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SINGLE SCALE ANALYSIS OF MANY FERMION SYSTEMS PART 3: SECTORIZED NORMS

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The generic renormalization group map associated to a weakly coupled system of fermions at temperature zero is treated by supplementing the methods of Part 1. The interplay between position and momentum space is captured by "sectors". It is shown that the difference between the complete four-legged vertex and its "ladder" part is irrelevant for the sequence of renormalization group maps.

Keywords: Fermi liquid; renormalization; fermionic functional integral; Euclidean Green's functions.

Contents

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XI. Introduction to Part 3

We use "sectors" to construct norms that allow nonperturbative control of renormalization group maps for two-dimensional many fermion systems. Thus from Sec. XII on, we assume that the dimension d of our system is two. Notation tables are provided at the end of the paper.

We assume that the dispersion relation $e(\mathbf{k})$ is $r+d+1$ times differentiable, with $r \geq 2$, and that its gradient does not vanish on the Fermi surface

$$F = \{(k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d \,|\, k_0 = 0,\ e(\mathbf{k}) = 0\}\,.$$

It follows from these hypotheses that the gradient of the dispersion relation $e(\mathbf{k})$ does not vanish in a neighborhood of F and that there is an $r+d+1$ times differentiable projection π_F to F in a neighborhood of the Fermi surface. We assume that the scale parameter M of Sec. VIII has been chosen so big that the "second doubly extended neighborhood" $\{k \in \mathbb{R} \times \mathbb{R}^2 \,|\, \bar{\nu}^{(\geq 2)}(k) \neq 0\}$ is contained in the two above-mentioned neighborhoods.

XII. Sectors and Sectorized Norms

From now on we consider only $d = 2$, so that the Fermi "surface" is a curve in $\mathbb{R} \times \mathbb{R}^2$.

Definition XII.1. (Sectors and sectorizations)

(i) Let I be an interval on the Fermi surface F and $j \geq 2$. Then

$$s = \{k \text{ in the } j\text{th neighborhood} \,|\, \pi_F(k) \in I\}$$

is called a sector of length $|I|$ at scale j. Recall that $\pi_F(k)$ is the projection of k on the Fermi surface. Two different sectors s and s' are called neighbors if $s' \cap s \neq \emptyset$.

(ii) If s is a sector at scale j, its extension is

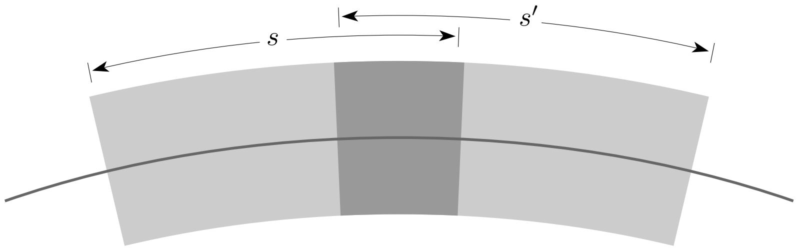
$$\tilde{s} = \{k \text{ in the } j\text{th extended neighborhood} \,|\, \pi_F(k) \in s\}\,.$$

(iii) A sectorization of length $\mathfrak{l}$ at scale j is a set Σ of sectors of length $\mathfrak{l}$ at scale j that obeys the following:

- the set Σ of sectors covers the Fermi surface
- each sector in Σ has precisely two neighbors in Σ, one to its left and one to its right
- if $s,\ s' \in \Sigma$ are neighbors then $\frac{1}{16}\mathfrak{l} \leq |s \cap s' \cap F| \leq \frac{1}{8}\mathfrak{l}$

Observe that there are at most 2 length $(F)/\mathfrak{l}$ sectors in Σ.

We will need partitions of unity for the sectors, as well as functions that envelope the sectors — i.e. that are identically one on a sector and are supported near the sector. Their L_1–L_∞ norm will be typical for a function with the specified support. To measure it we generalize Definition IV.10.



Definition XII.2. The element $\mathfrak{c}_j$ of $\mathfrak{N}_{d+1}$ is defined as

$$\mathfrak{c}_j = \sum_{\substack{|\boldsymbol{\delta}|\le r \\ |\delta_0|\le r_0}} M^{j|\delta|} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r \\ \text{or } |\delta_0|>r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}\,.$$

Lemma XII.3. *Let Σ be a sectorization of length $\frac{1}{M^{j-3/2}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$ at scale $j \ge 2$. Then there exist $\chi_s(k)$, $\tilde\chi_s(k)$, $s \in \Sigma$ that take values in $[0,1]$ such that*

(i) *χ_s is supported in the extended sector $\tilde s$ and*

$$\sum_{s\in\Sigma} \chi_s(k) = 1 \quad \textit{for } k \textit{ in the jth neighborhood}\,.$$

(ii) *$\tilde\chi_s$ is identically one on the extended sector $\tilde s$, is supported on the jth doubly extended neighborhood and $\tilde\chi_s(k)\cdot\tilde\chi_{s'}(k) = 0$ if $s \cap s' = \emptyset$. Furthermore, $\int d^3k \frac{\tilde\chi_s(k)}{|\imath k_0 - e(\mathbf{k})|} \le \mathrm{const}\,\frac{\mathfrak{l}}{M^j}$.*

(iii)

$$\|\hat\chi_s\|_{1,\infty}\,, \quad \|\hat{\tilde\chi}_s\|_{1,\infty} \le \mathit{const}\,\mathfrak{c}_{j-1} \le \mathit{const}\,\mathfrak{c}_j$$

with a constant const that does not depend on M, j, Σ or s.

The proof of this lemma is postponed to Sec. XIII.

Definition XII.4 (Sectorized representatives). Let Σ be a sectorization at scale j, and let $m, n \ge 0$.

(i) The antisymmetrization of a function φ on $\mathcal{B}^m \times (\mathcal{B}\times\Sigma)^n$ is

$$\begin{aligned}&\mathrm{Ant}\,\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n)) \\ &\quad = \frac{1}{m!n!}\sum_{\substack{\pