1}{M^{\aleph j}}$ *at scale* j*, respectively. Let* $\vec{\rho} = (\rho_{m;n})$ *be a system of positive real numbers obeying* (XV.1) *and set*

$$\rho'_{m;n} = \begin{cases} \rho_{m;n} & \text{if } m = 0 \\ \sqrt[4]{\dfrac{\mathfrak{l}_j M^j}{\mathfrak{l}_{j+1} M^{j+1}}}\ \rho_{m;n} = \dfrac{1}{M^{(1-\aleph)/4}}\ \rho_{m;n} & \text{if } m > 0\,. \end{cases}$$

Let

$$w(\phi,\psi) = \sum_{\substack{m,n \\ m+n \text{ even}}} \sum_{s_1,\ldots,s_n \in \Sigma_{j+1}} \int d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n$$
$$\times\, w_{m,n}(\eta_1,\ldots,\eta_m(\xi_1,s_1),\ldots,(\xi_n,s_n))$$
$$\times\, \phi(\eta_1)\cdots\phi(\eta_m)\psi((\xi_1,s_1))\cdots\psi((\xi_n,s_n))$$

with $w_{m,n} \in \mathcal{F}_m(n;\Sigma_j)$*, be an even* Σ_j*-sectorized particle number conserving Grassmann function with* $w_{0,2} = 0$ *and* $w_{m,0} = 0$ *for all* m*. If* M *is big enough, then*

$$N_{j+1}(w_{\Sigma_{j+1}}; 64\alpha; X, \Sigma_{j+1}, \vec{\rho}) \leq {\scriptstyle const}\, \mathfrak{e}_{j+1}(X) N_j\left(w; \frac{\alpha}{2}; X, \Sigma_j, \vec{\rho}\,'\right)$$

with the constant ${\scriptstyle const}$ *independent of* M, j, Σ_j *and* Σ_{j+1}*. If, in addition* $w_{0,4} = 0$*, then*

$$N_{j+1}(w_{\Sigma_{j+1}}; 64\alpha; X, \Sigma_{j+1}, \vec{\rho}) \leq \frac{1}{M^{(1-\aleph)/8}}\, \mathfrak{e}_{j+1}(X) N_j\left(w; \frac{\alpha}{2}; X, \Sigma_j, \vec{\rho}\,'\right).$$

Proof. We apply Proposition XIX.4 with j replaced by $j+1$, $i = j$, $\mathfrak{l} = \mathfrak{l}_{j+1}$ and $\mathfrak{l}' = \mathfrak{l}_j$. Observe that the hypotheses of part (ii) are fulfilled. In this proof, use $|\cdot|_{\Sigma,\vec{\rho}}$ to designate the norm of Definition XV.1 using the indicated $\vec{\rho}$.

If $m,n \geq 1$, by Proposition XIX.4(i),

$$\frac{M^{2(j+1)}}{\mathfrak{l}_{j+1}}\, \mathfrak{e}_{j+1}(X)(64\alpha)^n \left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2} |(w_{m,n})_{\Sigma_{j+1}}|_{\Sigma_{j+1},\vec{\rho}}$$
$$= \mathfrak{e}_{j+1}(X)(64\alpha)^n \left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2} \rho_{m;n}|(w_{m,n})_{\Sigma_{j+1}}|_{1,\Sigma_{j+1}}$$

$$\le \mathit{const}^n\ \mathfrak{c}_j\ \mathfrak{e}_{j+1}(X) \left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^n \left(\frac{2^{14}}{M}\frac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}\right)^{n/2} \frac{\rho_{m;n}}{\rho'_{m;n}}$$

$$\times \left(\frac{\alpha}{2}\right)^n \left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2} \rho'_{m;n}|w_{m,n}|_{1,\Sigma_j}$$

$$\le \mathit{const}^n\ \mathfrak{e}_{j+1}(X) \left(\frac{1}{M^{1-\aleph}}\right)^{(2n-1)/4} \frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{c}_j$$

$$\times \left(\frac{\alpha}{2}\right)^n \left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2} |w_{m,n}|_{\Sigma_j,\vec\rho\,'}$$

$$\le \frac{1}{M^{(1-\aleph)/8}}\,\mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{e}_j(X)\left(\frac{\alpha}{2}\right)^n \left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2} |w_{m,n}|_{\Sigma_j,\vec\rho\,'}$$

if M is large enough. If $m = 0$ and $n \ge 4$, by Proposition XIX.4(ii) and Remark XVI.5,

$$\frac{M^{2(j+1)}}{\mathfrak{l}_{j+1}}\,\mathfrak{e}_{j+1}(X)(64\alpha)^n \left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2} |(w_{m,n})_{\Sigma_{j+1}}|_{\Sigma_{j+1},\vec\rho}$$

$$= \frac{M^{2(j+1)}}{\mathfrak{l}_{j+1}}\,\mathfrak{e}_{j+1}(X)(64\alpha)^n \left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2} \rho_{0;n}$$

$$\times \Bigg[|(w_{0,n})_{\Sigma_{j+1}}|_{1,\Sigma_{j+1}} + \frac{1}{\mathfrak{l}_{j+1}}|(w_{0,n})_{\Sigma_{j+1}}|_{3,\Sigma_{j+1}}$$

$$+ \frac{1}{\mathfrak{l}_{j+1}^2}|(w_{0,n})_{\Sigma_{j+1}}|_{5,\Sigma_{j+1}}\Bigg]$$

$$\le \mathit{const}^n\ \mathfrak{c}_j\ \mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\left(\frac{\alpha}{2}\right)^n \left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2} \frac{1}{M^{(n-4)/2}}\left(\frac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}\right)^{(n-2)/2} \rho_{0;n}$$

$$\times \Bigg[\left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{n-3} |w_{0,n}|_{1,\Sigma_j} + \left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{n-3} \frac{1}{\mathfrak{l}_j}|w_{0,n}|_{3,\Sigma_j}$$

$$+ \left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{n-4} \frac{1}{\mathfrak{l}_j^2}|w_{0,n}|_{5,\Sigma_j}\Bigg]$$

$$\le \mathit{const}^n\ \mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{c}_j\left(\frac{\alpha}{2}\right)^n \left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2} \frac{1}{M^{(n-4)/2}}\left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{(n-4)/2} \rho'_{0;n}$$

$$\times \left[|w_{0,n}|_{1,\Sigma_j} + \frac{1}{\mathfrak{l}_j}|w_{0,n}|_{3,\Sigma_j} + \frac{1}{\mathfrak{l}_j^2}|w_{0,n}|_{5,\Sigma_j}\right]$$

$$
\begin{aligned}
&= \mathit{const}^n\ \mathfrak{e}_{j+1}(X)\left(\frac{1}{M^{1-\aleph}}\right)^{(n-4)/2}\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{c}_j\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}|w_{0,n}|_{\Sigma_j,\vec\rho\,'}\\
&\le \left(\frac{1}{M^{(1-\aleph)/8}}+\mathit{const}\ \delta_{n,4}\right)\mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{e}_j(X)\\
&\quad\times\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}|w_{0,n}|_{\Sigma_j,\vec\rho\,'}\,. \qquad\square
\end{aligned}
$$

The analog of Corollary XIX.8 for the $N_j^{\sim}$ norms is

Corollary XIX.9. *Fix $\frac{1}{2}<\aleph<\frac{2}{3}$ and let $j\ge\frac{3}{2-3\aleph}$. Let Σ_{j+1} and Σ_j be sectorizations of length $\mathfrak{l}_{j+1}=\frac{1}{M^{\aleph(j+1)}}$ at scale $j+1$ and $\mathfrak{l}_j=\frac{1}{M^{\aleph j}}$ at scale j, respectively. Let $\vec\rho=(\rho_{m;n})$ be a system of positive real numbers obeying* (XVII.1). *Let*

$$
w(\phi,\psi)=\sum_n\int_{\mathfrak{X}_\Sigma^n}dx_1\cdots dx_n\,f_n(x_1,\ldots,x_n)\Psi(x_1)\cdots\Psi(x_n)
$$

with $f_n\in\check{\mathcal{F}}_{n;\Sigma}$ antisymmetric, be an even Σ_j-sectorized particle number conserving Grassmann function with $f_2=0$. If M is big enough, then

$$
N_{j+1}^{\sim}(w_{\Sigma_{j+1}};64\alpha;X,\Sigma_{j+1},\vec\rho)\le \mathit{const}\ \mathfrak{e}_{j+1}(X)N_j^{\sim}\left(w;\frac{\alpha}{2};X,\Sigma_j,\vec\rho\right)
$$

with the constant const independent of M, j, Σ_j and Σ_{j+1}. If, in addition $f_4=0$, then

$$
N_{j+1}^{\sim}(w_{\Sigma_{j+1}};\ 64\alpha;\ X,\Sigma_{j+1},\vec\rho)\le\frac{1}{M^{(1-\aleph)/8}}\,\mathfrak{e}_{j+1}(X)N_j^{\sim}\left(w;\frac{\alpha}{2};X,\Sigma_j,\vec\rho\right).
$$

Proof. If $n\ge4$, by Proposition XIX.4(ii) with j replaced by $j+1$, $i=j$, $\mathfrak{l}=\mathfrak{l}_{j+1}$ and $\mathfrak{l}'=\mathfrak{l}_j$,

$$
\begin{aligned}
&\frac{M^{2(j+1)}}{\mathfrak{l}_{j+1}}\,\mathfrak{e}_{j+1}(X)(64\alpha)^n\left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2}|(f_n)_{\Sigma_{j+1}}|^{\sim}_{\Sigma_{j+1}}\\
&\quad\le\frac{M^{2(j+1)}}{\mathfrak{l}_{j+1}}\,\mathfrak{e}_{j+1}(X)(64\alpha)^n\left(\frac{\mathfrak{l}_{j+1}\,\mathrm{B}}{M^{j+1}}\right)^{n/2}\\
&\qquad\times\left\{|(f_n)_{\Sigma_{j+1}}|^{\sim}_{1,\Sigma_{j+1},\vec\rho}+\sum_{p=2}^{6}\frac{1}{\mathfrak{l}_{j+1}^{[(p-1)/2]}}|(f_n)_{\Sigma_{j+1}}|^{\sim}_{p,\Sigma_{j+1},\vec\rho}\right\}\\
&\quad\le\mathit{const}^n\ \mathfrak{c}_j\ \mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}\frac{1}{M^{(n-4)/2}}\left(\frac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}\right)^{(n-2)/2}\\
&\qquad\times\left\{\left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{n-3}\left(|f_n|^{\sim}_{1,\Sigma_j,\vec\rho}+\frac{1}{\mathfrak{l}_j}|f_n|^{\sim}_{3,\Sigma_j,\vec\rho}\right)\right.\\
&\qquad\left.+\sum_{p=2}^{6}\frac{1}{\mathfrak{l}_{j+1}^{[(p-1)/2]}}\left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{n-p-1}|f_n|^{\sim}_{p,\Sigma_j,\vec\rho}\right\}
\end{aligned}
$$

$$\le const^n \ \mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{c}_j\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}\frac{1}{M^{(n-4)/2}}\left(\frac{\mathfrak{l}_j}{\mathfrak{l}_{j+1}}\right)^{(n-4)/2}$$

$$\times\left\{|f_n|^{\sim}_{1,\Sigma_j,\vec\rho}+\frac{1}{\mathfrak{l}_j}|f_n|^{\sim}_{3,\Sigma_j,\vec\rho}+\sum_{p=2}^{6}\left(\frac{\mathfrak{l}_{j+1}}{\mathfrak{l}_j}\right)^{p-2-[(p-1)/2]}\frac{1}{\mathfrak{l}_j^{[(p-1)/2]}}|f_n|^{\sim}_{p,\Sigma_j,\vec\rho}\right\}$$

$$\le const^n \ \mathfrak{e}_{j+1}(X)\left(\frac{1}{M^{1-\aleph}}\right)^{(n-4)/2}\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{e}_j(X)\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}|f_n|^{\sim}_{\Sigma_j}$$

$$\le\left(\frac{1}{M^{(1-\aleph)/8}}+const\ \delta_{n,4}\right)\mathfrak{e}_{j+1}(X)\frac{M^{2j}}{\mathfrak{l}_j}\,\mathfrak{e}_j(X)\left(\frac{\alpha}{2}\right)^n\left(\frac{\mathfrak{l}_j\,\mathrm{B}}{M^j}\right)^{n/2}|f_n|^{\sim}_{\Sigma_j}\,. \qquad \square$$

The positive power counting of $|w_{0,2}|_{1,\Sigma}$ is achieved by renormalization. That is, we choose the counterterm in such a way that, at each scale, the restriction of the Fourier transform of $w_{0,2}$ to the Fermi surface is small. The following proposition ensures then that $|w_{0,2}|_{1,\Sigma}$ is also small.

Definition XIX.10. The function $u\in\mathcal{F}_0(2;\Sigma)$ is said to vanish at $k_0=0$ if

$$\check u(((0,\mathbf{k}),\sigma,a,s),((0,\mathbf{k}),\sigma',a',s'))=0$$

for all $a,a'\in\{0,1\}$, $\sigma,\sigma'\in\{\uparrow,\downarrow\}$ and $s,s'\in\Sigma$.

Proposition XIX.11. *There is a constant* $const$, *independent of* M, *such that the following holds*: *let* $j\ge i\ge 2$ *and* Σ *and* Σ' *be sectorizations at scale* j *and* i, *respectively. If* $i=j$ *assume that* $\Sigma=\Sigma'$. *Let* $u\in\mathcal{F}_0(2;\Sigma')$ *be a function that vanishes at* $k_0=0$. *Then*

(i)

$$|u_\Sigma|_{1,\Sigma}\le const\ \frac{1}{M^{j-1}}\,|\mathfrak{c}_{j-1}\mathcal{D}^{(1,0,0)}_{1,2}u|_{1,\Sigma'}+\sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\ne0}}\infty t^\delta\,.$$

(ii)

$$|\mathcal{D}^{(1,0,0)}_{1,2}u_\Sigma|_{1,\Sigma}\le const\ \mathfrak{c}_{j-1}|\mathcal{D}^{(1,0,0)}_{1,2}u|_{1,\Sigma'}\,.$$

Proof. (i) Fix $s_1,s_2\in\Sigma$. If $i<j$, by Lemmas II.7, IX.6(i), XIII.3 and (XIII.4)

$$\|u_\Sigma((\xi_1,s_1),(\xi_2,s_2))\|_{1,\infty}$$

$$\le const\max_{s_1',s_2'\in\Sigma'}\left\|\int d\eta_1d\eta_2u((\eta_1,s_1'),(\eta_2,s_2'))\hat\chi_{s_1}(\eta_1,\xi_1)\hat\chi_{s_2}(\eta_2,\xi_2)\right\|_{1,\infty}$$

$$\le const\|\hat\chi_{s_2}\|_{1,\infty}\max_{s_1',s_2'\in\Sigma'}\left\|\int d\eta_1u((\eta_1,s_1'),(\cdot\,,s_2'))\hat\chi_{s_1}(\eta_1,\cdot)\right\|_{1,\infty}$$

$$\le const\ \mathfrak{c}_{j-1}\left\|\frac{\partial\chi'_{s_1}}{\partial x_0}\right\|_{L^1}\max_{s_1',s_2'\in\Sigma'}\|\mathcal{D}^{(1,0,0)}_{1,2}u((\cdot\,,s_1'),(\cdot\,,s_2'))\|_{1,\infty}+\sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\ne0}}\infty t^\delta$$

$$\leq const\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}^2|\mathcal{D}_{1,2}^{(1,0,0)}u|_{1,\Sigma'} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\neq 0}} \infty t^\delta$$

$$\leq const\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}|\mathcal{D}_{1,2}^{(1,0,0)}u|_{1,\Sigma'} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\neq 0}} \infty t^\delta\,.$$

Similarly, if $i=j$ and $\Sigma=\Sigma'$, then setting

$$\chi_s^{(e)}(k) = \varphi\left(\frac{1}{2}M^{2j-2}(k_0^2+e(\mathbf{k})^2)\right)\Theta_s(\mathbf{k}'(k))$$

(which just differs by a $\frac{1}{2}$ from the definition of $\chi_s(k)$ in (XIII.2)), we have, using the support property of Definition XII.4(ii),

$$\begin{aligned}
&\|u_\Sigma((\xi_1,s_1),(\xi_2,s_2))\|_{1,\infty}\\
&\quad = \|u((\xi_1,s_1),(\xi_2,s_2))\|_{1,\infty}\\
&\quad = \left\|\sum_{s_1',s_2'\in\Sigma}\int d\eta_1 d\eta_2 u((\eta_1,s_1),(\eta_2,s_2))\hat{\chi}_{s_1'}^{(e)}(\eta_1,\xi_1)\hat{\chi}_{s_2'}^{(e)}(\eta_2,\xi_2)\right\|_{1,\infty}\\
&\quad \leq const\ \max_{s_1',s_2'\in\Sigma}\left\|\int d\eta_1 d\eta_2 u((\eta_1,s_1),(\eta_2,s_2))\hat{\chi}_{s_1'}^{(e)}(\eta_1,\xi_1)\hat{\chi}_{s_2'}^{(e)}(\eta_2,\xi_2)\right\|_{1,\infty}\\
&\quad \leq const\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}|\mathcal{D}_{1,2}^{(1,0,0)}u|_{1,\Sigma} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\neq 0}} \infty t^\delta\,.
\end{aligned}$$

Since for every $s_1\in\Sigma$ there are at most three sectors s_2 with $\tilde{s}_1\cap\tilde{s}_2\neq\emptyset$, in both cases

$$|u_\Sigma|_{1,\Sigma} \leq const\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}|\mathcal{D}_{1,2}^{(1,0,0)}u|_{1,\Sigma'} + \sum_{\substack{\delta\in|\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0\neq 0}} \infty|t^\delta\,.$$

(ii) If $i=j$ and $\Sigma=\Sigma'$ the statement is trivial. So assume that $i<j$. Fix $s_1,s_2\in\Sigma$. By Lemma IX.6(ii) (twice), Lemma XIII.3 and (XIII.4)

$$\begin{aligned}
&\|\mathcal{D}_{1,2}^{(1,0,0)}u_\Sigma((\xi_1,s_1),(\xi_2,s_2))\|_{1,\infty}\\
&\quad \leq const\ \max_{s_1',s_2'\in\Sigma'}\left\|\mathcal{D}_{1,2}^{(1,0,0)}\int d\eta_1 d\eta_2 u((\eta_1,s_1'),(\eta_2,s_2'))\hat{\chi}_{s_1}(\eta_1,\xi_1)\hat{\chi}_{s_2}(\eta_2,\xi_2)\right\|_{1,\infty}\\
&\quad \leq const\left(\|\hat{\chi}_{s_2}\|_{1,\infty} + \left\|x_0\frac{\partial\chi_{s_2}'}{\partial x_0}(x)\right\|_{L^1}\right)\\
&\qquad \times \max_{s_1',s_2'\in\Sigma'}\left\|\mathcal{D}_{1,2}^{(1,0,0)}\int d\eta_1 u((\eta_1,s_1'),(\,\cdot\,,s_2'))\hat{\chi}_{s_1}(\eta_1,\cdot)\right\|_{1,\infty} + \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0> r_0}} \infty t^\delta
\end{aligned}$$

$$\le \text{const}\ \mathfrak{c}_{j-1}\left(\|\hat{\chi}_{s_1}\|_{1,\infty} + \left\| x_0 \frac{\partial \chi'_{s_1}}{\partial x_0}(x)\right\|_{L^1}\right)$$

$$\times \max_{s'_1,s'_2\in\Sigma'} \|\mathcal{D}^{(1,0,0)}_{1,2} u((\cdot\,,s'_1),(\cdot\,,s'_2))\|_{1,\infty}$$

$$\le \text{const}\ \mathfrak{c}_{j-1}|\mathcal{D}^{(1,0,0)}_{1,2} u|_{1,\Sigma'}\,. \qquad \square$$

Corollary XIX.12. *There is a constant* const, *independent of* M, *such that the following holds*: *let* Σ *be a sectorization of scale* $j \ge 2$ *and* $u \in \mathcal{F}_0(2;\Sigma)$ *be a function that vanishes at* $k_0 = 0$. *Let* $X \in \mathfrak{N}_{d+1}$. *If*

$$|\mathcal{D}^{(1,0,0)}_{1,2} u|_{1,\Sigma} \le \mathfrak{c}_{j-1} X + \sum_{\delta_0 = r_0} \infty t^\delta$$

then

$$|u|_{1,\Sigma} \le \text{const}\ \frac{M}{M^j}\,\mathfrak{c}_{j-1} X\,.$$

Proof. By Proposition XIX.11(i) and (XIII.4)

$$|u|_{1,\Sigma} \le \text{const}\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}|\mathcal{D}^{(1,0,0)}_{1,2} u|_{1,\Sigma} + \sum_{\delta_0\ne 0} \infty t^\delta$$

$$\le \text{const}\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1} X + \sum_{\delta_0\ne 0} \infty t^\delta\,.$$

Also

$$|u|_{1,\Sigma} \le t_0 \frac{\partial}{\partial t_0} |u|_{1,\Sigma} + \sum_{\delta_0=0} \infty t^\delta$$

$$= t_0 |\mathcal{D}^{(1,0,0)}_{1,2} u|_{1,\Sigma} + \sum_{\delta_0=0} \infty t^\delta$$

$$\le t_0\,\mathfrak{c}_{j-1} X + \sum_{\delta_0=r_0+1} \infty t^\delta + \sum_{\delta_0=0} \infty t^\delta$$

$$\le \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1} X + \sum_{\delta_0=0} \infty t^\delta$$

since $t_0\,\mathfrak{c}_{j-1} \le \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1}$. The corollary now follows by taking the minimum of the two estimates on $|u|_{1,\Sigma}$. $\square$

Corollary XIX.13. *Let* $j > i \ge 2$ *and* Σ *and* Σ' *be sectorizations at scale* j *and* i, *respectively. Let* $u \in \mathcal{F}_0(2;\Sigma')$ *vanish at* $k_0 = 0$. *Assume that* $|u|_{1,\Sigma'} \le \lambda\,\mathfrak{c}_i$ *for some* $\lambda > 0$. *Then*

$$|u_\Sigma|_{1,\Sigma} \le \text{const}\ \lambda M \frac{M^i}{M^j}\,\mathfrak{c}_{j-1}\,.$$

Proof. By hypothesis

$$|\mathcal{D}_{1,2}^{(1,0,0)} u|_{1,\Sigma'} \le {\scriptstyle const}\ \lambda M^i\, \mathfrak{c}_i + \sum_{\delta_0 = r_0} \infty t^\delta\,.$$

Therefore, by Proposition XIX.11(i) and (XIII.4)

$$\begin{aligned} |u_\Sigma|_{1,\Sigma} &\le {\scriptstyle const}\ \frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1} |\mathcal{D}_{1,2}^{(1,0,0)} u|_{1,\Sigma'} + \sum_{\delta_0 \ne 0} \infty t^\delta \\ &\le {\scriptstyle const}\ \lambda \frac{M^i}{M^{j-1}}\,\mathfrak{c}_{j-1} + \sum_{\delta_0 \ne 0} \infty t^\delta\,. \end{aligned}$$

Also, by Proposition XIX.11(ii)

$$\begin{aligned} |u_\Sigma|_{1,\Sigma} &\le t_0 |\mathcal{D}_{1,2}^{(1,0,0)} u_\Sigma|_{1,\Sigma} + \sum_{\delta_0 = 0} \infty t^\delta \\ &\le {\scriptstyle const} (t_0\, \mathfrak{c}_{j-1}) |\mathcal{D}_{1,2}^{(1,0,0)} u|_{1,\Sigma'} + \sum_{\delta_0 = 0} \infty t^\delta \\ &\le {\scriptstyle const} \left(\frac{1}{M^{j-1}}\,\mathfrak{c}_{j-1} \right) \lambda M^i\, \mathfrak{c}_i + \sum_{\delta_0 = r_0 + 1} \infty t^\delta + \sum_{\delta_0 = 0} \infty t^\delta \\ &\le {\scriptstyle const}\ \lambda \frac{M^i}{M^{j-1}}\,\mathfrak{c}_{j-1} + \sum_{\delta_0 = 0} \infty t^\delta\,. \end{aligned}$$

Again, the corollary follows by taking the minimum of the two estimates on $|u_\Sigma|_{1,\Sigma}$. □

When we start the multi scale analysis in [2], the effective potential after integrating out the first scales does not have a natural sectorized representative (see also Theorem VIII.6). Therefore we need analogs of Definition XIX.2 and Proposition XIX.4 that pass from unsectorized functions to sectorized functions (see also Example XII.5).

Definition XIX.14. Let Σ be a sectorization of scale $j \ge 2$. For a function f on $\mathcal{B}^m \times \mathcal{B}^n$ define the function f_Σ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma)^n$ by

$$\begin{aligned} &f_\Sigma(\eta_1, \ldots, \eta_m; (\xi_1, s_1), \ldots, (\xi_n, s_n)) \\ &\quad = \int \prod_{i=1}^n (d\xi_i' \hat{\chi}_{s_i}(\xi_i', \xi_i)) f(\eta_1, \ldots, \eta_m; \xi_1', \ldots, \xi_n') \end{aligned}$$

where χ_s is the partition of unity of Lemma XII.3.

Proposition XIX.15. *Let Σ be a sectorization of scale $j \ge 0$ and $f \in \mathcal{F}_m(n)$, $f' \in \check{\mathcal{F}}_m(n)$ particle number conserving functions that are antisymmetric in their ξ-variables.*

(i) *If $m = 0$ and f is translation invariant, then for all $p < n$*

$$|f_\Sigma|_{p,\Sigma} \le const^n \frac{1}{\mathfrak{l}^{n-p-1}}\, \mathfrak{c}_{j-1} \|f\|_{1,\infty}\,.$$

(ii) *If $m \ne 0$*

$$|f_\Sigma|_{1,\Sigma} \le \left[\frac{const}{\mathfrak{l}}\right]^n \mathfrak{c}_{j-1} \|f\|_{1,\infty}\,.$$

(iii) *If $0 < m \le p \le m+n$*

$$|f'_\Sigma|^{\sim}_{p,\Sigma} \le \left[\frac{const}{\mathfrak{l}}\right]^{m+n-p} \mathfrak{c}_{j-1}\, \|f\|^{\sim}\,.$$

The proof of part (i) of this proposition is analogous to that of Proposition XIX.4, and part (ii) was already proven in Example XII.10. The proof of part (iii) is similar to that of part (ii).

XX. Sums of Momenta and ϵ-Separated Sets

In the next section we shall exploit conservation of momentum to prove Proposition XIX.1, relating the 1- and 3-norms of a four-legged kernel, and Proposition XIX.4, concerning the behavior of norms under change of sectorization. Conservation of momentum is equivalent to translation invariance in position space. Recall that we assume that the Fermi surface is strictly convex and either symmetric about a point or strictly asymmetric in the sense of Definition XVIII.3.

The following definition is motivated by of [6, Definition B.1.N], of conservation of particle number, and Definition XVI.7(i), of the spaces $\check{\mathcal{F}}_m(n;\Sigma)$.

Definition XX.1. A configuration $(\check\eta_1,\ldots,\check\eta_m; s_1,\ldots,s_n) \in \check{\mathcal{B}}^m \times \Sigma^n$, where Σ is a sectorization of some scale j, is consistent with conservation of momentum for the sequence $(a_1,\ldots,a_n)$ of creation–annihilation indices if there are $k_1,\ldots,k_n \in \mathbb{R}\times\mathbb{R}^2$, with k_i in the extended sector $\tilde{s}_i$ for each $i = 1,\ldots,n$, such that

$$\sum_{i=1}^{m}(-1)^{b_i}p_i + \sum_{i=1}^{n}(-1)^{a_i}k_i = 0$$

where $\check\eta_i = (p_i,\sigma_i,b_i)$.

We say that the configuration $(\check\eta_1,\ldots,\check\eta_m; s_1,\ldots,s_n)$ is consistent with conservation of momentum if it is consistent with conservation of momentum for some sequence $(a_1,\ldots,a_n) \in \{0,1\}^n$ of creation–annihilation indices such that

$$\#\{i|a_i = 0\} + \#\{\ell|b_\ell = 0\} = \#\{i|a_i = 1\} + \#\{\ell|b_\ell = 1\}\,.$$

Remark XX.2. Let Σ be a sectorization of scale j.

(i) If $f \in \check{\mathcal{F}}_m(n;\Sigma)$ preserves particle number then

$$f(\check\eta_1,\ldots,\check\eta_m; (\,\cdot\,,s_1),\ldots,(\,\cdot\,,s_n)) = 0$$

unless the configuration $(\check{\eta}_1, \ldots, \check{\eta}_m; s_1, \ldots, s_n)$ is consistent with conservation of momentum.

(ii) If a configuration $(\check{\eta}_1, \ldots, \check{\eta}_m; s_1, \ldots, s_n)$ is consistent with conservation of momentum for the sequence $(a_1, \ldots, a_n)$ of creation–annihilation indices then there are $\mathbf{k}_1, \ldots, \mathbf{k}_n \in \mathbb{R}^2$ such that

$$\sum_{i=1}^{m}(-1)^{b_i}\mathbf{p}_i + \sum_{i=1}^{n}(-1)^{a_i}\mathbf{k}_i = \mathbf{0}$$

and

$$\pi_F((0, \mathbf{k}_i)) \in s\,, \qquad |e(\mathbf{k}_i)| \le \frac{\sqrt{2}}{M^{j-1}}$$

for $i = 1, \ldots, n$.

The comparison of the 1- and 3-norms of a four-legged kernel (Proposition XIX.1) uses an estimate on the maximal number of triples (s_2, s_3, s_4) of sectors that complete a given sector s_1 to a quadruple (s_1, s_2, s_3, s_4) that is consistent with conservation of momentum (Proposition XX.10). Similarly, the estimate on the behavior of norms under change of sectorization (Proposition XIX.4) is based on estimates of the number of $(2n)$-tuples $(s_1, \ldots, s_{2n})$ of sectors that are consistent with conservation of momentum and such that each s_i intersects a given bigger sector from another sectorization (Proposition XX.11).

We reduce these counting problems to problems of estimating volumes of sets in momentum space that are characterized by the geometric constraints that the sectors are required to satisfy. To pass from volume estimates to sector counting we use the concept of ϵ-separated sets (see also [7, p. 22]).

ϵ-separated sets

Let M be a Riemannian manifold of dimension n. For any two points $x, y \in M$ we denote by $d(x, y)$ the distance between x and y in M. For $x \in M$ and $r > 0$ let

$$B_r(x) = \{y \in M | d(x, y) < r\}$$

be the open ball of radius r around x. We set, for $\epsilon > 0$,

$$V_{M,\epsilon} = \inf_{x \in M,\, 0 < r \le \epsilon} \frac{1}{r^n} \text{ vol } B_{r/2}(x)$$

where vol denotes the volume with respect to the Riemannian metric on M. For a subset $A \subset M$ and $\delta > 0$ we call

$$A_\delta = \left\{ x \in M \,\middle|\, \inf_{y \in A} d(x, y) \le \delta \right\}$$

the (closed) δ-neighborhood of A. If X is a tangent vector to M at the point x we denote by $\|X\|$ the length of X with respect to the Riemannian metric on M.

If f is a differentiable map from M to another Riemannian manifold N we denote by $Df(x)$ the derivative of f at the point $x \in M$. The point x is said to be a critical point of f if $Df(x)$ has rank strictly less than the dimension of N.

Definition XX.3. Let $\epsilon > 0$. A subset Γ of M is called *ϵ-separated* if for any two different $\gamma,\ \gamma' \in \Gamma$

$$d(\gamma,\gamma') \geq \epsilon\,.$$

Example. Let Σ be a sectorization of length $\mathfrak{l}$ and let Γ be the set of centers of the intervals $s \cap F,\ s \in \Sigma$. If $\epsilon < 7\,\mathfrak{l}/8$, then Γ is an ϵ-separated subset of the Fermi curve F and more generally Γ^n is an ϵ-separated subset of F^n.

We wish to count, for example, for a given sector s_4, the number of triples of sectors (s_1, s_2, s_3) such that there exist $\mathbf{k}_i \in s_i$ obeying $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0$. If (s_1, s_2, s_3) are such sectors, then the map

$$\begin{aligned} f\colon F \times F \times F &\longrightarrow \mathbb{R}^2 \\ (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &\longmapsto \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 \end{aligned}$$

maps $F^3 \cap (s_1 \times s_2 \times s_3)$ to a neighborhood of s_4. We start with an abstract lemma counting the number of points of an ϵ-separated set Γ in the preimage of a specified set A under a specified map f.

Lemma XX.4. *Let M be a Riemannian manifold of dimension n and $f\colon M \mapsto \mathbb{R}^d$ a differentiable map. For $x \in M$ denote by $Df(x)$ the derivative of f at the point x. Let $\vec{n}_1, \ldots, \vec{n}_d$ be an orthonormal basis of $\mathbb{R}^d$. Set, for $i = 1, \ldots, d$*

$$C_i = \sup_{x \in M} \sup\{|\vec{n}_i \cdot Df(x)v| \;|v \text{ is a unit tangent vector to } M \text{ at } x\}\,.$$

Furthermore, for any subset A of $\mathbb{R}^d$ and any $\epsilon > 0$ set

$$A'(\epsilon) = \left\{ y \in \mathbb{R}^d | \exists (t_1, \ldots, t_d) \in (-\epsilon, \epsilon)^d \text{ such that } y + \sum_{i=1}^{d} t_i C_i \vec{n}_i \in A \right\}.$$

Then for all $A \subset \mathbb{R}^d$, $\epsilon_0 > 0$, $0 < \epsilon < \epsilon_0$ and all ϵ-separated subsets Γ of M

$$\#(f^{-1}(A) \cap \Gamma) \leq \frac{1}{\epsilon^n V_{M,\epsilon_0}} \operatorname{vol}(f^{-1}(A'(\epsilon)))\,.$$

Proof. If $\gamma \in f^{-1}(A) \cap \Gamma$ and $x \in M$ with $d(x,\gamma) < \epsilon/2$, then, by the assumption on the derivative of f for $i = 1, \ldots, d$

$$|\vec{n}_i \cdot (f(x) - f(\gamma))| < C_i \epsilon$$

so that $f(x) \in A'(\epsilon)$. Obviously the sets $B_{\epsilon/2}(\gamma),\ \gamma \in f^{-1}(A) \cap \Gamma$ are pairwise disjoint. Consequently, by the definition of V_{M,ϵ_0}

$$\begin{aligned} V_{M,\epsilon_0}\, \epsilon^n\, \#(f^{-1}(A) \cap \Gamma) &\leq \sum_{\gamma \in f^{-1}(A) \cap \Gamma} \operatorname{vol}(B_{\epsilon/2}(\gamma)) \\ &= \operatorname{vol}\left(\bigcup_{\gamma \in f^{-1}(A) \cap \Gamma} B_{\epsilon/2}(\gamma) \right) \\ &\leq \operatorname{vol}(f^{-1}(A'(\epsilon)))\,. \end{aligned}$$

□

Sums of momenta

For the proofs of Propositions XX.10 and XX.11 we shall apply the discussion of the previous subsection with Γ being the set of centers of sectors of a given sectorization. The proofs of these propositions then lead to the problem of estimating the number of points $(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{k}_{n+1}, \dots, \mathbf{k}_{2n-1}) \in \Gamma^{2n-1}$ such that $\mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}_{n+1} - \cdots - \mathbf{k}_{2n-1}$ is close to $\mathbf{q}$. Thus we are led to studying the maps

$$F^{2n-1} \longrightarrow \mathbb{R}^2$$

$$(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{k}_{n+1}, \dots, \mathbf{k}_{2n-1}) \longmapsto \mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}_{n+1} - \cdots - \mathbf{k}_{2n-1}$$

and the intersection of preimages of sets in $\mathbb{R}^2$ with Γ^{2n-1}. Outside a neighborhood of the set of critical points of this map this can usually be done using Lemma XX.4. The critical points of the map are exactly those points $(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \in F^{2n-1}$ for which the tangent lines of F at $\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}$ are all parallel. This is the case if and only all the points $\mathbf{k}_2, \dots, \mathbf{k}_{2n-1}$ coincide either with $\mathbf{k}_1$ or its antipode $a(\mathbf{k}_1)$.

For $\mathbf{k} \in F$ and $0 < \Lambda \leq \mathfrak{l}$, we call

$$s_{\Lambda,\mathfrak{l}}(\mathbf{k}) = \{\mathbf{q} \in \mathbb{R}^2 |\ |e(\mathbf{q})| \leq \Lambda,\ d_F(\mathbf{q}', \mathbf{k}) \leq \mathfrak{l}/2\}$$

the two-dimensional sector of length l and width Λ around $\mathbf{k}$. Here $\mathbf{q} \mapsto \mathbf{q}'$ is the projection to the Fermi curve introduced in Sec. XI and used in [8, Definition XII.1(i)] and d_F is the intrinsic metric on F.

Near critical points of the map discussed above we shall use

Proposition XX.5. *Let $\mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_{2n-3} \in F$ and $\omega > 0$ be such that for $i = 1, \dots, 2n-3$ one has $\|\mathbf{k}_i - \mathbf{k}\| < \omega$ or $\|\mathbf{k}_i - a(\mathbf{k})\| < \omega$. Let $\epsilon_1, \dots, \epsilon_{2n-3} \in \{\pm 1\}$ and set*

$$\mathbf{q} = \epsilon_1 \mathbf{k}_1 + \cdots + \epsilon_{2n-3} \mathbf{k}_{2n-3} \,.$$

Furthermore, let $0 < \Lambda \leq \mathfrak{l} \leq \omega$ and let $\vec{\mathrm{n}}$ respectively $\vec{\mathrm{t}}$ be unit normal respectively tangent vectors of F at $\mathbf{k}$. Then

$$\{\epsilon_1 x_1 + \cdots + \epsilon_{2n-3} x_{2n-3} | x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)\}$$

is contained in the rectangle

$$\{\mathbf{q} + t_1 \vec{\mathrm{n}} + t_2 \vec{\mathrm{t}} |\ |t_1| \leq n \ {\scriptstyle const}(\Lambda + \mathfrak{l}\omega),\ |t_2| \leq 4n\mathfrak{l}\} \,.$$

The constant ${\scriptstyle const}$ depends only on the geometry of F.

Proof. Without loss of generality we may assume that $\vec{\mathrm{n}} = (0, 1)$ and $\vec{\mathrm{t}} = (1, 0)$. The angle between F and the k_1 direction at a point $\mathbf{q} \in F$ is bounded by ${\scriptstyle const}\ \|\mathbf{q} - \mathbf{k}\|$ and ${\scriptstyle const}\|\mathbf{q} - a(\mathbf{k})\|$. Therefore $s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)$ is contained in a rectangle that is centered at $\mathbf{k}_i$ and has two edges parallel to the k_1 axis of length $2\mathfrak{l}$ and two edges parallel to the k_2 axis of length ${\scriptstyle const}(\Lambda + \mathfrak{l}\ \min\{\|\mathbf{k}_i - \mathbf{k}\|, \|\mathbf{k}_i - a(\mathbf{k})\|\}) \leq {\scriptstyle const}(\Lambda + \mathfrak{l}\omega)$. The claim now follows. $\square$

Proposition XX.6. *Let $\mathbf{k}, \mathbf{k}_1, \ldots, \mathbf{k}_{2n-1} \in F$, $I \subset \{1, \ldots, n\}$, $J \subset \{n+1, \ldots, 2n-1\}$ and ω a positive real number smaller than the diameter of F such that $\|\mathbf{k}_i - \mathbf{k}\| < \omega$ for $i \in I \cup J$ and $\|\mathbf{k}_j - a(\mathbf{k})\| < \omega$ for $j \notin I \cup J$. Furthermore, let $\mathbf{p} \in \mathbb{R}^2$ and $0 < \Lambda \leq \mathfrak{l} \leq \omega$. Assume that there are points $x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)$, $i = 1, \ldots, 2n-1$ such that*

$$\|x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} - \mathbf{p}\| \leq 2\mathfrak{l}\,.$$

Then

(i)

$$\|\mathbf{p} - (\#I - \#J)\mathbf{k} + (\#I - \#J - 1)a(\mathbf{k})\| \leq \mathit{const}\; n\omega\,.$$

(ii) *If $\mathbf{p} \in F$ then*

$$\|\mathbf{p} - \mathbf{k}\| \leq \mathit{const}\; n\omega \quad \text{or} \quad \|\mathbf{p} - a(\mathbf{k})\| \leq \mathit{const}\; n\omega\,.$$

The constants const depend only on the geometry of F.

Proof. By Proposition XX.5,

$$\begin{aligned} &\|x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} - (\mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}_{n+1} - \cdots - \mathbf{k}_{2n-1})\| \\ &\quad \leq (n+1)\; \mathit{const}(\Lambda + \mathfrak{l}\omega) + 4(n+1)\mathfrak{l} \leq \mathit{const}\; n\omega\,. \end{aligned}$$

Since

$$\begin{aligned} &\|\mathbf{k}_1 + \cdots + \mathbf{k}_n - \mathbf{k}_{n+1} - \cdots - \mathbf{k}_{2n-1} - (\#I)\mathbf{k} - (n - \#I)a(\mathbf{k}) \\ &\quad + (\#J)\mathbf{k} + (n - 1 - \#J)a(\mathbf{k})\| \leq \mathit{const}\; n\omega \end{aligned}$$

part (i) follows. To prove part (ii) assume that $\mathbf{p} \in F$. Set $r = \#I, s = \#J$. By possibly interchanging $\mathbf{k}$ with its antipode, we may assume that $r \geq s$. If $r = s$ or $r = s+1$, it follows directly from part (i) that $\|\mathbf{p} - \mathbf{k}\| \leq \mathit{const}\; n\omega$. So we assume that $r - s \geq 2$.

Let $x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)$, $i = 1, \ldots, 2n-1$ such that

$$y = x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1}$$

has distance at most $2\mathfrak{l}$ from $\mathbf{p}$. Let $\vec{n}$ be the outward pointing unit normal vector of F at $\mathbf{k}$. Then $(\mathbf{k} - a(\mathbf{k})) \cdot \vec{n} \geq \mathit{const}_1$ and

$$|(x_i - \mathbf{k}) \cdot \vec{n}| \leq \|x_i - \mathbf{k}_i\| + \|\mathbf{k}_i - \mathbf{k}\| \leq \mathit{const}(\Lambda + \mathfrak{l}) + \omega \leq \mathit{const}\; \omega$$

for $i \in I \cup J$. Similarly for $j \notin I \cup J$

$$|(x_j - a(\mathbf{k})) \cdot \vec{n}| \leq \mathit{const}\; \omega\,.$$

Consequently

$$\begin{aligned} &|(x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} - \mathbf{k}) \cdot \vec{n} \\ &\quad - (r - s - 1)(\mathbf{k} - a(\mathbf{k})) \cdot \vec{n}| \leq (2n-1)\; \mathit{const}\; \omega \end{aligned}$$

and therefore

$$(y - \mathbf{k}) \cdot \vec{n} \geq A((r - s - 1) - \mathit{const}\ n\omega)$$

with strictly positive constants A, const.

The tangent line to F at $\mathbf{k}$ is a "supporting hyperplane" for the convex hull of F. Therefore

$$F \cap \{x \in \mathbb{R}^2 | (x - \mathbf{k}) \cdot \vec{n} > 0\} = \emptyset\,.$$

So

$$\begin{aligned} 0 \geq (\mathbf{p} - \mathbf{k}) \cdot \vec{n} &= (\mathbf{p} - y) \cdot \vec{n} + (y - \mathbf{k}) \cdot \vec{n} \\ &\geq (\mathbf{p} - y) \cdot \vec{n} + A((r - s - 1) - n\ \mathit{const}\ \omega)\,. \end{aligned}$$

As $|(\mathbf{p} - y) \cdot \vec{n}| \leq 2\mathfrak{l}$ and hence $(\mathbf{p} - y) \cdot \vec{n} \geq -2\mathfrak{l} \geq -2\omega \geq -n\ \mathit{const}\ \omega$, this shows that

$$n \geq \mathit{const}' \frac{r - s - 1}{\omega} \geq \mathit{const}' \frac{1}{\omega}$$

since $r - s \geq 2$. Thus $n\omega$ is larger than some strictly positive constant and the estimate of part (ii) holds. □

Pairs of momenta

Lemma XX.7. *Let $\epsilon_1,\ \epsilon_2 \in \{\pm 1\}$. There is a subset X of $F \times F$ and a constant C such that*

(i) *For every $\mathbf{p} \in \mathbb{R}^2$*

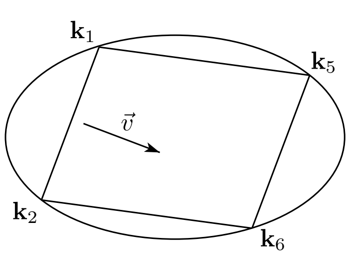
$$\#\{(\mathbf{k}_1, \mathbf{k}_2) \in X | \epsilon_1 \mathbf{k}_1 + \epsilon_2 \mathbf{k}_2 = \mathbf{p}\} \leq C\,.$$

(ii) *$(F \times F) \setminus X$ has measure zero.*

Proof. We may assume that $\epsilon_1 = +1$. If $\epsilon_2 = -1$, then we claim that, for every $\mathbf{p} \neq 0$

$$\#\{(\mathbf{k}_1, \mathbf{k}_2) \in F \times F | \mathbf{k}_1 - \mathbf{k}_2 = \mathbf{p}\} \leq 2\,. \tag{XX.1}$$

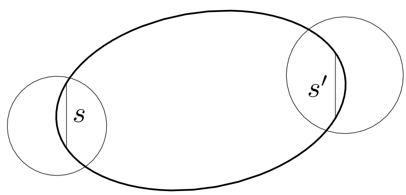
In this case, the lemma with $C = 2$ and $X = \{(\mathbf{k}_1, \mathbf{k}_2) | \mathbf{k}_1, \mathbf{k}_2 \in F,\ \mathbf{k}_1 \neq \mathbf{k}_2\}$ follows directly from (XX.1). To prove (XX.1), choose a nonzero vector $\vec{v}$ perpendicular



to $\mathbf{p}$. Assume that there are distinct pairs $(\mathbf{k}_1, \mathbf{k}_2)$, $(\mathbf{k}_3, \mathbf{k}_4)$, $(\mathbf{k}_5, \mathbf{k}_6)$ such that $\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_3 - \mathbf{k}_4 = \mathbf{k}_5 - \mathbf{k}_6 = \mathbf{p}$. Without loss of generality we assume that $\mathbf{k}_1 \cdot \vec{v} < \mathbf{k}_3 \cdot \vec{v} < \mathbf{k}_5 \cdot \vec{v}$. By convexity, the parallelogram, P, with vertices $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_5, \mathbf{k}_6$ is contained in the convex hull of F. The segment joining $\mathbf{k}_3$ and $\mathbf{k}_4$ must cross this parallelogram. Therefore $\mathbf{k}_3$ and $\mathbf{k}_4$ lie on the edges of P. This contradicts the strict convexity of F.

Formula (XX.1) may be phrased in more geometrical terms as follows. Let s be a secant of F (that is, a straight line segment joining two different points $\mathbf{k}_1, \mathbf{k}_2 \in F$). Then there is at most one other secant s' for F that is parallel to s and has the same length. In the case that there is no such second secant, we set $s' = s$. Clearly there are $r, R > 0$ such that for all $\mathbf{k} \in F$

(i) $B_r(\mathbf{k}) \cap B_R(a(\mathbf{k})) = \emptyset$.
(ii) For any secant $s \subset B_r(\mathbf{k})$ one has $s' \subset B_R(a(\mathbf{k}))$.



Now consider $\epsilon_2 = +1$. If F is invariant under inversion in some point $\mathbf{p}_0 \in \mathbb{R}^2$, then, for $\mathbf{p} \neq 2\mathbf{p}_0$,

$$\begin{aligned} &\#\{(\mathbf{k}_1, \mathbf{k}_2) \in F \times F | \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{p}\} \\ &\quad = \#\{(\mathbf{k}_1, \mathbf{k}_2') \in F \times F | \mathbf{k}_1 + (2\mathbf{p}_0 - \mathbf{k}_2') = \mathbf{p}\} \leq 2 \end{aligned}$$

by the case $\epsilon_2 = -1$.

Now we discuss the case that F is strongly asymmetric in the sense of Definition XVIII.3. Since $F \times F$ is compact, it suffices to show that for each point $(\mathbf{k}_1, \mathbf{k}_2) \in F \times F$ there exists a neighborhood U of $(\mathbf{k}_1, \mathbf{k}_2)$ in $F \times F$, a subset U' of U whose complement $U \setminus U'$ has measure zero and a number m such that, for all $\mathbf{p} \in \mathbb{R}^2$,

$$\#\{(\mathbf{q}_1, \mathbf{q}_2) \in U' | \mathbf{q}_1 + \mathbf{q}_2 = \mathbf{p}\} \leq m\,.$$

If $\mathbf{k}_2 \neq \mathbf{k}_1, a(\mathbf{k}_1)$, then the map $(\mathbf{q}_1, \mathbf{q}_2) \mapsto \mathbf{q}_1 + \mathbf{q}_2$ has rank 2 at $(\mathbf{k}_1, \mathbf{k}_2)$. By the inverse function theorem, there is a neighborhood U of $(\mathbf{k}_1, \mathbf{k}_2)$ such that for all $\mathbf{p} \in \mathbb{R}^2$ $\#\{(\mathbf{q}_1, \mathbf{q}_2) \in U | \mathbf{q}_1 + \mathbf{q}_2 = \mathbf{p}\} \leq 1$.

Next assume that $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}$. Let $U_r = \{(\mathbf{q}_1, \mathbf{q}_2) \in F^2 | \, \|\mathbf{q}_1 - \mathbf{k}\|, \|\mathbf{q}_2 - \mathbf{k}\| < r\}$, where r is defined in the discussion of secants above. We claim that for $(\mathbf{q}_1, \mathbf{q}_2), (\mathbf{q}_1', \mathbf{q}_2') \in U_r$

$$\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_1' + \mathbf{q}_2' \iff (\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{q}_1', \mathbf{q}_2') \text{ or } (\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{q}_2', \mathbf{q}_1')\,.$$

Assume that $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_1' + \mathbf{q}_2'$ and $(\mathbf{q}_1, \mathbf{q}_2) \neq (\mathbf{q}_1', \mathbf{q}_2'), (\mathbf{q}_2', \mathbf{q}_1')$. Then $\mathbf{q}_1 - \mathbf{q}_1' = \mathbf{q}_2' - \mathbf{q}_2 \neq 0$, so that the sector s of F joining $\mathbf{q}_1$ to $\mathbf{q}_1'$ is parallel to and of the same length as, but disjoint from the sector $\tilde{s}$ joining $\mathbf{q}_2'$ to $\mathbf{q}_2$. Therefore, $\tilde{s} = s'$ where, as above, s' is the unique second secant parallel to and of the same length as s. But this is impossible, as $\tilde{s} \subset B_r(\mathbf{k})$, $s' \subset B_R(a(\mathbf{k}))$ and $B_r(\mathbf{k}) \cap B_R(a(\mathbf{k})) = \emptyset$.

Finally, assume that $\mathbf{k}_1 = \mathbf{k}$ and $\mathbf{k}_2 = a(\mathbf{k})$ for some $\mathbf{k} \in F$. We may assume, without loss of generality, that the oriented unit tangent vector to F at $\mathbf{k}$ is (1, 0) and that the inward pointing unit normal vector to F at $\mathbf{k}$ is (0, 1). Then in the notation of Definition XVIII.3,

$$t \longmapsto \mathbf{k} + (t, \varphi_{\mathbf{k}}(t))$$

is a parametrization of F near $\mathbf{k}$ and

$$t \longmapsto a(\mathbf{k}) - (t, \varphi_{a(\mathbf{k})}(t))$$

is a parametrization of F near $a(\mathbf{k})$. Then

$$(t_1, t_2) \longmapsto (\mathbf{k} + (t_1, \varphi_{\mathbf{k}}(t_1)), a(\mathbf{k}) - (t_2, \varphi_{a(\mathbf{k})}(t_2)))$$

is a parametrization of $F \times F$ near $(\mathbf{k}_1, \mathbf{k}_2)$. With respect to these coordinates, the map $(\mathbf{q}_1, \mathbf{q}_2) \longmapsto \mathbf{q}_1 + \mathbf{q}_2 - (\mathbf{k} + a(\mathbf{k}))$ from $F \times F$ to $\mathbb{R}^2$ is

$$\tilde{f} : (t_1, t_2) \longmapsto (t_1 - t_2, \varphi_{\mathbf{k}}(t_1) - \varphi_{a(\mathbf{k})}(t_2)) .$$

Since F is strongly asymmetric, there is $2 \le n \le n_0$ such that

$$\varphi_{\mathbf{k}}^{(n)}(0) \neq \varphi_{a(\mathbf{k})}^{(n)}(0) .$$

We show that, if ϵ is small enough, then, for any $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$, the equation

$$\tilde{f}(t_1, t_2) = \mathbf{p} \tag{XX.2}$$

has at most n solutions in $(-\epsilon, \epsilon)^2$. Fix $\mathbf{p}$ and set $g(t) = \varphi_{\mathbf{k}}(p_1 + t) - \varphi_{a(\mathbf{k})}(t) - p_2$. Then (t_1, t_2) is a solution of (XX.2) if and only if $(t_1, t_2) = (p_1 + t, t)$ with t a zero of $g(t)$. Hence it suffices to prove that $g(t)$ has at most n zeros. But, since $\varphi_{\mathbf{k}}^{(n)}(0) - \varphi_{a(\mathbf{k})}^{(n)}(0) \neq 0$, the nth derivative $g^{(n)}(t) = \varphi_{\mathbf{k}}^{(n)}(p_1 + t) - \varphi_{a(\mathbf{k})}^{(n)}(t)$ never vanishes for $|p_1 + t|, |t| < \epsilon$, for ϵ sufficiently small. Consequently g can have at most n zeros on this set. $\square$

Lemma XX.8. *Let* $\mathbf{p} \in F$, $0 < \omega_1 < \frac{1}{2}\omega_2$ *and set*

$$M = \{(\mathbf{k}_1, \mathbf{k}_2) \in F \times F |\, \min[d(\mathbf{k}_1, \mathbf{k}_2), d(a(\mathbf{k}_1), \mathbf{k}_2)] \ge \omega_1$$
$$\text{and } \min[d(\mathbf{k}_i, \mathbf{p}), d(a(\mathbf{k}_i), \mathbf{p})] \le \omega_2 \text{ for } i = 1, 2\} .$$

Let $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ *and let* f *be the map from* $F \times F$ *to* $\mathbb{R}^2$ *given by*

$$f(\mathbf{k}_1, \mathbf{k}_2) = \epsilon_1 \mathbf{k}_1 + \epsilon_2 \mathbf{k}_2 .$$

There are constants that depend only on the geometry of F *such that*

(i) *for all measurable subsets A of* $\mathbb{R}^2$

$$\mathrm{vol}(f^{-1}(A) \cap M) \le \frac{\mathit{const}}{\omega_1}\ \mathrm{vol}(A)\,.$$

(ii) $V_{M,\omega_1} \ge \mathit{const}$.

(iii) *Let $\vec{n}$ a unit normal vector to F at* $\mathbf{p}$. *Then for all* $(\mathbf{k}_1, \mathbf{k}_2) \in M$

$$\sup_{\substack{\vec{v}\in T_{(\mathbf{k}_1,\mathbf{k}_2)}M \\ \|\vec{v}\|\le 1}} |\vec{n}\cdot Df(\mathbf{k}_1,\mathbf{k}_2)\vec{v}| \le \mathit{const}\ \omega_2\,.$$

(iv) *Let $0 < \epsilon < \omega_1/4$ and let Γ be an ϵ-separated subset of F. Furthermore let R be a rectangle in $\mathbb{R}^2$ having one pair of sides parallel to $\vec{n}$ of length $A > 0$ and a second pair of sides perpendicular to $\vec{n}$ of length $B > 0$. Then*

$$\# f^{-1}(R) \cap M \cap \Gamma^2 \le \frac{\mathit{const}}{\omega_1 \epsilon^2}(A + \epsilon\omega_2)(B + \epsilon)\,.$$

Proof. (i) For $\mathbf{k}_1, \mathbf{k}_2 \in F$ let $\theta(\mathbf{k}_1, \mathbf{k}_2)$ be the angle between the normal vectors to F at $\mathbf{k}_1$ and at $\mathbf{k}_2$. Then the Jacobian determinant of f at $(\mathbf{k}_1, \mathbf{k}_2)$ is

$$|\sin\theta(\mathbf{k}_1,\mathbf{k}_2)| \ge \mathit{const}\ \min(d(\mathbf{k}_1,\mathbf{k}_2), d(\mathbf{k}_1, a(\mathbf{k}_2))) \ge \mathit{const}\ \omega_1\,.$$

The claim follows from the rule for the change of variables in integrals and Lemma XX.7.

(ii) is trivial.

(iii) For $\mathbf{q} \in F$, let $\vartheta(\mathbf{q})$ be the angle between $\vec{n}$ and the normal vector to F at $\mathbf{q}$. Then

$$\begin{aligned}\sup_{\substack{\vec{v}\in T_{(\mathbf{k}_1,\mathbf{k}_2)}M \\ \|\vec{v}\|\le 1}} |\vec{n}\cdot Df(\mathbf{k}_1,\mathbf{k}_2)\vec{v}| &\le 2\max(|\sin\vartheta(\mathbf{k}_1)|, |\sin\vartheta(\mathbf{k}_2)|)\\ &\le \mathit{const}\ \max(\min(\|\mathbf{p}-\mathbf{k}_1\|, \|\mathbf{p}-a(\mathbf{k}_1)\|)\,,\\ &\qquad \min(\|\mathbf{p}-\mathbf{k}_2\|, \|\mathbf{p}-a(\mathbf{k}_2)\|))\\ &\le \mathit{const}\ \omega_2\,.\end{aligned}$$

(iv) Let $\vec{t}$ be the tangent vector to F at $\mathbf{p}$. Obviously

$$\sup_{\substack{\vec{v}\in T_{(\mathbf{k}_1,\mathbf{k}_2)}M \\ \|\vec{v}\|\le 1}} |\vec{t}\cdot Df(\mathbf{k}_1,\mathbf{k}_2)\vec{v}| \le 2$$

for all $(\mathbf{k}_1, \mathbf{k}_2) \in M$. So by parts (ii) and (iii) of this lemma and Lemma XX.4

$$\# f^{-1}(R) \cap M \cap \Gamma^2 \le \frac{\mathit{const}}{\epsilon^2}\ \mathrm{vol}(f^{-1}(R'))$$

where R' is a rectangle of side lengths $A + \mathit{const}\,\epsilon\omega_2$ and $B + 4\epsilon$. By part (i),

$$\mathrm{vol}(f^{-1}(R')) \le \frac{1}{\omega_1}(A + \mathit{const}\ \epsilon\omega_2)(B + 4\epsilon)\,.$$

□

Sectors that are compatible with conservation of momentum

Let $0 \le \Lambda \le \mathfrak{l}$, $\Lambda \ge \mathfrak{l}^2$, let $\mathbf{p} \in F$, $\mathbf{q} \in \mathbb{R}^2$ and let Γ be a discrete subset of F. We define for $n \ge 2$

$$\begin{aligned} Mom_{2n-1}(\Gamma, \mathbf{p}) = \{(\mathbf{k}_1, \ldots, \mathbf{k}_{2n-1}) \in \Gamma^{2n-1} | \exists\, x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i),\ i = 1, \ldots, 2n-1 \\ \text{such that } x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} \in s_{\Lambda,\mathfrak{l}}(\mathbf{p})\} \\ Mom^{\sim}_{2n-1}(\Gamma, \mathbf{q}) = \{(\mathbf{k}_1, \ldots, \mathbf{k}_{2n-1}) \in \Gamma^{2n-1} | \exists\, x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i),\ i = 1, \ldots, 2n-1 \\ \text{such that } x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} = \mathbf{q}\}\,. \end{aligned}$$

Lemma XX.9. *Let $\mathbf{p} \in F$, $\mathbf{q} \in \mathbb{R}^2$ and Γ an $\mathfrak{l}$-separated subset of F.*

(i) *If $\omega \ge \Lambda/\mathfrak{l}$ then*

$$\#\Big\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p}) | \omega \le \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 2\omega\Big\} \le \mathit{const}\ \frac{\omega}{\mathfrak{l}}$$

$$\#\Big\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom^{\sim}_3(\Gamma, \mathbf{q}) | \omega \le \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 2\omega\Big\} \le \mathit{const}\ \frac{\omega}{\mathfrak{l}}\,.$$

(ii) *If $4\mathfrak{l} \le \omega \le \Lambda/\mathfrak{l}$ then*

$$\#\Big\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p}) | \omega \le \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 2\omega\Big\} \le \mathit{const}\ \frac{\Lambda}{\mathfrak{l}^2}$$

$$\#\Big\{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom^{\sim}_3(\Gamma, \mathbf{q}) | \omega \le \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 2\omega\Big\} \le \mathit{const}\ \frac{\Lambda}{\mathfrak{l}^2}\,.$$

The constants const above depend only on the geometry of F.

Proof. If $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p})$ and $\max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 2\omega$, then by Proposition XX.6(ii), with $n = 2$, ω replaced by 2ω and $\mathbf{k} = \mathbf{k}_i$ or $a(\mathbf{k}_i)$, $i = 1, 2, 3$

$$\min[d(\mathbf{k}_i, \mathbf{p}), d(a(\mathbf{k}_i), \mathbf{p})] \le 4\, \mathit{const}_0\ \omega \quad \text{for } 1 \le i \le 3 \tag{XX.3}$$

where const_0 is the constant of Proposition XX.6. Therefore we set for $1 \le \mu \ne \nu \le 3$

$$\mathcal{S}_{\mu,\nu} = \Big\{ (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in Mom_3(\Gamma,\mathbf{p}) | \omega \le \min[d(\mathbf{k}_\mu,\mathbf{k}_\nu), d(a(\mathbf{k}_\mu),\mathbf{k}_\nu)] \le 2\omega$$
$$\max_{1\le\alpha\ne\beta\le 3} \min[d(\mathbf{k}_\alpha,\mathbf{k}_\beta), d(a(\mathbf{k}_\alpha),\mathbf{k}_\beta)] \le 2\omega$$
$$\text{and } \min[d(\mathbf{k}_i,\mathbf{p}), d(a(\mathbf{k}_i),\mathbf{p})] \le 4\,{\scriptstyle const_0}\;\omega \quad \text{for } 1\le i\le 3 \Big\}.$$

The discussion above shows that

$$\Big\{ (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in Mom_3(\Gamma,\mathbf{p}) | \omega \le \max_{1\le\mu\ne\nu\le 3} \min[d(\mathbf{k}_\mu,\mathbf{k}_\nu), d(a(\mathbf{k}_\mu),\mathbf{k}_\nu)] \le 2\omega \Big\}$$
$$\subset \bigcup_{1\le\mu\ne\nu\le 3} \mathcal{S}_{\mu,\nu}\,. \tag{XX.4}$$

Similarly, if $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in Mom_3^\sim(\Gamma,\mathbf{q})$ and $\min[d(\mathbf{k}_\mu,\mathbf{k}_\nu), d(a(\mathbf{k}_\mu),\mathbf{k}_\nu)] \le 2\omega$ for all $1\le\mu\ne\nu\le 3$, then by Proposition XX.6(i), we have for $i = 1, 2, 3$

$$\begin{aligned} d(\mathbf{k}_i,\mathbf{p}) \le 4\,{\scriptstyle const_0}\;\omega \quad &\text{or} \quad d(a(\mathbf{k}_i),\mathbf{p}) \le 4\,{\scriptstyle const_0}\;\omega \quad \text{or} \\ d(2\mathbf{k}_i - a(\mathbf{k}_i),\mathbf{p}) \le 4\,{\scriptstyle const_0}\;\omega \quad &\text{or} \quad d(2a(\mathbf{k}_i) - \mathbf{k}_i,\mathbf{p}) \le 4\,{\scriptstyle const_0}\;\omega\,. \end{aligned} \tag{XX.5}$$

Setting for $1\le\mu\ne\nu\le 3$

$$\mathcal{S}^\sim_{\mu,\nu} = \Big\{ (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in Mom_3^\sim(\Gamma,\mathbf{q}) | \omega \le \min[d(\mathbf{k}_\mu,\mathbf{k}_\nu), d(a(\mathbf{k}_\mu),\mathbf{k}_\nu)] \le 2\omega$$
$$\times \max_{1\le\alpha\ne\beta\le 3} \min[d(\mathbf{k}_\alpha,\mathbf{k}_\beta), d(a(\mathbf{k}_\alpha),\mathbf{k}_\beta)] \le 2\omega \text{ and (XX.5) holds} \Big\}$$

we get

$$\Big\{ (\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in Mom_3^\sim(\Gamma,\mathbf{q}) | \omega \le \max_{1\le\mu\ne\nu\le 3} \min[d(\mathbf{k}_\mu,\mathbf{k}_\nu), d(a(\mathbf{k}_\mu),\mathbf{k}_\nu)] \le 2\omega \Big\}$$
$$\subset \bigcup_{1\le\mu\ne\nu\le 3} \mathcal{S}^\sim_{\mu,\nu}\,. \tag{XX.6}$$

We show that for $1\le\mu\ne\nu\le 3$ one has $\#\mathcal{S}_{\mu,\nu},\ \#\mathcal{S}^\sim_{\mu,\nu} \le {\scriptstyle const}\ \frac{\omega}{\mathfrak{l}}$ in case (i), and that $\#\mathcal{S}_{\mu,\nu},\ \#\mathcal{S}^\sim_{\mu,\nu} \le {\scriptstyle const}\ \frac{\Lambda}{\mathfrak{l}^2}$ in case (ii). We only discuss the case $\mu = 1, \nu = 2$, the other cases are similar.

Set $\mathcal{S} = \mathcal{S}_{1,2}$ or $\mathcal{S} = \mathcal{S}^\sim_{1,2}$. By construction, if $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in \mathcal{S}$,

$$\min[\|\mathbf{k}_1 - \mathbf{k}_3\|, \|\mathbf{k}_1 - a(\mathbf{k}_3)\|], \min[\|\mathbf{k}_2 - \mathbf{k}_3\|, \|\mathbf{k}_2 - a(\mathbf{k}_3)\|] \le {\scriptstyle const}\ \omega \tag{XX.7}$$

and (XX.3) respectively (XX.5) hold for $i = 3$. Since the maps $\mathbf{k} \mapsto \mathbf{k}$, $\mathbf{k} \mapsto a(\mathbf{k})$, $\mathbf{k} \mapsto 2\mathbf{k} - a(\mathbf{k})$ and $\mathbf{k} \mapsto 2a(\mathbf{k}) - \mathbf{k}$ are embeddings of F, there are at most ${\scriptstyle const}\ \frac{\omega}{\mathfrak{l}}$ choices of $\mathbf{k}_3 \in \Gamma$ for which (XX.3) or (XX.5) are satisfied. Fix such a $\mathbf{k}_3$. Let $\vec{\mathrm{n}}$ be a unit normal vector to F at $\mathbf{k}_3$ and $\vec{\mathrm{t}}$ be a unit tangent vector to F at $\mathbf{k}_3$. If $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \in \mathcal{S}$, by (XX.7), the sectors $s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)$, $i = 1, 2, 3$ are each contained in a rectangle two of whose edges are parallel to $\vec{\mathrm{t}}$ and have length at most ${\scriptstyle const}\,\mathfrak{l}$, and two of whose edges are parallel to $\vec{\mathrm{n}}$ and have length at most

$$ {\scriptstyle const}(\Lambda + \mathfrak{l}\,\omega) \le \begin{cases} {\scriptstyle const}\ \mathfrak{l}\,\omega & \text{in case (i)} \\ {\scriptstyle const}\ \Lambda & \text{in case (ii)}\,. \end{cases} $$

The same holds for $s_{\Lambda,\mathfrak{l}}(\mathbf{p})$ when $\mathcal{S} = \mathcal{S}_{1,2}$. In particular, if $\mathcal{S} = \mathcal{S}_{1,2}$, the set

$$ \{x_3 + y | x_3 \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_3), y \in s_{\Lambda,\mathfrak{l}}(\mathbf{p})\} $$

is contained in a rectangle R whose one pair of edges is parallel to $\mathbf{t}$ and has length at most ${\scriptstyle const}\,\mathfrak{l}$, and whose other pair of edges is parallel to $\mathbf{n}$ and has length at most ${\scriptstyle const}\,\mathfrak{l}\omega$ in case (i) and length at most ${\scriptstyle const}\,\Lambda$ in case (ii). If $\mathcal{S} = \mathcal{S}^{\sim}_{1,2}$, the set

$$ \{x_3 + \mathbf{q} | x_3 \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_3)\} $$

is contained in such a rectangle R.

Let

$$ \mathcal{M}(\mathbf{k}_3) = \{(\mathbf{k}_1, \mathbf{k}_2) \in \Gamma^2 | (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathcal{S}\}\,. $$

By definition, if $(\mathbf{k}_1, \mathbf{k}_2) \in \mathcal{M}(\mathbf{k}_3)$, there are $x_1 \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_1)$, $x_2 \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_2)$ such that $x_1 + x_2 \in R$. The shape of $s_{\Lambda,\mathfrak{l}}(\mathbf{k}_1)$, $s_{\Lambda,\mathfrak{l}}(\mathbf{k}_2)$ and R determined above implies that the map $f : (\mathbf{k}_1, \mathbf{k}_2) \mapsto \mathbf{k}_1 + \mathbf{k}_2$ maps $\mathcal{M}(\mathbf{k}_3)$ to a rectangle R' that contains R and has one pair of edges parallel to $\vec{\mathrm{t}}$ and of length at most ${\scriptstyle const}'\ \mathfrak{l}$, and a second pair of edges parallel to $\vec{\mathrm{n}}$ and of length at most ${\scriptstyle const}'\ \mathfrak{l}\omega$ in case (i) and ${\scriptstyle const}'\ \Lambda$ in case (ii). Observe that

$$ \begin{aligned} \mathcal{M}(\mathbf{k}_3) \subset \{(\mathbf{k}_1, \mathbf{k}_2) \in \Gamma^2 | \omega \le \min[d(\mathbf{k}_1, \mathbf{k}_2), d(a(\mathbf{k}_1), \mathbf{k}_2)] \\ \text{and } \min[d(\mathbf{k}_i, \mathbf{k}_3), d(a(\mathbf{k}_i), \mathbf{k}_3)] \le {\scriptstyle const}\ \omega \text{ for } i = 1, 2\}\,. \end{aligned} $$

It follows from part (iv) of Lemma XX.8, with $\mathbf{p} = \mathbf{k}_3$, $A = {\scriptstyle const}'\ \mathfrak{l}\omega$ or ${\scriptstyle const}'\ \Lambda$, $B = {\scriptstyle const}'\ \mathfrak{l}$, $\omega_1 = \omega$, $\omega_2 = {\scriptstyle const}\ \omega$ and $\epsilon = \mathfrak{l}$ that

$$ \#\mathcal{M}(\mathbf{k}_3) \le \begin{cases} \dfrac{\scriptstyle const}{\omega \mathfrak{l}^2}(\mathfrak{l}\,\omega)(\mathfrak{l}) = {\scriptstyle const} & \text{in case (i)} \\[2ex] \dfrac{\scriptstyle const}{\omega \mathfrak{l}^2}\Lambda\,\mathfrak{l} = {\scriptstyle const}\ \dfrac{\Lambda}{\omega \mathfrak{l}} & \text{in case (ii)}\,. \end{cases} $$

Together with the observation made above, that there are at most ${\scriptstyle const}\ \frac{\omega}{\mathfrak{l}}$ choices of $\mathbf{k}_3 \in \Gamma$ for which there exist $(\mathbf{k}_1, \mathbf{k}_2) \in \Gamma^2$ with $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathcal{S}$, this completes the proof of the lemma. □

Proposition XX.10. *For all $\mathfrak{l}$-separated subsets Γ of F and all $\mathbf{p} \in F$, $\mathbf{q} \in \mathbb{R}^2$*

$$ \#Mom_3(\Gamma, \mathbf{p}) \le \frac{\scriptstyle const}{\mathfrak{l}} \left(1 + \frac{\Lambda}{\mathfrak{l}} \log \frac{\Lambda}{\mathfrak{l}^2}\right) $$

$$ \#Mom^{\sim}_3(\Gamma, \mathbf{q}) \le \frac{\scriptstyle const}{\mathfrak{l}} \left(1 + \frac{\Lambda}{\mathfrak{l}} \log \frac{\Lambda}{\mathfrak{l}^2}\right) $$

with a constant ${\scriptstyle const}$ that depends only on the geometry of F.

Proof. We give the proof for $Mom_3(\Gamma, \mathbf{p})$, the proof for $Mom_3^{\sim}(\Gamma, \mathbf{q})$ is similar. Applying part (i) of Lemma XX.9 successively to

$$\omega = \Lambda/\mathfrak{l},\ 2\Lambda/\mathfrak{l},\ 4\Lambda/\mathfrak{l}, \ldots, {\scriptstyle const}$$

one sees that

$$\#\left\{ (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p}) \,\middle|\, \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \ge \Lambda/\mathfrak{l} \right\}$$

$$\le \sum_{j=1}^{\ln_2(const\frac{\mathfrak{l}}{\Lambda})} {\scriptstyle const}\, 2^j \frac{\Lambda}{\mathfrak{l}^2} \le {\scriptstyle const}\ \frac{\mathfrak{l}}{\Lambda} \frac{\Lambda}{\mathfrak{l}^2} \le \frac{const}{\mathfrak{l}} \,.$$

Similarly, if $4\mathfrak{l} \le \frac{\Lambda}{\mathfrak{l}}$ and one applies part (ii) of Lemma XX.9 successively to

$$\omega = 4\mathfrak{l}, 8\mathfrak{l}, 16\mathfrak{l}, \ldots, 2^{1+[\log_2 \frac{\Lambda}{\mathfrak{l}^2}]}\mathfrak{l}$$

one sees that

$$\#\left\{ (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p}) | 4\mathfrak{l} \le \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le \Lambda/\mathfrak{l} \right\}$$

$$\le \frac{const\, \Lambda}{\mathfrak{l}^2} \left(1 + \log_2 \frac{\Lambda}{\mathfrak{l}^2} \right) .$$

Finally, it is obvious that

$$\#\left\{ (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in Mom_3(\Gamma, \mathbf{p}) \,\middle|\, \max_{1 \le \mu \ne \nu \le 3} \min[d(\mathbf{k}_\mu, \mathbf{k}_\nu), d(a(\mathbf{k}_\mu), \mathbf{k}_\nu)] \le 4\mathfrak{l} \right\}$$

$$\le \frac{const}{\mathfrak{l}} \,.$$

□

Proposition XX.11. *Let $n \ge 2$, $\delta \ge \mathfrak{l}$ and let $I_1, \ldots, I_{2n-1}$ be intervals of length δ in F. Assume that*

$$\frac{1}{3}\omega = \max_{1 \le i \ne j \le 2n-1} \min(\mathrm{dist}(I_i, I_j), \mathrm{dist}(I_i, a(I_j))) > \max(\delta, 4\mathfrak{l}) \,.$$

Then for all $\mathfrak{l}$-separated subsets Γ of F, all $\mathbf{p} \in F$ and all $\mathbf{q} \in \mathbb{R}^2$

$$\# Mom_{2n-1}(\Gamma, \mathbf{p}) \cap (I_1 \times \cdots \times I_{2n-1}) \le {\scriptstyle const}\ n^2 \left(\frac{\delta}{\mathfrak{l}} + 1 \right)^{2n-3} \left(1 + \frac{\Lambda}{\mathfrak{l}\omega} \right)$$

$$\# Mom_{2n-1}^{\sim}(\Gamma, \mathbf{q}) \cap (I_1 \times \cdots \times I_{2n-1}) \le {\scriptstyle const}\ n^2 \left(\frac{\delta}{\mathfrak{l}} + 1 \right)^{2n-3} \left(1 + \frac{\Lambda}{\mathfrak{l}\omega} \right)$$

with a constant ${\scriptstyle const}$ that depends only on the geometry of F.

Proof. The proof is similar to the proof of Proposition XX.10. Set

$$\epsilon_i = \begin{cases} +1 & \text{for } 1 \le i \le n \\ -1 & \text{for } n+1 \le i \le 2n-1 \end{cases} .$$

Choose a point $\mathbf{k} \in I_1$. Then for all $x \in \bigcup_{i=1}^{2n-1} I_i$

$$d(x, \mathbf{k}) \le \omega \quad \text{or} \quad d(x, a(\mathbf{k})) \le \omega\,.$$

Choose $1 \le i_0 < j_0 \le 2n-1$ such that

$$\min(\mathrm{dist}(I_{i_0}, I_{j_0}), \mathrm{dist}(I_{i_0}, a(I_{j_0}))) = \frac{1}{3}\omega\,.$$

Since Γ is $\mathfrak{l}$-separated and each I_j is of length δ,

$$\# \prod_{\substack{i=1 \\ i \ne i_0, j_0}}^{2n-1} \Gamma \cap I_i \le \left(\frac{\delta}{\mathfrak{l}} + 1\right)^{2n-3}\,. \tag{XX.8}$$

Fix $\mathbf{k}_i \in \Gamma \cap I_i$ for $i = 1, \ldots, 2n-1,\ i \ne i_0, j_0$. Let $\vec{\mathfrak{n}}$ be a unit normal vector to F at $\mathbf{k}$ and $\vec{\mathfrak{t}}$ be a unit tangent vector to F at $\mathbf{k}$.

By Proposition XX.5

$$-\left\{ \sum_{\substack{i=1 \\ i \ne i_0, j_0}}^{2n-1} \epsilon_i x_i - \mathbf{q} \,\middle|\, x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i) \right\}$$

is contained in a rectangle $R^{\sim}$ having one pair of edges parallel to $\vec{\mathfrak{t}}$ and of length at most $_{const}\ n\mathfrak{l}$ and a second pair of edges parallel to $\vec{\mathfrak{n}}$ and of length at most $_{const}\ n[\Lambda + \mathfrak{l}\,\omega]$. As each $s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i)$ is contained in a rectangle having one pair of edges parallel to $\vec{\mathfrak{t}}$ and of length at most $_{const}\ \mathfrak{l}$ and a second pair of edges parallel to $\vec{\mathfrak{n}}$ and of length at most $_{const}[\Lambda + \mathfrak{l}\,\omega]$, the map $f\colon (\mathbf{k}_{i_0}, \mathbf{k}_{j_0}) \mapsto \epsilon_{i_0}\mathbf{k}_{i_0} + \epsilon_{j_0}\mathbf{k}_{j_0}$ maps the set

$$\mathcal{M}^{\sim} = \{(\mathbf{k}_{i_0}, \mathbf{k}_{j_0}) \in \Gamma^2 \cap (I_{i_0} \times I_{j_0}) | \exists x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i),\ i = 1, \ldots, 2n-1$$
$$\text{such that } x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n-1} = \mathbf{q}\}$$

to a rectangle R' having one pair of edges parallel to $\vec{\mathfrak{t}}$ and of length at most $B = n\ _{const'}\ \mathfrak{l}$, and a second pair of edges parallel to $\vec{\mathfrak{n}}$ and of length at most

$$A = {}_{const'}\ n[\Lambda + \mathfrak{l}\,\omega]\,.$$

By part (iv) of Lemma XX.8, with $\mathbf{p}$ replaced by $\mathbf{k}$, $\omega_1 = \frac{1}{3}\omega,\ \omega_2 = \omega$ and $\epsilon = \mathfrak{l}$

$$\#\mathcal{M}^{\sim} \le \frac{const}{\mathfrak{l}^2\omega}(n\Lambda + n\mathfrak{l}\omega)(n\mathfrak{l}) \le {}_{const}\ n^2\left(1 + \frac{\Lambda}{\mathfrak{l}\omega}\right).$$

This, together with (XX.8), proves

$$\#Mom^{\sim}_{2n-1}(\Gamma, \mathbf{q}) \cap (I_1 \times \cdots \times I_{2n-1}) \le {}_{const}\left(\frac{\delta}{\mathfrak{l}} + 1\right)^{2n-3} n^2\left(1 + \frac{\Lambda}{\mathfrak{l}\omega}\right).$$

By Proposition XX.6(ii)

$$\|\mathbf{p} - \mathbf{k}\| \le {}_{const}\ n\omega \quad \text{or} \quad \|\mathbf{p} - a(\mathbf{k})\| \le {}_{const}\ n\omega\,.$$

Therefore $s_{\Lambda,\mathfrak{l}}(\mathbf{p})$ is contained in a rectangle, two of whose edges are parallel to $\vec{\mathfrak{t}}$ and have length at most ${\scriptstyle const}\ \mathfrak{l}$ and two of whose edges are parallel to $\vec{\mathfrak{n}}$ and have length at most

$$ {\scriptstyle const}(\Lambda + \mathfrak{l}\|\mathbf{p}-\mathbf{k}\|) \le {\scriptstyle const}(\Lambda + n\,\mathfrak{l}\,\omega)\,. $$

This and Proposition XX.5 imply that

$$ -\left\{ \sum_{\substack{i=1\\ i\ne i_0,j_0}}^{2n-1} \epsilon_i x_i - y \middle| x_i \in s_{\Lambda,\mathfrak{l}}(\mathbf{k}_i),\ y \in s_{\Lambda,\mathfrak{l}}(\mathbf{p}) \right\} $$

is contained in a rectangle R having one pair of edges parallel to $\vec{\mathfrak{t}}$ and of length at most ${\scriptstyle const}\ n\,\mathfrak{l}$ and a second other pair of edges parallel to $\vec{\mathfrak{n}}$ and of length at most

$$ {\scriptstyle const}[n\Lambda + n\,\mathfrak{l}\,\omega + n\,\mathfrak{l}\,\omega] \le {\scriptstyle const}\ n[\Lambda + \mathfrak{l}\,\omega]\,. $$

As above, this implies that

$$ \begin{aligned} &\#Mom_{2n-1}(\Gamma,\mathbf{p}) \cap (I_1 \times \cdots \times I_{2n-1}) \\ &\quad \le {\scriptstyle const}\left(\frac{\delta}{\mathfrak{l}}+1\right)^{2n-3} n^2 \left(1+\frac{\Lambda}{\mathfrak{l}\omega}\right). \end{aligned} $$

□

XXI. Sectors Compatible with Conservation of Momentum

Comparison of the 1-norm and the 3-norm for four-legged Kernels

Lemma XXI.1. *There is a constant* ${\scriptstyle const}$ *independent of* M *such that the following holds. Let* Σ *be a sectorization of length* $\mathfrak{l}$ *at scale* j *with* $\frac{2}{M^{j-1}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$. *Furthermore let* $\varphi \in \mathcal{F}_0(4;\Sigma)$ *and* $f \in \check{\mathcal{F}}_1(3;\Sigma)$ *be particle number conserving functions. Then*

$$ |\varphi|_{1,\Sigma} \le \frac{const}{\mathfrak{l}}\left(1+\frac{1}{\mathfrak{l}M^{j-1}}\log\frac{1}{\mathfrak{l}^2M^{j-1}}\right)|\varphi|_{3,\Sigma} $$

$$ |f|_{1,\Sigma}^{\sim} \le \frac{const}{\mathfrak{l}}\left(1+\frac{1}{\mathfrak{l}M^{j-1}}\log\frac{1}{\mathfrak{l}^2M^{j-1}}\right)|f|_{3,\Sigma}^{\sim}\,. $$

Proof. By [8, Definition XII.9] and Remark XX.2(i)

$$ |\varphi|_{1,\Sigma} = \max_{1\le i_1\le 4}\ \max_{s_{i_1}\in\Sigma} \sum_{\substack{s_i\in\Sigma \text{ for } i\ne i_1\\ s_1,s_2,s_3,s_4 \text{ compatible with}\\ \text{conservation of momentum}}} \|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_4))\|_{1,\infty}\,. $$

Let $1 \le i_1 \le 4$ and $s_{i_1} \in \Sigma$. Choose i_2,i_3,i_4 such that $\{1,2,3,4\} = \{i_1,i_2,i_3,i_4\}$. By Remark XX.2(ii) and Proposition XX.10, with $\Lambda = \frac{\sqrt{2}}{M^{j-1}}$, there are at most $\frac{const}{\mathfrak{l}}(1+\frac{M}{\mathfrak{l}M^j}\log\frac{M}{\mathfrak{l}^2M^j})$ triples $(s_{i_2},s_{i_3},s_{i_4})$ such that (s_1,s_2,s_3,s_4) is compatible with conservation of momentum. Consequently

$$
\begin{aligned}
|\varphi|_{1,\Sigma} &\le \frac{const}{\mathfrak{l}}\left(1+\frac{M}{\mathfrak{l}M^j}\log\frac{M}{\mathfrak{l}^2M^j}\right)\max_{s_1,s_2,s_3,s_4\in\Sigma}\|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_4))\|_{1,\infty}\\
&\le \frac{const}{\mathfrak{l}}\left(1+\frac{1}{\mathfrak{l}M^{j-1}}\log\frac{1}{\mathfrak{l}^2M^{j-1}}\right)\\
&\quad\times\max_{\substack{1\le i_1<i_2<i_3\le 4\\ s_{i_1},s_{i_2},s_{i_3}\in\Sigma}}\ \sum_{\substack{s_i\in\Sigma\\ \text{for } i\ne i_1,i_2,i_3}}\|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_4))\|_{1,\infty}\\
&= \frac{const}{\mathfrak{l}}\left(1+\frac{1}{\mathfrak{l}M^{j-1}}\log\frac{1}{\mathfrak{l}^2M^{j-1}}\right)|\varphi|_{3,\Sigma}\,.
\end{aligned}
$$

The argument for $|f|\widetilde{}_{1,\Sigma}$ is analogous. □

Proof of Proposition XIX.1. Under the hypotheses of this proposition, the term $\frac{1}{\mathfrak{l}M^{j-1}}\log\frac{1}{\mathfrak{l}^2M^{j-1}}$ is bounded by an M-independent constant and the first inequality, $|\varphi|_{1,\Sigma}\le const\ \frac{1}{\mathfrak{l}}|\varphi|_{3,\Sigma}$, follows. If $f\in\check{\mathcal{F}}_{4;\Sigma}$ and $\vec{i}\in\{0,1\}^4$, then $|f|_{\vec{i}}\widetilde{|}_{1,\Sigma}=0$ unless $m(\vec{i})\le 1$. Therefore the second inequality, $|f|\widetilde{}_{1,\Sigma}\le const\ \frac{1}{\mathfrak{l}}|f|\widetilde{}_{3,\Sigma}$, also follows from Lemma XXI.1. □

Auxiliary norms

Let Σ be a sectorization of scale j and length $\mathfrak{l}\ge\frac{1}{M^{j-1}}$. For $\omega>0$ we define auxiliary norms on functions φ in $\mathcal{F}_0(n;\Sigma)$ and $f\in\mathcal{F}_1(n;\Sigma)$ that are antisymmetric in their (ξ,s) arguments by

$$
|\varphi|_{1,\Sigma,\omega}=\max_{s_1\in\Sigma}\sum_{\substack{s_2,\ldots,s_n\in\Sigma\\ \mathrm{dist}(s_k,s_\ell)\ge\omega\text{ and}\\ \mathrm{dist}(s_k,a(s_\ell))\ge\omega\\ \text{for some } 2\le k\ne\ell\le n}}\|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_n))\|_{1,\infty}
$$

$$
\begin{aligned}
|f|\widetilde{}_{1,\Sigma,\omega}=\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\ \sup_{\check{\eta}\in\mathcal{B}}\ \sum_{\substack{s_1,\ldots,s_n\in\Sigma\\ \mathrm{dist}(s_k,s_\ell)\ge\omega\text{ and}\\ \mathrm{dist}(s_k,a(s_\ell))\ge\omega\\ \text{for some } 1\le k\ne\ell\le n}}\ \max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}}\\
\times|||\mathrm{D}f(\check{\eta};(\xi_1,s_1),\ldots,(\xi_n,s_n))|||_{1,\infty}\,\frac{t^\delta}{\delta!}\,.
\end{aligned}
$$

The norm $|||\cdot|||_{1,\infty}$ of Example II.6 refers to the variables $\xi_1,\ldots,\xi_n$. Furthermore, maxima, like $\max_{s_1\in\Sigma}$, that act on a formal power series $\sum_\delta a_\delta t^\delta$ are to be applied separately to each coefficient a_δ.

Lemma XXI.2. *Let* $\omega\ge\max\{\mathfrak{l},\frac{1}{M^{(j-1)/2}}\}$, $n\ge 3$ *and let* $\varphi\in\mathcal{F}_0(n;\Sigma)$ *and* $f\in\mathcal{F}_1(n;\Sigma)$ *be particle number conserving functions that are antisymmetric in their* (ξ,s) *arguments. Then*

$$
|\varphi|_{1,\Sigma}\le|\varphi|_{1,\Sigma,\omega}+const\ n\frac{\omega^2}{\mathfrak{l}^2}|\varphi|_{3,\Sigma}
$$

$$|f|\widetilde{}_{1,\Sigma} \le |f|\widetilde{}_{1,\Sigma,\omega} + {\scriptstyle const}\ n^2 \frac{\omega^2}{\mathfrak{l}^2} |f|\widetilde{}_{3,\Sigma}\,.$$

Proof. By definition of $|\varphi|_{1,\Sigma}$ and the antisymmetry of φ

$$|\varphi|_{1,\Sigma} \le |\varphi|_{1,\Sigma,\omega} + \max_{s_1\in\Sigma} \sum_{\substack{s_2,\ldots,s_n\in\Sigma \\ \mathrm{dist}(s_k,s_\ell)\le\omega \text{ or} \\ \mathrm{dist}(s_k,a(s_\ell))\le\omega \\ \text{for all } 2\le k\ne\ell\le n}} \|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_n))\|_{1,\infty}\,.$$

Fix $s_1 \in \Sigma$. If $s_2,\ldots,s_n \in \Sigma$ are such that for all $2 \le k \ne \ell \le n$ one has $\mathrm{dist}(s_k,s_\ell) \le \omega$ or $\mathrm{dist}(s_k,a(s_\ell)) \le \omega$ and such that $s_1,\ldots,s_n$ are compatible with conservation of momentum for some choice of annihilation/creation indices $(b_1,\ldots,b_n)$, then, by Proposition XX.6(ii) with $\Lambda = \frac{\sqrt{2}}{M^{j-1}}$, $\mathbf{p}$ the center of s_1 and $\mathbf{k}$ the center of s_2,

$$\mathrm{dist}(s_1,s_k) \le {\scriptstyle const}\ n\omega \quad \text{or} \quad \mathrm{dist}(s_1,a(s_k)) \le {\scriptstyle const}\ n\omega$$

for $2 \le k \le n$. Set

$$\mathrm{Sect} = \big\{(s_2,s_3)\in\Sigma^2 \big| \mathrm{dist}(s_1,s_2) \le {\scriptstyle const}\ n\omega \text{ or } \mathrm{dist}(s_1,a(s_2)) \le {\scriptstyle const}\ n\omega$$
$$\text{and } \mathrm{dist}(s_2,s_3) \le {\scriptstyle const}\ \omega \text{ or } \mathrm{dist}(s_2,a(s_3)) \le {\scriptstyle const}\ \omega\big\}\,.$$

Clearly $\#\mathrm{Sect} \le {\scriptstyle const}\ n\frac{\omega^2}{\mathfrak{l}^2}$. Consequently

$$\sum_{\substack{s_2,\ldots,s_n\in\Sigma \\ \mathrm{dist}(s_k,s_\ell)\le\omega \text{ or} \\ \mathrm{dist}(s_k,a(s_\ell))\le\omega \\ \text{for all } 2\le k\ne\ell\le n}} \|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_n))\|_{1,\infty}$$

$$\le \sum_{s_2,s_3\in\mathrm{Sect}} \sum_{s_4,\ldots,s_n\in\Sigma} \|\varphi((\cdot\,,s_1),\ldots,(\cdot\,,s_n))\|_{1,\infty} \le {\scriptstyle const}\ n\frac{\omega^2}{\mathfrak{l}^2}|\varphi|_{3,\Sigma}\,.$$

Similarly,

$$|f|\widetilde{}_{1,\Sigma} \le |f|\widetilde{}_{1,\Sigma,\omega} + \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \max_{\check\eta\in\check{\mathcal{B}}} \sum_{\substack{s_1,\ldots,s_n\in\Sigma \\ \mathrm{dist}(s_k,s_\ell)\le\omega \text{ or} \\ \mathrm{dist}(s_k,a(s_\ell))\le\omega \\ \text{for all } 1\le k\ne\ell\le n}}$$
$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta;(\xi_1,s_1),\ldots,(\xi_n,s_n))|||_{1,\infty}\,\frac{t^\delta}{\delta!}\,.$$

Fix $\check\eta = (p_0,\mathbf{p},\sigma,b) \in \check{\mathcal{B}}$. If $s_1,\ldots,s_n \in \Sigma$ are such that for all $1 \le k \ne \ell \le n$ one has $\mathrm{dist}(s_k,s_\ell) \le \omega$ or $\mathrm{dist}(s_k,a(s_\ell)) \le \omega$ and such that the configuration $(\check\eta;\ s_1,\ldots,s_n)$ is compatible with conservation of momentum, then, by Proposition XX.6(i) with $\Lambda = \frac{\sqrt{2}}{M^{j-1}}$ and $\mathbf{k}$ the center of s_1, there is an integer r with $|r| \le n$ such that

$$\|\mathbf{p} - r\mathbf{k} + (r-1)a(\mathbf{k})\| \le {\scriptstyle const}\ n\omega\,. \tag{XXI.1}$$

The maps $F \longrightarrow \mathbb{R}^2,\ \mathbf{k} \mapsto r\mathbf{k} - (r-1)a(\mathbf{k})$ are embeddings. Therefore there are at most ${\scriptstyle const}\ n\omega/\mathfrak{l}$ sectors s_1 containing a $\mathbf{k}$ such that (XXI.1) holds. Set

$$\begin{aligned}\mathrm{Sect} = \{(s_1,s_2) \in \Sigma^2 | s_1 &\text{ contains a point } \mathbf{k} \text{ for which} \\ &\text{(XXI.1) holds with some } |r| \le n \\ &\text{and } \mathrm{dist}(s_1,s_2) \le {\scriptstyle const}\ \omega \text{ or } \mathrm{dist}(s_1,a(s_2)) \le {\scriptstyle const}\ \omega\}\,.\end{aligned}$$

Again $\#\mathrm{Sect} \le {\scriptstyle const}\ n^2 \frac{\omega^2}{\mathfrak{l}^2}$. Consequently

$$\begin{aligned}&\sum_{\substack{s_1,\ldots,s_n\in\Sigma \\ \mathrm{dist}(s_k,s_\ell)\le\omega \text{ or} \\ \mathrm{dist}(s_k,a(s_\ell))\le\omega \\ \text{for all } 1\le k\ne\ell\le n}} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\text{ dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta;(\xi_1,s_1),\ldots,(\xi_n,s_n))|||_{1,\infty} t^\delta \\ &\le \sum_{s_1,s_2\in\mathrm{Sect}} \sum_{s_3,\ldots,s_n\in\Sigma} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \\ &\quad\times \max_{\substack{\mathrm{D}\text{ dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta;(\xi_1,s_1),\ldots,(\xi_n,s_n))|||_{1,\infty}\, t^\delta \\ &\le {\scriptstyle const}\ n^2 \frac{\omega^2}{\mathfrak{l}^2} |f|_{3,\Sigma}^{\sim}\,.\end{aligned}$$

□

Change of sectorization

To prepare for the proof of Proposition XIX.4, we note

Lemma XXI.3. *Let $j > i \ge 2$, $\frac{1}{M^{j-3/2}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$ and $\frac{1}{M^{i-3/2}} \le \mathfrak{l}' \le \frac{1}{M^{(i-1)/2}}$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively. Suppose that $\mathfrak{l} < \mathfrak{l}'$. Let $\varphi \in \mathcal{F}_m(n;\Sigma')$ and $f \in \check{\mathcal{F}}_m(n;\Sigma')$ be particle number conserving.*

(i) *For $s_1,\ldots,s_n \in \Sigma$*

$$\begin{aligned}&\|\varphi_\Sigma(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))\|_{1,\infty} \\ &\quad\le {\scriptstyle const}^n\ \mathfrak{c}_{j-1} \sum_{\substack{s'_1,\ldots,s'_n\in\Sigma' \\ \tilde s'_i\cap\tilde s_i\ne\emptyset}} \|\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s'_1),\ldots,(\xi_n,s'_n))\|_{1,\infty}\end{aligned}$$

and for $\check\eta_1,\ldots,\check\eta_m \in \check{\mathcal{B}}$ and $s_1,\ldots,s_n \in \Sigma$

$$\begin{aligned}&\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\text{ dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f_\Sigma(\check\eta_1,\ldots,\check\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))|||_{1,\infty}\, t^\delta \\ &\quad\le {\scriptstyle const}^n\ \mathfrak{c}_{j-1} \sum_{\substack{s'_1,\ldots,s'_n\in\Sigma' \\ \tilde s'_i\cap\tilde s_i\ne\emptyset}} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \\ &\qquad\times \max_{\substack{\mathrm{D}\text{ dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1,s'_1),\ldots,(\xi_n,s'_n))|||_{1,\infty}\, t^\delta\,.\end{aligned}$$

(ii) *If f is antisymmetric in its (ξ, s) arguments*

$$|f_\Sigma|^{\sim}_{p,\Sigma} \le const^n\ \mathfrak{c}_{j-1} |f|^{\sim}_{p,\Sigma'} \sup_{\check\eta_1,\dots,\check\eta_m \in \check{\mathcal{B}}_m} \max_{\substack{s_1,\dots,s_{p-m}\in\Sigma \\ s'_{p-m+1},\dots,s'_n\in\Sigma'}} \times \#\mathrm{Cons}(\check\eta_1,\dots,\check\eta_m; s_1,\dots,s_{p-m}; s'_{p-m+1},\dots,s'_n)$$

where $\mathrm{Cons}(\check\eta_1,\dots,\check\eta_m;\ s_1,\dots,s_{p-m};\ s'_{p-m+1},\dots,s'_n)$ *denotes the set of all* $(s_{p-m+1},\dots,s_n) \in \Sigma^{m+n-p}$ *such that* $\tilde s_i \cap \tilde s'_i \ne \emptyset$ for $i = p-m+1,\dots,n$ *and the configuration* $(\check\eta_1,\dots,\check\eta_m;\ s_1,\dots,s_n)$ *is consistent with conservation of momentum in the sense of Definition* XX.1.

(iii) *If $m = 0$, $\omega \ge \mathfrak{l}'$ and φ is antisymmetric, then*

$$|\varphi_\Sigma|_{1,\Sigma,\omega} \le const^n\ \mathfrak{c}_{j-1} |\varphi|_{1,\Sigma'} \max_{s_1\in\Sigma} \max_{\substack{s'_2,\dots,s'_n\in\Sigma' \\ \mathrm{dist}(s'_k,s'_\ell)\ge\omega-2\mathfrak{l}' \text{ and} \\ \mathrm{dist}(s'_k,a(s'_\ell))\ge\omega-2\alpha\mathfrak{l}' \\ \text{for some } 2\le k\ne\ell\le n}} \times \#\mathrm{Cons}(s_1;\ s'_2,\dots,s'_n)\,.$$

Here, α is the supremum of the derivative of the antipodal map a on the Fermi curve F. If $m = 1$, $\omega \ge \mathfrak{l}'$ and f is antisymmetric in its (ξ, s) arguments, then

$$|f_\Sigma|^{\sim}_{1,\Sigma,\omega} \le const^n\ \mathfrak{c}_{j-1} |f|^{\sim}_{1,\Sigma'} \sup_{\check\eta\in\check{\mathcal{B}}} \max_{\substack{s'_1,\dots,s'_n\in\Sigma' \\ \mathrm{dist}(s'_k,s'_\ell)\ge\omega-2\mathfrak{l}' \text{ and} \\ \mathrm{dist}(s'_k,a(s'_\ell))\ge\omega-2\alpha\mathfrak{l}' \\ \text{for some } 1\le k\ne\ell\le n}} \times \#\mathrm{Cons}(\check\eta;\ s'_1,\dots,s'_n)\,.$$

Proof. (i)

$$\varphi_\Sigma(\eta_1,\dots,\eta_m;\ (\xi_1,s_1),\dots,(\xi_n,s_n)) = \sum_{\substack{s'_1,\dots,s'_n\in\Sigma' \\ \tilde s'_i\cap\tilde s_i\ne\emptyset}} \int d\xi'_1\cdots d\xi'_n \varphi(\eta_1,\dots,\eta_m;(\xi'_1,s'_1),\dots,(\xi'_n,s'_n)) \times \prod_{\ell=1}^n \hat\chi_{s_\ell}(\xi'_\ell,\xi_\ell)\,.$$

Hence, by [9, Lemma II.7] and [8, Lemma XII.3],

$$\|\varphi_\Sigma(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_n,s_n))\|_{1,\infty} \le const^n\ \mathfrak{c}^n_{j-1} \sum_{\substack{s'_1,\dots,s'_n\in\Sigma' \\ \tilde s'_i\cap\tilde s_i\ne\emptyset}} \|\varphi(\eta_1,\dots,\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n))\|_{1,\infty}\,.$$

The proof of the second inequality is similar.

(ii) By part (i) and Remark XX.2(i)

$$
\begin{aligned}
|f_\Sigma|^{\sim}_{p,\Sigma} \le \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{s_1,\dots,s_{p-m}\in\Sigma\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}_m}} \sum_{s_{p-m+1},\dots,s_n\in\Sigma} \\
\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D} f_\Sigma(\check\eta_1,\dots,\check\eta_m;(\cdot,s_1),\dots,(\cdot,s_n))|||_{1,\infty}\frac{t^\delta}{\delta!} \\
\le const^n\ \mathfrak{c}_{j-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{s_1,\dots,s_{p-m}\in\Sigma\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}_m}} \sum_{s_{p-m+1},\dots,s_n\in\Sigma} \sum_{\substack{s'_1,\dots,s'_n\in\Sigma'\\ \tilde s'_i\cap\tilde s_i\ne\emptyset}} \\
\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D} f(\check\eta_1,\dots,\check\eta_m;(\cdot,s'_1),\dots,(\cdot,s'_n))|||_{1,\infty}\frac{t^\delta}{\delta!} \\
= const^n\ \mathfrak{c}_{j-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{s_1,\dots,s_{p-m}\in\Sigma\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}_m}} \\
\times \sum_{\substack{s'_1,\dots,s'_{p-m}\in\Sigma'\\ \tilde s'_i\cap\tilde s_i\ne\emptyset\\ i=1,\dots,p-m}} \sum_{s'_{p-m+1},\dots,s'_n\in\Sigma'} \sum_{\substack{s_{p-m+1},\dots,s_n\in\Sigma\\ \tilde s'_i\cap\tilde s_i\ne\emptyset\\ i=p-m+1,\dots,n}} \\
\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D} f(\check\eta_1,\dots,\check\eta_m;(\cdot,s'_1),\dots,(\cdot,s'_n))|||_{1,\infty}\frac{t^\delta}{\delta!} \\
\le const^n\ \mathfrak{c}_{j-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{s_1,\dots,s_{p-m}\in\Sigma\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}_m}} \sum_{\substack{s'_1,\dots,s'_{p-m}\in\Sigma'\\ \tilde s'_i\cap\tilde s_i\ne\emptyset\\ i=1,\dots,p-m}} \sum_{s'_{p-m+1},\dots,s'_n\in\Sigma'} \\
\times \#\mathrm{Cons}(\check\eta_1,\dots,\check\eta_m;s_1,\dots,s_{p-m};s'_{p-m+1},\dots,s'_n) \\
\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D} f(\check\eta_1,\dots,\check\eta_m;(\cdot,s'_1),\dots,(\cdot,s'_n))|||_{1,\infty}\frac{t^\delta}{\delta!} \\
\le const^n\ \mathfrak{c}_{j-1}|f|^{\sim}_{p,\Sigma'} \sup_{\check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}_m} \max_{\substack{s_1,\dots,s_{p-m}\in\Sigma\\ s'_{p-m+1},\dots,s'_n\in\Sigma'}} \\
\times \#\mathrm{Cons}(\check\eta_1,\dots,\check\eta_m;s_1,\dots,s_{p-m};s'_{p-m+1},\dots,s'_n)
\end{aligned}
$$

since, for $i=1,\dots,p-m$, there are at most three sectors $s'_i\in\Sigma'$ with $\tilde s'_i\cap\tilde s_i \ne\emptyset$.

(iii) If $s_k,s_\ell\in\Sigma$ with $\mathrm{dist}(s_k,s_\ell)\ge\omega$, $\mathrm{dist}(s_k,a(s_\ell))\ge\omega$ and $s'_k,s'_\ell\in\Sigma'$ with $\tilde s_k\cap\tilde s'_k\ne\emptyset$, $\tilde s_\ell\cap\tilde s'_\ell\ne\emptyset$ then $\mathrm{dist}(s'_k,s'_\ell)\ge\omega-2\mathfrak{l}'$ and $\mathrm{dist}(s'_k,a(s'_\ell))\ge\omega-2\alpha\mathfrak{l}'$. Using this observation, the proof of (iii) is analogous to the proof of (ii). □

Lemma XXI.4. *Let $j > i \geq 2$, $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$ and $\frac{1}{M^{i-3/2}} \leq \mathfrak{l}' \leq \frac{1}{M^{(i-1)/2}}$ with $\mathfrak{l} < \frac{1}{4}\mathfrak{l}'$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively.*

(i) *There is a constant const independent of M such that for every $s' \in \Sigma'$*

$$\#\{s \in \Sigma | \tilde{s} \cap \tilde{s}' \neq \emptyset\} \leq \mathrm{const}\, \frac{\mathfrak{l}'}{\mathfrak{l}} .$$

(ii) *Let $m \geq 0$, $p \geq m$, $n \geq p - m + 1$, $\check{\eta}_1, \ldots, \check{\eta}_m \in \mathcal{B}$, $s_1, \ldots, s_{p-m} \in \Sigma$ and $s'_{p-m+1}, \ldots, s'_n \in \Sigma'$. Then*

$$\#\mathrm{Cons}(\check{\eta}_1, \ldots, \check{\eta}_m; s_1, \ldots, s_{p-m}; s'_{p-m+1}, \ldots, s'_n) \leq \mathrm{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-p-1} .$$

(iii) *Let $\omega' \geq 4\mathfrak{l}'$, and let $s_1 \in \Sigma$ and $s'_2, \ldots, s'_n \in \Sigma'$ such that $\mathrm{dist}(s'_k, s'_\ell) \geq \omega'$ and $\mathrm{dist}(s'_k, a(s'_\ell)) \geq \omega'$ for some $2 \leq k \neq \ell \leq n$. Then*

$$\#\mathrm{Cons}(s_1; s'_2, \ldots, s'_n) \leq \mathrm{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n-3} \left(1 + \frac{1}{M^{j-1}\mathfrak{l}\omega'}\right).$$

(iv) *Let $\omega' \geq 4\mathfrak{l}'$, $\check{\eta} = (q_0, \mathbf{q}, \sigma, a) \in \check{B}$ and $s'_1, \ldots, s'_n \in \Sigma'$ such that $\mathrm{dist}(s'_k, s'_\ell) \geq \omega'$ and $\mathrm{dist}(s'_k, a(s'_\ell)) \geq \omega'$ for some $1 \leq k \neq \ell \leq n$. Then*

$$\#\mathrm{Cons}(\check{\eta}; s'_1, \ldots, s'_n) \leq \mathrm{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n-2} \left(1 + \frac{1}{M^{j-1}\mathfrak{l}\omega'}\right).$$

Proof. (i) is trivial.

(ii) By part (i), there are at most $\mathrm{const}^n(\frac{\mathfrak{l}'}{\mathfrak{l}})^{n+m-p-1}$ $(n+m-p-1)$-tuples $(s_{p-m+1}, \ldots, s_{n-1})$ of sectors in Σ such that $\tilde{s}_i \cap \tilde{s}'_i \neq \emptyset$ for $i = p-m+1, \ldots, n-1$. Given such an $(n+m-p-1)$-tuple $(s_{p-m+1}, \ldots, s_{n-1})$ and a particle number preserving sequence $(a_1, \ldots, a_n)$ of creation–annihilation indices, the set

$$\{-(-1)^{a_n}(\check{\eta}_1 + \cdots + \check{\eta}_m + (-1)^{a_1}\mathbf{k}_1 + \cdots + (-1)^{a_{n-1}}\mathbf{k}_{n-1}) | \mathbf{k}_i \in \tilde{s}_i \text{ for } i = 1, \ldots, n-1\}$$

has diameter at most $\mathrm{const}(n-1)\mathfrak{l}$ and therefore meets at most $\mathrm{const}(n-1)$ extended sectors of Σ. This shows that there are at most const^n sectors $s_n \in \Sigma$ such that $(s_1, \ldots, s_n)$ is consistent with conservation of momentum.

(iii) Let $(a_1, \ldots, a_n)$ be a particle number preserving sequence of creation–annihilation indices. For $i = 1, \ldots, n-1$ let $I_i = \{\mathbf{k} \in F | \mathrm{dist}(\mathbf{k}, s'_{i+1}) \leq \mathfrak{l}\}$. We apply the first inequality of Proposition XX.11 with $\delta = \mathfrak{l}' + 2\mathfrak{l}$, $\Lambda = \frac{\sqrt{2}}{M^{j-1}}$, Γ the set of centers of the intervals $s \cap F$, $s \in \Sigma$ and $\mathbf{p}$ the center of $s_1 \cap F$. It follows that

$$\#\mathrm{Cons}(s_1;\ s'_2, \ldots, s'_n) \leq \mathrm{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n-3} \left(1 + \frac{1}{M^{j-1}\mathfrak{l}\omega'}\right).$$

(iv) is similar to (iii), using the second inequality of Proposition XX.11 instead. □

Proof of Proposition XIX.4. (i) As $m \neq 0$, by Lemmas XXI.3(i) and XXI.4(i)

$$
\begin{aligned}
|\varphi_\Sigma|_{1,\Sigma} &= \sum_{s_1,\ldots,s_n \in \Sigma} \|\varphi_\Sigma(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))\|_{1,\infty} \\
&\le \mathit{const}^n \; \mathfrak{c}_{j-1} \sum_{\substack{s_1,\ldots,s_n \in \Sigma \\ s'_1,\ldots,s'_n \in \Sigma' \\ \tilde{s}'_i \cap \tilde{s}_i \neq \emptyset}} \|\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s'_1),\ldots,(\xi_n,s'_n))\|_{1,\infty} \\
&\le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^n \sum_{s'_1,\ldots,s'_n \in \Sigma'} \|\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s'_1),\ldots,(\xi_n,s'_n))\|_{1,\infty} \\
&= \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^n |\varphi|_{1,\Sigma'} \,.
\end{aligned}
$$

(ii) By Lemmas XXI.3(ii) and XXI.4(ii)

$$
|f_\Sigma\widetilde{\rceil}_{p,\Sigma} \le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-p-1} |f\widetilde{\rceil}_{p,\Sigma'} \,. \tag{XXI.2}
$$

Now assume that $\mathfrak{l} \ge \frac{1}{M^{2/3(j-1)}}$, $\mathfrak{l}' \le \frac{1}{16}$ and $n \ge 3$. Observe that $|f_\Sigma\widetilde{\rceil}_{1,\Sigma}$ vanishes for $m \ge 2$, so it suffices to consider $m = 0, 1$. Set $\omega = \alpha\sqrt{\mathfrak{l}}$. By Lemma XXI.2

$$
|f_\Sigma\widetilde{\rceil}_{1,\Sigma} \le |f_\Sigma\widetilde{\rceil}_{1,\Sigma,\omega} + \mathit{const}\; n^2 \frac{\omega^2}{\mathfrak{l}^2} |f_\Sigma\widetilde{\rceil}_{3,\Sigma} \,.
$$

By (XXI.2),

$$
\begin{aligned}
n^2 \frac{\omega^2}{\mathfrak{l}^2} |f_\Sigma\widetilde{\rceil}_{3,\Sigma} &\le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-4} \frac{\omega^2}{\mathfrak{l}^2} |f\widetilde{\rceil}_{3,\Sigma'} \\
&\le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-4} \frac{1}{\mathfrak{l}} |f\widetilde{\rceil}_{3,\Sigma'} \\
&= \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-3} \frac{1}{\mathfrak{l}'} |f\widetilde{\rceil}_{3,\Sigma'} \,.
\end{aligned}
$$

If $m = 0$, by Lemmas XXI.3(iii) and XXI.4(iii), with $\omega' = \omega - 2\alpha\mathfrak{l}'$,

$$
\begin{aligned}
|f_\Sigma\widetilde{\rceil}_{1,\Sigma,\omega} &\le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-3} \left(1 + \frac{1}{M^{j-1}\mathfrak{l}(\omega - 2\alpha\mathfrak{l}')}\right) |f\widetilde{\rceil}_{1,\Sigma'} \\
&\le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-3} |f\widetilde{\rceil}_{1,\Sigma'}
\end{aligned}
$$

since $M^{j-1}\mathfrak{l}(\omega - 2\alpha\mathfrak{l}') \ge M^{j-1}\mathfrak{l}(\alpha\sqrt{\mathfrak{l}} - \frac{\alpha}{3}\sqrt{\mathfrak{l}}) = \frac{2}{3}\alpha M^{j-1}\mathfrak{l}^{3/2} \ge \frac{2}{3}\alpha$. Similarly one sees, using Lemma XXI.4(iv), that also in the case $m = 1$

$$
|f_\Sigma\widetilde{\rceil}_{1,\Sigma,\omega} \le \mathit{const}^n \; \mathfrak{c}_{j-1} \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{n+m-3} |f\widetilde{\rceil}_{1,\Sigma'} \,.
$$

(iii) Write

$$f_{\Sigma'}(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n)) = \sum_{s_1,\dots,s_n\in\Sigma} g(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n);s_1,\dots,s_n)$$

with

$$g(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n);s_1,\dots,s_n) = \int d\xi'_1\cdots d\xi'_n f(\check\eta_1,\dots,\check\eta_m;(\xi'_1,s_1),\dots,(\xi'_n,s_n))\prod_{\ell=1}^{n}\hat\chi_{s'_\ell}(\xi'_\ell,\xi_\ell)\,.$$

Then

$$\begin{aligned}|f_{\Sigma'}|^{\sim}_{p,\Sigma'} &= \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\ \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n\\ s'_{i_1},\dots,s'_{i_{p-m}}\in\Sigma'\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}}}\ \sum_{\substack{s'_i\in\Sigma'\text{ for}\\ i\ne i_1,\dots,i_{p-m}}}\\ &\quad\times\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}f_\Sigma(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n))|||_{1,\infty}\frac{t^\delta}{\delta!}\\ &\le \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\ \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n\\ s'_{i_1},\dots,s'_{i_{p-m}}\in\Sigma'\\ \check\eta_1,\dots,\check\eta_m\in\check{\mathcal{B}}}}\ \sum_{\substack{s'_i\in\Sigma'\text{ for}\\ i\ne i_1,\dots,i_{p-m}}}\ \sum_{\substack{s_1,\dots,s_n\in\Sigma\\ s_i\cap s'_i\ne\emptyset\text{ for }1\le i\le n}}\\ &\quad\times\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}g(\check\eta_1,\dots,\check\eta_m;\ (\xi_1,s'_1),\dots,(\xi_n,s'_n);\ s_1,\dots,s_n)|||_{1,\infty}\frac{t^\delta}{\delta!}\,.\end{aligned}$$

For each fixed $\check\eta_1,\dots,\check\eta_m,s'_1,\dots,s'_n,s_1,\dots,s_n$,

$$\begin{aligned}&\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}g(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n);s_1,\dots,s_n)|||_{1,\infty}\,t^\delta\\ &\quad\le \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\dots,\check\eta_m;(\xi'_1,s_1),\dots,(\xi'_n,s_n))|||_{1,\infty}\\ &\qquad\times t^\delta\prod_{\ell=1}^{n}\|\hat\chi_{s'_\ell}\|_{1,\infty}\end{aligned}$$

as in [9, Lemma II.7]. Hence, by [8, Lemma XII.3] and [9, Example A.3],

$$\begin{aligned}&\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}g(\check\eta_1,\dots,\check\eta_m;(\xi_1,s'_1),\dots,(\xi_n,s'_n);s_1,\dots,s_n)|||_{1,\infty}\,t^\delta\\ &\quad\le \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\mathrm{D}\text{ dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\dots,\check\eta_m;(\xi'_1,s_1),\dots,(\xi'_n,s_n))|||_{1,\infty}\end{aligned}$$

$$\times\, t^\delta \prod_{i=1}^{n} const\ \mathfrak{c}_{i-1} \le const^n\ \mathfrak{c}_{i-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!}$$

$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\ \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1',s_1),\ldots,(\xi_n',s_n))|||_{1,\infty}\, t^\delta$$

uniformly in $s_1',\ldots,s_n',s_1,\ldots,s_n,\check\eta_1,\ldots,\check\eta_m$. So

$$|f_{\Sigma'}|^{\sim}_{p,\Sigma'} \le const^n\ \mathfrak{c}_{i-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n\\ s_{i_1}',\ldots,s_{i_{p-m}}'\in\Sigma'\\ \check\eta_1,\ldots,\check\eta_m\in\check{\mathcal{B}}}} \sum_{\substack{s_i'\in\Sigma'\ \text{for}\\ i\ne i_1,\ldots,i_{p-m}}} \sum_{\substack{s_1,\ldots,s_n\in\Sigma\\ s_i\cap s_i'\ne\emptyset\ \text{for}\ 1\le i\le n}}$$

$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\ \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1',s_1),\ldots,(\xi_n',s_n))|||_{1,\infty}\, \frac{t^\delta}{\delta!}$$

$$\le const^n\ \mathfrak{c}_{i-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n\\ s_{i_1}',\ldots,s_{i_{p-m}}'\in\Sigma'\\ \check\eta_1,\ldots,\check\eta_m\in\check{\mathcal{B}}}}$$

$$\times \sum_{\substack{s_i\in\Sigma\\ s_i\cap s_i'\ne\emptyset\\ \text{for}\ i=i_1,\ldots,i_{p-m}}} \sum_{\substack{s_i\in\Sigma\ \text{for}\\ i\ne i_1,\ldots,i_{p-m}}} \sum_{\substack{s_i'\in\Sigma'\\ s_i\cap s_i'\ne\emptyset\\ \text{for}\ i\ne i_1,\ldots,i_{p-m}}}$$

$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\ \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1',s_1),\ldots,(\xi_n',s_n))|||_{1,\infty}\, \frac{t^\delta}{\delta!}$$

$$\le const^n\ \mathfrak{c}_{i-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n\\ s_{i_1}',\ldots,s_{i_{p-m}}'\in\Sigma'\\ \check\eta_1,\ldots,\check\eta_m\in\check{\mathcal{B}}}} \sum_{\substack{s_i\in\Sigma\\ s_i\cap s_i'\ne\emptyset\\ \text{for}\ i=i_1,\ldots,i_{p-m}}} \sum_{\substack{s_i\in\Sigma\ \text{for}\\ i\ne i_1,\ldots,i_{p-m}}}$$

$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\ \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1',s_1),\ldots,(\xi_n',s_n))|||_{1,\infty}\, \frac{t^\delta}{\delta!}$$

since the set $\{s_i'\in\Sigma', i\ne i_1,\ldots,i_{p-m} | s_i\cap s_i'\ne\emptyset \text{ for } i\ne i_1,\ldots,i_{p-m}\}$ contains at most 3^n terms. Finally, applying

$$\sup_{s_{i_1}',\ldots,s_{i_{p-m}}'\in\Sigma'} \sum_{\substack{s_i\in\Sigma\\ s_i\cap s_i'\ne\emptyset\\ \text{for}\ i=i_1,\ldots,i_{p-m}}} h(s_1,\ldots,s_{i_{p-m}})$$

$$\le \left(const\ \frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{p-m} \sup_{s_{i_1},\ldots,s_{i_{p-m}}\in\Sigma} h(s_1,\ldots,s_{i_{p-m}})$$

yields

$$|f_{\Sigma'}|^{\sim}_{p,\Sigma'} \le \mathit{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{p-m} \mathfrak{c}_{i-1} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \sup_{\substack{1\le i_1<\cdots<i_{p-m}\le n \\ s_{i_1},\ldots,s_{i_{p-m}}\in\Sigma \\ \check\eta_1,\ldots,\check\eta_m\in\check{\mathcal{B}}}} \sum_{\substack{s_i\in\Sigma \text{ for} \\ i\ne i_1,\ldots,i_{p-m}}}$$

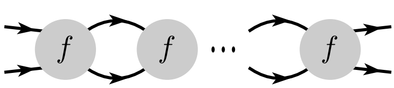
$$\times \max_{\substack{\mathrm{D}\ \text{dd-operator} \\ \text{with } \delta(\mathrm{D})=\delta}} |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_m;(\xi_1',s_1),\ldots,(\xi_n',s_n))|||_{1,\infty}\frac{t^\delta}{\delta!}$$

$$\le \mathit{const}^n \left(\frac{\mathfrak{l}'}{\mathfrak{l}}\right)^{p-m} \mathfrak{c}_{i-1}|f|^{\sim}_{p,\Sigma}\,.$$

□

XXII. Sector Counting for Particle–Particle Ladders

In this section we prove that, when the Fermi surface F is strongly asymmetric in the sense of Definition XVIII.3, particle–particle ladders obey bounds that are stronger than those given by standard power counting. The precise formulation of this result is given in Theorem XXII.7. It bounds the $|\cdot|_{3,\Sigma}$ norm of a particle–particle ladder $L_\ell(f;C,D)$ with $\ell+1$ vertices f and propagators determined by C and D. Such a ladder looks like



Each line of this ladder is either C or D with at least one C in each of the ℓ rungs.[a] The detailed definition of $L_\ell(f;C,D)$ is given in Definition XIV.1. When applied to four-legged kernels, the $|\cdot|_{3,\Sigma}$ norm measures roughly the suprmemum in momentum space of the kernel and its derivatives. Naive power counting, as in Appendix D, leads to a bound on $|L_\ell(f;C,D)|^{\sim}_{3,\Sigma}$ of order $|f|^{\sim\,\ell+1}_{3,\Sigma}$. In this section, we use sector counting to implement the geometric argument outlined in [1, Sec. II, Subsec. 5] exploiting the asymmetry of the Fermi surface to improve the bound to one of order $(\mathfrak{l}^{1/n_0})^\ell\ |f|^{\sim\,\ell+1}_{3,\Sigma}$. The main sector counting result is

Proposition XXII.1. *Assume that the Fermi surface F is strongly asymmetric. There is a constant const independent of M such that for all sectorizations of scale $j\ge 2$ and length $\mathfrak{l}\ge\frac{1}{M^{j-1}}$ and all $s_1',s_2'\in\Sigma$ and all $k_1,k_2\in\mathbb{R}\times\mathbb{R}^2$*

$$\sharp\{(s_1,s_2)\in\Sigma\times\Sigma\,|\,(\tilde s_1+\tilde s_2)\cap(\tilde s_1'+\tilde s_2')\ne\emptyset\}\le \mathit{const}\ \frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}$$

$$\sharp\{(s_1,s_2)\in\Sigma\times\Sigma\,|\,(\tilde s_1+\tilde s_2)\cap(k_1+\tilde s_1')\ne\emptyset\}\le \mathit{const}\ \frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}$$

$$\sharp\{(s_1,s_2)\in\Sigma\times\Sigma\,|\,k_1+k_2\in\tilde s_1+\tilde s_2\}\le \mathit{const}\ \frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}\,.$$

[a]The reader should think of C as a "hard" propagator and D as a "soft" propagator arising from Wick ordering.

The proof of this proposition, which is given after Proposition XXII.4, is based on the following three lemmas.

Lemma XXII.2. *Assume that F is strongly asymmetric. There exists a constant* $_{const}$ *such that for all $\varepsilon > 0$ and all disks D in $\mathbb{R}^2$ of radius ε*

$$\text{length}\,\{\mathbf{k} \in F | \mathbf{k} + a(\mathbf{k}) \in D\} \le {\scriptstyle const}\; \varepsilon^{1/(n_0-1)}$$

where n_0 is the constant of Definition XVIII.3 *and $a(\mathbf{k})$ is the antipode of* $\mathbf{k}$.

Proof. Since F is compact, it suffices to show that each point $\mathbf{p} \in F$ has a neighborhood U in F for which there exists $1 \le n \le n_0 - 1$ and a constant $_{const}$ such that, for all $\varepsilon > 0$ and all disks D in $\mathbb{R}^2$ of radius ε,

$$\text{length}\,\{\mathbf{k} \in U | \mathbf{k} + a(\mathbf{k}) \in D\} \le {\scriptstyle const}\; \varepsilon^{1/n}\,.$$

Fix $\mathbf{p} \in F$. Without loss of generality, we may assume that the oriented unit tangent vector to F at $\mathbf{p}$ is $(1, 0)$ and that the unit inward pointing normal vector to F at $\mathbf{p}$ is $(0, 1)$. Let $\varphi(t) = \varphi_{\mathbf{p}}(t)$, $\bar\varphi(t) = \varphi_{a(\mathbf{p})}(t)$, where $\varphi_{\mathbf{p}}$ is the parameterizing map of Definition XVIII.3. Precisely, $t \mapsto \mathbf{k}(t) = \mathbf{p} + (t, \varphi(t))$ is a parameterization of F near $\mathbf{p}$ and $t \mapsto \bar{\mathbf{k}}(t) = a(\mathbf{p}) - (t, \bar\varphi(t))$ is a parameterization of F near $a(\mathbf{p})$.

By strict convexity, the slopes $\dot\varphi(t)$ and $\dot{\bar\varphi}(t)$ for the Fermi curve at $\mathbf{k}(t)$ and $\bar{\mathbf{k}}(t)$, respectively, are strictly increasing with t. Hence there is a strictly increasing function $\bar t(t)$ such that

$$\dot{\bar\varphi}(\bar t(t)) = \dot\varphi(t) \tag{XXII.1}$$

and hence

$$\bar{\mathbf{k}}(\bar t(t)) = a(\mathbf{k}(t))$$

so that

$$\mathbf{k}(t) + a(\mathbf{k}(t)) = \mathbf{p} + a(\mathbf{p}) + (t - \bar t(t), \varphi(t) - \bar\varphi(\bar t(t)))\,.$$

By construction, $\varphi(0) = \bar\varphi(0) = \dot\varphi(0) = \dot{\bar\varphi}(0) = 0$. Since F is strongly asymmetric, there is a minimal $1 \le n \le n_0 - 1$ such that $\bar\varphi^{(n+1)}(0) \ne \varphi^{(n+1)}(0)$. We may assume, without loss of generality, that

$$|\bar\varphi^{(n+1)}(0)| < |\varphi^{(n+1)}(0)|\,. \tag{XXII.2}$$

Since the curvature of F is assumed to be bounded away from zero, the second derivatives of both φ and $\bar\varphi$ are nonzero. Thus

$$\begin{aligned} &\dot\varphi(0) = \dot{\bar\varphi}(0) = 0\,, && \ddot\varphi(0), \ddot{\bar\varphi}(0) \ne 0\,,\\ &\varphi^{(i)}(0) = \bar\varphi^{(i)}(0) \quad \text{for } 1 \le i \le n\,, && \varphi^{(n+1)}(0) \ne \bar\varphi^{(n+1)}(0)\,. \end{aligned}$$

Using (XXII.1) we conclude that $\bar t(t)$ is C^n and obeys

$$\bar t^{(i)}(0) = \left\{\begin{matrix} 1 & \text{if } i = 1 \\ 0 & \text{if } 1 < i \le n-1 \\ \tilde b \ne 0 & \text{if } i = n \end{matrix}\right\} \quad \text{if } n > 1 \quad \text{and} \quad \dot{\bar t}(0) = \tilde b \ne 1 \quad \text{if } n = 1\,.$$

Consequently, there is a neighborhood U' of 0 and a $b > 0$ such that for all $t \in U'$

$$\left| \frac{d^n}{dt^n}(t - \bar{t}(t)) \right| \geq b\,. \tag{XXII.3}$$

Set $U = \{\mathbf{p} + (t, \varphi(t)) | t \in U'\}$. If D is a disk of radius ε, then its projection to the x-axis is an interval $\mathcal{J}$ of length 2ε and

$$\text{length}\,\{\mathbf{k} \in U | \mathbf{k} + a(\mathbf{k}) \in D\} \leq_{const} \text{length}\,\{t \in U' | x_0 + t - \bar{t}(t) \in \mathcal{J}\}$$

where x_0 is the x-component of $\mathbf{p} + a(\mathbf{p})$. Therefore, by (XXII.3), this lemma follows from Lemma XXII.3 below. □

Lemma XXII.3. *Let b be a strictly positive real number and n be a strictly positive integer. Let $\mathcal{I} \subset \mathbb{R}$ be an interval (not necessarily compact) and f a C^n function on $\mathcal{I}$ obeying*

$$|f^{(n)}(x)| \geq b \quad \textit{for all } x \in \mathcal{I}$$

Then for all $\varepsilon > 0$ and all intervals $\mathcal{J}$ of length 2ε,

$$\text{length}\,\{x \in \mathcal{I} | f(x) \in \mathcal{J}\} \leq 2^{n+1} \left(\frac{\varepsilon}{b}\right)^{1/n}\,.$$

Proof. Set $\alpha = (\frac{\varepsilon}{b})^{1/n}$ and $g(x) = f(x) - y_0$, where y_0 is the midpoint of $\mathcal{J}$. We must show

$$|g^{(n)}(x)| \geq \frac{\varepsilon}{\alpha^n} \quad \text{for all } x \in \mathcal{I} \Longrightarrow \text{length}\,\{x \in \mathcal{I} |\ |g(x)| \leq \varepsilon\} \leq 2^{n+1}\alpha\,.$$

Define c_n inductively by $c_1 = 2$ and $c_n = 2 + 2c_{n-1}$. Because $d_n = 2^{-n}c_n$ obeys $d_1 = 1$ and $d_n = 2^{-n+1} + d_{n-1}$ we have $d_n \leq 2$ and hence $c_n \leq 2^{n+1}$. We shall prove

$$|g^{(n)}(x)| \geq \frac{\varepsilon}{\alpha^n} \quad \text{for all } x \in \mathcal{I} \Longrightarrow \text{length}\,\{x \in \mathcal{I} |\ |g(x)| \leq \varepsilon\} \leq c_n\alpha$$

by induction on n.

Suppose that $n = 1$ and let x and y be any two points in $\{x \in \mathcal{I} |\, |g(x)| \leq \varepsilon\}$. Then

$$|x - y| = \frac{|x-y|}{|g(x) - g(y)|}|g(x) - g(y)| = \frac{|g(x) - g(y)|}{|g'(\zeta)|} \leq \frac{2\varepsilon}{|g'(\zeta)|}$$

for some $\zeta \in \mathcal{I}$. As $|g'(\zeta)| \geq \frac{\varepsilon}{\alpha}$ we have $|x - y| \leq 2\alpha$. Thus $\{x \in \mathcal{I} |\, |g(x)| \leq \varepsilon\}$ is contained in an interval of length at most 2α as desired.

Now suppose that $|g^{(n)}(x)| \geq \frac{\varepsilon}{\alpha^n}$ on $\mathcal{I}$ and that the induction hypothesis is satisfied for $n-1$. As in the last paragraph, the set $\{x \in \mathcal{I} |\, |g^{(n-1)}(x)| \leq \frac{\varepsilon}{\alpha^{n-1}}\}$ is contained in a subinterval $\mathcal{I}_0$ of $\mathcal{I}$ of length at most 2α. Then $\mathcal{I} \setminus \mathcal{I}_0$ is the union of at most two other intervals $\mathcal{I}_+, \mathcal{I}_-$ on which $|g^{(n-1)}(x)| \geq \frac{\varepsilon}{\alpha^{n-1}}$. By the inductive hypothesis

$$\begin{aligned}\text{length}\,\{x \in \mathcal{I} |\ |g(x)| \leq \varepsilon\} &\leq \text{length}\,(\mathcal{I}_0) + \sum_{i=\pm} \text{length}\,\{x \in \mathcal{I}_i |\ |g(x)| \leq \varepsilon\} \\ &\leq 2\alpha + 2c_{n-1}\alpha = c_n\alpha\,.\end{aligned}$$

□

Proposition XXII.4. *Assume that F is strongly asymmetric. Let Γ be an ε-separated set in F and R a square of side length 8ε in $\mathbb{R}^2$. Then*

$$\#\{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma|\gamma_1+\gamma_2\in R\}\le {\scriptstyle const}\ \frac{\varepsilon^{1/n_0}}{\varepsilon}$$

with ${\scriptstyle const}$ depending only on the geometry of F. Here n_0 is the constant of Definition XVIII.3.

Proof. Let $\omega_1=\varepsilon^{1-\frac{1}{n_0}}$ and

$$X_1=\{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma|\gamma_1+\gamma_2\in R,\ \min\{d(\gamma_1,\gamma_2),d(a(\gamma_1),\gamma_2)\}\ge\omega_1\}$$

$$X_2=\{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma|\gamma_1+\gamma_2\in R,\ d(\gamma_1,\gamma_2)\le\omega_1\}$$

$$X_3=\{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma|\gamma_1+\gamma_2\in R,\ d(a(\gamma_1),\gamma_2)\le\omega_1\}\,.$$

By Lemma XX.8, part (iv), with an arbitrary point $\mathbf{p}$ and ω_2 large enough,

$$\#X_1\le\frac{{\scriptstyle const}}{\omega_1}={\scriptstyle const}\ \frac{\varepsilon^{1/n_0}}{\varepsilon}\,.$$

Next observe that, for any given $\gamma_1\in\Gamma$, the length of $\{\gamma_1+\mathbf{k}|\mathbf{k}\in F\}\cap R$ is bounded by ${\scriptstyle const}\ \varepsilon$, so that

$$\#\{\gamma_2\in\Gamma|\gamma_1+\gamma_2\in R\}\le{\scriptstyle const}\,. \tag{XXII.4}$$

If, for some $\gamma_1\in\Gamma$, there exists $\gamma_2\in\Gamma$ such that $(\gamma_1,\gamma_2)\in X_2$, then $2\gamma_1=\gamma_1+\gamma_2+(\gamma_1-\gamma_2)$ lies in the disk D of radius $8\varepsilon+\omega_1$ centered at the center of R. Since

$$\text{length}\,\{\mathbf{k}\in F|2\mathbf{k}\in D\}\le{\scriptstyle const}\ \omega_1$$

there are at most ${\scriptstyle const}\ \frac{\omega_1}{\varepsilon}$ choices of $\gamma_1\in\Gamma$ with $2\gamma_1\in D$. By (XXII.4) this implies that

$$\#X_2\le{\scriptstyle const}\ \frac{\omega_1}{\varepsilon}={\scriptstyle const}\ \varepsilon^{-1/n_0}\le{\scriptstyle const}\ \frac{\varepsilon^{1/n_0}}{\varepsilon}$$

since $n_0\ge2$. If, for some $\gamma_1\in\Gamma$, there exists $\gamma_2\in\Gamma$ such that $(\gamma_1,\gamma_2)\in X_3$, then $\gamma_1+a(\gamma_1)\in D$. By Lemma XXII.2

$$\text{length}\,\{\mathbf{k}\in F|\mathbf{k}+a(\mathbf{k})\in D\}\le{\scriptstyle const}\ \omega_1^{\frac{1}{n_0-1}}\,.$$

Consequently

$$\#X_3\le{\scriptstyle const}\ \frac{\omega_1^{\frac{1}{n_0-1}}}{\varepsilon}={\scriptstyle const}\ \frac{\varepsilon^{1/n_0}}{\varepsilon}\,.$$

□

Proof of Proposition XXII.1. For each sector $s\in\Sigma_j$ let γ_s be the center of $s\cap F$. Then $\Gamma=\{\gamma_s|s\in\Sigma_j\}$ is a $\frac{3}{4}\mathfrak{l}_j$ separated set. Clearly $\tilde{s}'_1+\tilde{s}'_2$ is contained in the disk of radius ${\scriptstyle const}'\ \mathfrak{l}_j$ around $\gamma_{s'_1}+\gamma_{s'_2}$. Therefore $(\tilde{s}_1+\tilde{s}_2)\cap(\tilde{s}'_1+\tilde{s}'_2)\ne\emptyset$ only if $\gamma_{s_1}+\gamma_{s_2}$ is contained in the disk of radius $2\,{\scriptstyle const}'\ \mathfrak{l}_j$ around $\gamma_{s'_1}+\gamma_{s'_2}$. So the first part of the proposition follows directly from Proposition XXII.4, applied $(\frac{4\,const'}{8}\times\frac{4}{3})^2$ times. The other two parts are similar. □

Definition XXII.5. (i) The creation/annihilation index of $z\in\check{\mathcal{B}}\cup(\mathcal{B}\times\Sigma)$ is

$$b(z)=\begin{cases} b & \text{if } z=(k,\sigma,b)\in\check{\mathcal{B}}\\ b & \text{if } z=(x,\sigma,b,s)\in\mathcal{B}\times\Sigma\,.\end{cases}$$

(ii) Let $f\in\check{\mathcal{F}}_{4;\Sigma}$. We say that f is of particle–particle type if

$$f(z_1,z_2,z_3,z_4)=0\quad\text{unless } b(z_1)=b(z_2)=0\,,\quad b(z_3)=b(z_2)=1\,.$$

Lemma XXII.6. *Let $f\in\check{\mathcal{F}}_{4;\Sigma}$ be of particle–particle type. Then,*

$$|f|^{\sim}_{ch,\Sigma}\le const\,\frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma}$$

with the channel norm $|\cdot|^{\sim}_{ch,\Sigma}$ of [8, *Definition* D.1].

Proof. It suffices to consider $f\in\check{\mathcal{F}}_r(4-r,\Sigma)$ with $r\le 2$. As in the proof of [8, Lemma D.2], set

$$\begin{aligned}&F(\check{\eta}_1,\ldots,\check{\eta}_r;\ s_1,\ldots,s_{4-r})\\&\quad=\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\text{D dd-operator}\\ \text{with }\delta(\text{D})=\delta}}|||\text{D}f(\check{\eta}_1,\ldots,\check{\eta}_r;\ (\xi_1,s_1),\ldots,(\xi_{4-r},s_{4-r}))|||_{1,\infty}\,t^{\delta}\,.\end{aligned}$$

Then, by Proposition XXII.1,

$$\begin{aligned}|f|^{\sim}_{ch,\Sigma}&=\sup_{\substack{\check{\eta}_1,\ldots,\check{\eta}_r\in\check{\mathcal{B}}\\ s_1,\ldots,s_{2-r}\in\Sigma}}\ \sum_{s_{3-r},s_{4-r}\in\Sigma}F(\check{\eta}_1,\ldots,\check{\eta}_r;\ s_1,\ldots,s_{4-r})\\&\qquad\times const\,\frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}\sup_{\substack{\check{\eta}_1,\ldots,\check{\eta}_r\in\check{\mathcal{B}}\\ s_1,\ldots,s_{3-r}\in\Sigma}}\ \sum_{s_{4-r}\in\Sigma}F(\check{\eta}_1,\ldots,\check{\eta}_r;\ s_1,\ldots,s_{4-r})\\&\le const\,\frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}\sup_{\substack{1\le i_1<\cdots<i_{3-r}\le 4-r\\ s_{i_1},\ldots,s_{i_{3-r}}\in\Sigma\\ \check{\eta}_1,\ldots,\check{\eta}_r\in\check{\mathcal{B}}}}\ \sum_{\substack{s_i\in\Sigma\text{ for}\\ i\ne i_1,\ldots,i_{3-r}}}F(\check{\eta}_1,\ldots,\check{\eta}_r;\ s_1,\ldots,s_{4-r})\\&=const\,\frac{\mathfrak{l}^{1/n_0}}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma}\,.\end{aligned}$$

□

Theorem XXII.7. *Let Σ be a sectorization of scale $j\ge 2$ and length $\frac{1}{M^{j-3/2}}\le\mathfrak{l}\le\frac{1}{M^{(j-1)/2}}$. Let $u((\xi,s),(\xi',s')),v((\xi,s),(\xi',s'))\in\mathcal{F}_0(2;\Sigma)$ be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms obey $|\check{u}(k)|,|\check{v}(k)|\le\frac{1}{2}|\imath k_0-e(k)|$. Furthermore, let $X\in\mathfrak{N}_3$ and assume that $|u|_{1,\Sigma}\le\frac{1}{2}X$ and $M^jX_{\mathbf{0}}\le\min\{\tau_1,\tau_2\}$, where τ_1 and τ_2 are the constants of Proposition* XIII.5 *and* [8, *Lemma* XIII.6], *respectively. Set*

$$C(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)}\,, \qquad D(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}(k)}$$

and let $C(\xi,\xi')$, $D(\xi,\xi')$ *be the Fourier transforms of* $C(k)$, $D(k)$ *as in* [6, *Definition* IX.3]. *Furthermore, let* $f \in \check{\mathcal{F}}_{4;\Sigma}$ *be of particle–particle type. If the Fermi curve* F *is strongly asymmetric in the sense of Definition* XVIII.3, *then for all* $\ell \geq 1$

$$|L_\ell(f; C, D)|^{\sim}_{3,\Sigma} \leq (\text{const } \mathfrak{l}^{1/n_0}\, \mathfrak{e}_j(X))^\ell\ |f|^{\sim\,\ell+1}_{3,\Sigma}$$

where $\mathfrak{e}_j(X) = \frac{\mathfrak{c}_j}{1-M^jX}$.

Proof. By Proposition D.7, with X replaced by M^jX, and Lemma XXII.6

$$\begin{aligned} |L_\ell(f; C, D)|^{\sim}_{3,\Sigma} &\leq \left(\text{const } \frac{\mathfrak{l}\,\mathfrak{c}_j}{1-M^jX}\right)^\ell\ |f|^{\sim\,\ell}_{ch,\Sigma}\ |f|^{\sim}_{3,\Sigma} \\ &\leq (\text{const } \mathfrak{l}^{1/n_0}\, \mathfrak{e}_j(X))^\ell\ |f|^{\sim\,\ell+1}_{3,\Sigma}\,. \end{aligned}$$

□

Appendices

E. *Sectors for k_0 independent functions*

In [1–3] we shall implement a renormalization algorithm that uses counterterms for the dispersion relation $e(\mathbf{k})$ that are independent of k_0. In this appendix we adjust the discussion of sectorized norms in Sec. XII and the discussion of resectorization, following Definition XIX.2, to deal with such functions.

Definition E.1. Let $f(\mathbf{x}, \mathbf{x}')$ be a translation invariant function on $\mathbb{R}^2 \times \mathbb{R}^2$ and we define its extension $f_{\text{ext}}(\xi, \xi')$ by

$$\begin{aligned} &f_{\text{ext}}((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a')) \\ &\quad = \delta_{\sigma,\sigma'}\delta(x_0 - x_0') \begin{cases} f(\mathbf{x}, \mathbf{x}') & \text{if } a = 1,\ a' = 0 \\ -f(\mathbf{x}', \mathbf{x}) & \text{if } a = 0,\ a' = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and its Fourier transform as

$$\check{f}(\mathbf{k}) = \int d^2\mathbf{x}\ e^{-\imath(\mathbf{k}_1\mathbf{x}_1 + \mathbf{k}_2\mathbf{x}_2)}\, f(\mathbf{x}, \mathbf{0})\,.$$

Remark E.2. If $\check{f}_{\text{ext}}(k)$ is the Fourier transform of f_{ext} as in [6, Definition IX.1(i)], then

$$\check{f}_{\text{ext}}((k_0, \mathbf{k})) = \check{f}(\mathbf{k})\,.$$

Definition E.3. Let Σ be a sectorization at scale j and $K((\mathbf{x}, s), (\mathbf{x}', s'))$ a translation invariant function on $(\mathbb{R}^2 \times \Sigma)^2$.

(i) We define its extension $K_{\rm ext}((\xi,s),(\xi',s'))$ on $(\mathcal{B}\times\Sigma)^2$ by

$$K_{\rm ext}((x_0,\mathbf{x},\sigma,a,s),(x_0',\mathbf{x}',\sigma',a',s'))$$
$$=\delta_{\sigma,\sigma'}\delta(x_0-x_0')\begin{cases}K((\mathbf{x},s),(\mathbf{x}',s')) & \text{if } a=1,\ a'=0\\ -K((\mathbf{x}',s'),(\mathbf{x},s)) & \text{if } a=0,\ a'=1\\ 0 & \text{otherwise}\end{cases}.$$

(ii) The function $K((\mathbf{x},s),(\mathbf{x}',s'))$ is said to be sectorized if its Fourier transform

$$\int d^2\mathbf{x}d^2\mathbf{x}'e^{-\imath(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2)}e^{\imath(\mathbf{k}_1'\mathbf{x}_1'+\mathbf{k}_2'\mathbf{x}_2')}K((\mathbf{x},s),(\mathbf{x}',s'))$$

vanishes unless $(0,\mathbf{k})\in\tilde{s}$ and $(0,\mathbf{k}')\in\tilde{s}'$ where $\tilde{s}$ and $\tilde{s}'$ are the extensions of s and s' of [8, Definition XII.1(ii)].

(iii) We define $\check{K}(\mathbf{k})$ by

$$\check{K}(\mathbf{k})=\sum_{s,s'\in\Sigma}\int d^2\mathbf{x}\,e^{-\imath(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2)}K((\mathbf{x},s),(\mathbf{0},s')).$$

(iv) We set

$$\|K\|_{1,\Sigma}=|K_{\rm ext}|_{1,\Sigma}.$$

Remark E.4. (i) If $\check{K}_{\rm ext}(k)$ is the Fourier transform of $K_{\rm ext}$ as in Definition XII(iv), then

$$\check{K}_{\rm ext}((k_0,\mathbf{k}))=\check{K}(\mathbf{k}).$$

(ii) If K is sectorized, then $K((\mathbf{x},s),(\mathbf{x}',s'))$ and $K_{\rm ext}((x_0,\mathbf{x},\sigma,a,s),(x_0',\mathbf{x}',\sigma',a',s'))$ vanish unless $s\cap s'\neq\emptyset$.

(iii) Suppose that K is sectorized and write $\|K\|_{1,\Sigma}=\sum_{\delta\in\mathbb{N}\times\mathbb{N}^2}\frac{1}{\delta!}\kappa_\delta t^\delta$. Then κ_δ vanishes unless $\delta_0=0$ and otherwise is given by

$$\kappa_{0,\boldsymbol{\delta}}=\max\left\{\max_{s'\in\Sigma}\sum_{s\in\Sigma}\int d^2\mathbf{x}|\mathbf{x}^{\boldsymbol{\delta}}\,K((\mathbf{x},s),(\mathbf{0},s'))|,\right.$$
$$\left.\max_{s\in\Sigma}\sum_{s'\in\Sigma}\int d^2\mathbf{x}\,|\mathbf{x}^{\boldsymbol{\delta}}\,K((\mathbf{x},s),(\mathbf{0},s'))|\right\}$$

and obeys

$$\max_{s,s'\in\Sigma}\int d^2\mathbf{x}\,|\mathbf{x}^{\boldsymbol{\delta}}\,K((\mathbf{x},s),(\mathbf{0},s'))|$$
$$\le\kappa_{0,\boldsymbol{\delta}}\le 3\max_{s,s'\in\Sigma}\int d^2\mathbf{x}\,|\mathbf{x}^{\boldsymbol{\delta}}\,K((\mathbf{x},s),(\mathbf{0},s'))|.$$

Lemma E.5. *Let Σ be a sectorization of scale $j\ge 2$ and length $\frac{1}{M^{j-3/2}}\le\mathfrak{l}\le\frac{1}{M^{(j-1)/2}}$ and $K((\mathbf{x},s),(\mathbf{x}',s'))$ be a sectorized, translation invariant function on $(\mathbb{R}^2\times\Sigma)^2$. Let $\mu(t)$ be a C_0^∞ function on $\mathbb{R}$ and set, for each $\Lambda>0$*

$$\mu_\Lambda(k) = \mu(\Lambda^2[k_0^2 + e(\mathbf{k})^2])$$

$$(K_{\rm ext} * \hat\mu_\Lambda)((\xi,s),(\xi',s')) = \int_{\mathcal B} d\zeta\ K_{\rm ext}((\xi,s),(\zeta,s'))\hat\mu_\Lambda(\zeta,\xi')$$

$$(\hat\mu_\Lambda * K_{\rm ext})((\xi,s),(\xi',s')) = \int_{\mathcal B} d\zeta\ K_{\rm ext}((\zeta,s),(\xi',s'))\hat\mu_\Lambda(\zeta,\xi)$$

where $\hat\mu_\Lambda$ *was defined in* [6, *Definition* IX.4]. *Denote* $j(\Lambda) = \min\{i \in \mathbb{N} | M^i \geq \Lambda\}$. *Then, there is a constant* const, *depending on* μ, *but not on* M, j *or* Λ, *such that*

$$|K_{\rm ext} * \hat\mu_\Lambda|_{1,\Sigma}\,, \quad |\hat\mu_\Lambda * K_{\rm ext}|_{1,\Sigma} \leq \mathit{const}\ \mathfrak{c}_{j(\Lambda)}\ \|K\|_{1,\Sigma}\,.$$

This lemma is an immediate consequence of [8, Lemma XIII.7].

Remark E.6. In the notation of Lemma E.5,

$$(K_{\rm ext} * \hat\mu_\Lambda)\check{}(k) = (\hat\mu_\Lambda * K_{\rm ext})\check{}(k) = \check K(\mathbf{k})\mu_\Lambda(k)\,.$$

As in Definition XIX.2, we define a resectorization for functions on $(\mathbb{R}^2 \times \Sigma)^2$. For a function $\chi(k)$ on $\mathbb{R} \times \mathbb{R}^2$, set, as in [8, Lemma XIII.3],

$$\chi^0(\mathbf{x}) = \int e^{\imath \mathbf{k}\cdot\mathbf{x}}\chi(0,\mathbf{k})\frac{d^2\mathbf{k}}{(2\pi)^2} = \int dx_0 \hat\chi((x_0,\mathbf{x},\uparrow,0),(0,\mathbf{0},\uparrow,0))$$

and let

$$\hat\chi^0(\xi,\xi') = \delta_{\sigma,\sigma'}\delta_{a,a'}\delta(x_0 - x_0')\chi^0((-1)^a(\mathbf{x}-\mathbf{x}'))\,.$$

Then

$$\begin{aligned}\hat\chi^0((x_0,\mathbf{x},\sigma,a),(x_0',\mathbf{x}',\sigma',a')) &= \delta(x_0 - x_0')\int dt \hat\chi((t,\mathbf{x},\sigma,a),(0,\mathbf{x}',\sigma',a')) \\ &= \hat\chi((x_0,\mathbf{x},\sigma,a),(x_0',\mathbf{x}',\sigma',a'))\,.\end{aligned}$$

Definition E.7. Let $j, i \geq 2$. Let Σ and Σ' be sectorizations of scale j and i, respectively. If $i \neq j$, define, for each function K on $(\mathbb{R}^2 \times \Sigma')^2$,

$$K_\Sigma((\mathbf{x},s_1),(\mathbf{y},s_2)) = \sum_{s_1',s_2' \in \Sigma}\int d\mathbf{x}'d\mathbf{y}'\chi^0_{s_1}(\mathbf{x}-\mathbf{x}')K((\mathbf{x}',s_1'),(\mathbf{y}',s_2'))\chi^0_{s_2}(\mathbf{y}'-\mathbf{y})$$

where χ_s, $s \in \Sigma$ is the partition of unity of Lemma XII.3 and [8, (XIII.2)]. For $i = j$ and $\Sigma' = \Sigma$, define $K_\Sigma = K$.

Remark E.8. (i) If K is translation invariant, then

$$\check K_\Sigma(\mathbf{k},s_1,s_2) = \sum_{s_1',s_2' \in \Sigma}\check K(\mathbf{k},s_1',s_2')\chi_{s_1}(0,\mathbf{k})\chi_{s_2}(0,\mathbf{k})\,.$$

(ii) The resectorization K_Σ is sectorized.

Remark E.9. Let $K((\mathbf{x},s),(\mathbf{x}',s'))$ be a translation invariant sectorized function on $(\mathbb{R}^2 \times \Sigma')^2$. Then

$$(K_\Sigma)_{\rm ext}((\xi,s_1),(\eta,s_2)) = \sum_{\substack{s_1',s_2'\in\Sigma\\ s_1'\cap s_1\neq\emptyset\\ s_2'\cap s_2\neq\emptyset}} \int d\xi' d\eta' K_{\rm ext}((\xi',s_1'),(\eta',s_2'))\hat\chi^0_{s_1}(\xi',\xi)\chi^0_{s_2}(\eta',\eta)\,.$$

Proof. Let $\xi = (x_0, \mathbf{x}, \sigma, a),\ \eta = (y_0, \mathbf{y}, \tau, b) \in \mathcal{B}$. We consider the case $a = 1, b = 0$, the other cases are similar. Fix any $s_1', s_2' \in \Sigma'$. If $s_1' \cap s_1 = \emptyset$ or $s_2' \cap s_2 = \emptyset$.

$$\int d\xi' d\eta' K_{\rm ext}((\xi',s_1'),(\eta',s_2'))\hat\chi^0_{s_1}(\xi',\xi)\hat\chi^0_{s_2}(\eta',\eta) = 0 = \int d\mathbf{x}' d\mathbf{y}' \chi^0_{s_1}(\mathbf{x}-\mathbf{x}')K((\mathbf{x}',s_1),(\mathbf{y}',s_2'))\chi^0_{s_2}(\mathbf{y}'-\mathbf{y})\,.$$

Otherwise

$$\begin{aligned}
&\int d\xi' d\eta' K_{\rm ext}((\xi',s_1'),(\eta',s_2'))\hat\chi^0_{s_1}(\xi',\xi)\hat\chi^0_{s_2}(\eta',\eta)\\
&\quad= \delta_{\sigma,\tau}\int dx_0' dy_0' d\mathbf{x}' d\mathbf{y}' \delta(x_0'-y_0')\delta(x_0'-x_0)\delta(y_0'-y_0)\\
&\qquad\times \chi^0_{s_1}(\mathbf{x}-\mathbf{x}')K((\mathbf{x}',s_1'),(\mathbf{y}',s_2'))\chi^0_{s_2}(\mathbf{y}'-\mathbf{y})\\
&\quad= \delta_{\sigma,\tau}\delta(x_0-y_0)\int d\mathbf{x}' d\mathbf{y}' \chi^0_{s_1}(\mathbf{x}-\mathbf{x}')K((\mathbf{x}',s_1'),(\mathbf{y}',s_2'))\chi^0_{s_2}(\mathbf{y}'-\mathbf{y})\,.
\end{aligned}$$ □

Proposition E.10. *Let $j > i \geq 2$, $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$ and $\frac{1}{M^{i-3/2}} \leq \mathfrak{l}' \leq \frac{1}{M^{(i-1)/2}}$ with $4\mathfrak{l} < \mathfrak{l}'$. Let Σ and Σ' be sectorizations of length $\mathfrak{l}$ at scale j and length $\mathfrak{l}'$ at scale i, respectively.*

(i) *Let $K((\mathbf{x},s),(\mathbf{x}',s'))$ be a translation invariant sectorized function on $(\mathbb{R}^2 \times \Sigma')^2$. Then*

$$\|K_\Sigma\|_{1,\Sigma} \leq const\ \mathfrak{c}_{j-1}\|K\|_{1,\Sigma'}\,.$$

(ii) *Let $K((\mathbf{x},s),(\mathbf{x}',s'))$ be a translation invariant sectorized function on $(\mathbb{R}^2\times\Sigma)^2$. Then*

$$\|K_{\Sigma'}\|_{1,\Sigma'} \leq const \left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right] \mathfrak{c}_{i-1}\|K\|_{1,\Sigma}\,.$$

Proof. (i) By Definition E.3(iv), Remark E.9, [9, Lemma II.7], Lemma XIII.3 and [8, (XIII.4)],

$$\begin{aligned}
\|K_\Sigma\|_{1,\Sigma} &= |(K_\Sigma)_{\rm ext}|_{1,\Sigma}\\
&\leq const \max_{s_1,s_2\in\Sigma} \sum_{\substack{s_1',s_2'\in\Sigma\\ s_1'\cap s_1\neq\emptyset\\ s_2'\cap s_2\neq\emptyset}} \|K_{\rm ext}((\cdot\,,s_1'),(\cdot\,,s_2'))\|_{1,\infty}\,\|\hat\chi^0_{s_1}\|_{1,\infty}\,\|\chi^0_{s_2}\|_{1,\infty}
\end{aligned}$$

$$\le const \max_{s_1,s_2\in\Sigma} \|\chi^0_{s_1}\|_{L_1} \|\chi^0_{s_2}\|_{L_1} \max_{s'_1,s'_2\in\Sigma'} \|K_{\rm ext}((\cdot\,,s'_1),(\cdot\,,s'_2))\|_{1,\infty}$$

$$\le const\ \mathfrak{c}^2_{j-1} |K_{\rm ext}|_{1,\Sigma'} \le const\ \mathfrak{c}_{j-1}\|K\|_{1,\Sigma'}$$

since, for any $s\in\Sigma$, there are at most three sectors $s'\in\Sigma'$ with $s'\cap s\ne\emptyset$.

(ii) Similarly

$$\|K_{\Sigma'}\|_{1,\Sigma'} = |(K_{\Sigma'})_{\rm ext}|_{1,\Sigma'}$$

$$\le const \max_{s'_1,s'_2\in\Sigma'} \sum_{\substack{s_1,s_2\in\Sigma\\ s_1\cap s'_1\ne\emptyset\\ s_2\cap s'_2\ne\emptyset}} \|K_{\rm ext}((\cdot\,,s_1),(\cdot\,,s_2))\|_{1,\infty} \|\hat\chi^0_{s'_1}\|_{1,\infty} \|\chi^0_{s'_2}\|_{1,\infty}$$

$$\le const \max_{s'_1,s'_2\in\Sigma'} \|\chi^0_{s'_1}\|_{L_1} \|\chi^0_{s'_2}\|_{L_1} \sum_{\substack{s_1\in\Sigma\\ s_1\cap s'_1\ne\emptyset}} \sum_{s_2\in\Sigma} \|K_{\rm ext}((\cdot\,,s_1),(\cdot\,,s_2))\|_{1,\infty}$$

$$\le const\ \mathfrak{c}^2_{i-1}\left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right] \max_{s_1\in\Sigma} \sum_{s_2\in\Sigma} \|K_{\rm ext}((\cdot\,,s_1),(\cdot\,,s_2))\|_{1,\infty}$$

$$\le const\ \mathfrak{c}_{i-1}\left[\frac{\mathfrak{l}'}{\mathfrak{l}}\right] \|K\|_{1,\Sigma}$$

since, for any $s'_1\in\Sigma'$, there are at most $const[\frac{\mathfrak{l}'}{\mathfrak{l}}]$ sectors $s_1\in\Sigma$ with $s_1\cap s'_1 \ne\emptyset$. □

Notation

Norms

Norm	Characteristics	Reference
$\vert\vert\vert\cdot\vert\vert\vert_{1,\infty}$	no derivatives, external positions, acts on functions	Example II.6
$\|\cdot\|_{1,\infty}$	derivatives, external positions, acts on functions	Example II.6
$\|\cdot\|\check{}_{\infty}$	derivatives, external momenta, acts on functions	Definition IV.6
$\vert\vert\vert\cdot\vert\vert\vert_{\infty}$	no derivatives, external positions, acts on functions	Example III.4
$\|\cdot\|\check{}_{1}$	derivatives, external momenta, acts on functions	Definition IV.6
$\|\cdot\|\check{}_{\infty,B}$	derivatives, external momenta, $B\subset\mathbb{R}\times\mathbb{R}^d$	Definition IV.6
$\|\cdot\|\check{}_{1,B}$	derivatives, external momenta, $B\subset\mathbb{R}\times\mathbb{R}^d$	Definition IV.6
$\|\cdot\|$	$\rho_{m;n}\|\cdot\|_{1,\infty}$	Lemma V.1
$N(\mathcal{W};\ \mathfrak{c},{\rm b},\alpha)$	$\frac{1}{{\rm b}^2}\,\mathfrak{c}\sum_{m,n\ge0}\alpha^n{\rm b}^n\|\mathcal{W}_{m,n}\|$	Definition III.9
		Theorem V.2

Norm	Characteristics	Reference
$N_0(\mathcal{W};\ \beta;\ X,\vec{\rho})$	$\mathfrak{e}_0(X) \sum_{m+n\in 2\mathbb{N}} \beta^n \rho_{m;n} \|\mathcal{W}_{m,n}\|_{1,\infty}$	Theorem VIII.6
$\|\cdot\|_{L^1}$	derivatives, acts on functions on $\mathbb{R}\times\mathbb{R}^d$	before Lemma IX.6
$\|\cdot\|^{\sim}$	derivatives, external momenta, acts on functions	Definition X.4
$N_0^{\sim}(\mathcal{W};\ \beta;\ X,\vec{\rho})$	$\mathfrak{e}_0(X) \sum_{m+n\in 2\mathbb{N}} \beta^{m+n} \rho_{m;n} \|W^{\sim}_{m,n}\|^{\sim}$	before Lemma X.11
$\lvert\cdot\rvert^{\sim}$	like $\rho_{m;n}\|\cdot\|^{\sim}$ but acts on $\tilde{V}^{\otimes n}$	Theorem X.12
$N^{\sim}(\mathcal{W};\ \mathfrak{c},\mathrm{b},\alpha)$	$\frac{1}{\mathrm{b}^2}\,\mathfrak{c}\sum_{m,n}\alpha^{m+n}\mathrm{b}^{m+n}\lvert W^{\sim}_{m,n}\rvert^{\sim}$	Theorem X.12
$\lvert\cdot\rvert_{p,\Sigma}$	derivatives, external positions, all but p sectors summed	Definition XII.9
$\|\cdot\|_{1,\Sigma}$	like $\lvert\cdot\rvert_{1,\Sigma}$, but for functions on $(\mathbb{R}^2\times\Sigma)^2$	Definition E.3
$\lvert\varphi\rvert_{\Sigma}$	$\rho_{m;n}\begin{cases} \lvert\varphi\rvert_{1,\Sigma}+\frac{1}{\mathfrak{l}}\lvert\varphi\rvert_{3,\Sigma}+\frac{1}{\mathfrak{l}^2}\lvert\varphi\rvert_{5,\Sigma} & \text{if } m=0 \\ \frac{\mathfrak{l}}{M^{2j}}\lvert\varphi\rvert_{1,\Sigma} & \text{if } m\neq 0 \end{cases}$	Definition XV.1
$N_j(w;\ \alpha;\ X,\Sigma,\vec{\rho})$	$\frac{M^{2j}}{\mathfrak{l}}\,\mathfrak{e}_j(X)\sum_{m,n\ge 0}\alpha^n\left(\frac{\mathfrak{l}\mathfrak{B}}{M^j}\right)^{n/2}\lvert w_{m,n}\rvert_{\Sigma}$	Definition XV.1
$\lvert\cdot\rvert^{\sim}_{p,\Sigma}$	derivatives, external momenta, all but p sectors summed	Definition XVI.4
$\lvert\cdot\rvert^{\sim}_{p,\Sigma,\boldsymbol{\rho}}$	weighted variant of $\lvert\cdot\rvert^{\sim}_{p,\Sigma}$	Definition XVII.1(i)
$\lvert f\rvert^{\sim}_{\Sigma}$	$\rho_{m;n}\begin{cases} \lvert f\rvert^{\sim}_{1,\Sigma}+\frac{1}{\mathfrak{l}}\lvert f\rvert^{\sim}_{3,\Sigma}+\frac{1}{\mathfrak{l}^2}\lvert f\rvert^{\sim}_{5,\Sigma} & \text{if } m=0 \\ \sum_{p=1}^{6}\frac{1}{\mathfrak{l}^{\lceil (p-1)/2\rceil}}\lvert f\rvert^{\sim}_{p,\Sigma} & \text{if } m\neq 0 \end{cases}$	Definition XVII.1(ii)
$N_j^{\sim}(w;\ \alpha;\ X,\Sigma,\vec{\rho})$	$\frac{M^{2j}}{\mathfrak{l}}\,\mathfrak{e}_j(X)\sum_{n\ge 0}\alpha^n\left(\frac{\mathfrak{l}\mathfrak{B}}{M^j}\right)^{n/2}\lvert f_n\rvert^{\sim}_{\Sigma}$	Definition XVII.1(iii)
$\lvert\cdot\rvert^{\sim}_{ch,\Sigma}$	channel variant of $\lvert\cdot\rvert^{\sim}_{2,\Sigma}$ for ladders	Definition D.1
$\lvert\cdot\rvert_{ch,\Sigma}$	channel variant of $\lvert\cdot\rvert_{2,\Sigma}$ for ladders	Definition D.1
$\lvert\cdot\rvert_{1,\Sigma,\omega}$	like $\lvert\cdot\rvert_{1,\Sigma}$ but excludes almost degenerate sectors	Lemma XXI.2
$\lvert\cdot\rvert^{\sim}_{1,\Sigma,\omega}$	like $\lvert\cdot\rvert^{\sim}_{1,\Sigma}$ but excludes almost degenerate sectors	Lemma XXI.2

Other notation

Notation	Description	Reference
$\Omega_S(\mathcal{W})(\phi,\psi)$	$\log \frac{1}{Z}\int e^{\mathcal{W}(\phi,\psi+\zeta)}d\mu_S(\zeta)$	before (I.6)
J	particle/hole swap operator	(VI.1)
$\tilde\Omega_C(\mathcal{W})(\phi,\psi)$	$\log \frac{1}{Z}\int e^{\phi J\zeta}e^{\mathcal{W}(\phi,\psi+\zeta)}d\mu_C(\zeta)$	Definition VII.1
r_0	number of k_0 derivatives tracked	Sec. VI
r	number of $\mathbf{k}$ derivatives tracked	Sec. VI
M	scale parameter, $M>1$	before Definition VIII.1
const	generic constant, independent of scale	
const	generic constant, independent of scale and M	
$\nu^{(j)}(k)$	jth scale function	Definition VIII.1
$\tilde\nu^{(j)}(k)$	jth extended scale function	Definition VIII.4(i)
$\nu^{(\ge j)}(k)$	$\varphi(M^{2j-1}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.1
$\tilde\nu^{(\ge j)}(k)$	$\varphi(M^{2j-2}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.4(ii)
$\bar\nu^{(\ge j)}(k)$	$\varphi(M^{2j-3}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.4(iii)
n_0	degree of asymmetry	Definition XVIII.3
$\mathfrak{l}$	length of sectors	Definition XII.1
Σ	sectorization	Definition XII.1
$S(C)$	$\sup_m \sup_{\xi_1,\ldots,\xi_m\in\mathcal{B}} \left(\left\|\int \psi(\xi_1)\cdots\psi(\xi_m)d\mu_C(\psi)\right\|\right)^{1/m}$	Definition IV.1
B	j-independent constant	Definitions XV.1, XVII.1
$\mathfrak{c}_j$	$=\sum_{\substack{\|\boldsymbol\delta\|\le r\\ \|\delta_0\|\le r_0}} M^{j\|\delta\|}t^\delta + \sum_{\substack{\|\boldsymbol\delta\|>r\\ \text{or } \|\delta_0\|>r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}$	Definition XII.2
$\mathfrak{e}_j(X)$	$=\frac{\mathfrak{c}_j}{1-M^jX}$	Definition XV.1(ii)
$f_{\rm ext}$	extends $f(\mathbf{x},\mathbf{x}')$ to $f_{\rm ext}((x_0,\mathbf{x},\sigma,a),(x_0',\mathbf{x}',\sigma',a'))$	Definition E.1
$*$	convolution	before (XIII.6)
$\circ$	ladder convolution	Definition XIV.1(iv)
$\bullet$	ladder convolution	Definitions XIV.3, XVI.9
$\check f$	Fourier transform	Definition IX.1(i)
$\check u$	Fourier transform for sectorized u	Definition XII.4(iv)
$f^\sim$	partial Fourier transform	Definition IX.1(ii)
$\hat\chi$	Fourier transform	Definition IX.4

Notation	Description	Reference
$\mathcal{B}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0,1\}$ viewed as position space	beginning of Sec. II
$\check{\mathcal{B}}$	$\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0,1\}$ viewed as momentum space	beginning of Sec. IX
$\check{\mathcal{B}}_m$	$\{(\check{\eta}_1, \dots, \check{\eta}_m) \in \check{\mathcal{B}}^m \vert \check{\eta}_1 + \cdots + \check{\eta}_m = 0\}$	before Definition X.1
$\mathfrak{X}_\Sigma$	$\check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$	Definition XVI.1
$\mathcal{F}_m(n)$	functions on $\mathcal{B}^m \times \mathcal{B}^n$, antisymmetric in $\mathcal{B}^m$ arguments	Definition II.9
$\check{\mathcal{F}}_m(n)$	functions on $\check{\mathcal{B}}^m \times \mathcal{B}^n$, antisymmetric in $\check{\mathcal{B}}^m$ arguments	Definition X.8
$\mathcal{F}_m(n; \Sigma)$	functions on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma)^n$, internal momenta in sectors	Definition XII.4(ii)
$\check{\mathcal{F}}_m(n; \Sigma)$	functions on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^n$, internal momenta in sectors	Definition XVI.7(i)
$\check{\mathcal{F}}_{n;\Sigma}$	functions on $\mathfrak{X}_\Sigma^n$ that reorder to $\check{\mathcal{F}}_m(n-m; \Sigma)$'s	Definition XVI.7(iii)

References

[1] J. Feldman, H. Knörrer and E. Trubowitz, A two dimensional Fermi liquid, Part 1: overview, to appear in *Commun. Math. Phys.*

[2] J. Feldman, H. Knörrer and E. Trubowitz, A two dimensional Fermi liquid, Part 2: convergence, to appear in *Commun. Math. Phys.*

[3] J. Feldman, H. Knörrer and E. Trubowitz, A two dimensional Fermi liquid, Part 3: the Fermi surface, to appear in *Commun. Math. Phys.*

[4] J. Feldman, H. Knörrer and E. Trubowitz, Asymmetric Fermi surfaces for magnetic Schrödinger operators, *Comm. Partial Differential Equations* **25** (2000), 319–336.

[5] J. Feldman, H. Knörrer and E. Trubowitz, Particle–hole ladders, preprint.

[6] J. Feldman, H. Knörrer and E. Trubowitz, Single scale analysis of many Fermion systems, Part 2: the first scale, *Rev. Math. Phys.* **15** (2003) 995–1037.

[7] M. Gromov, *Asymptotic Invariants of Infinite Groups.*

[8] J. Feldman, H. Knörrer and E. Trubowitz, Single scale analysis of many Fermion systems, Part 3: sectorized norms, *Rev. Math. Phys.* **15** (2003) 1039–1120.

[9] J. Feldman, H. Knörrer and E. Trubowitz, Single scale analysis of many Fermion systems, Part 1: insulators, *Rev. Math. Phys.* **15** (2003) 949–993.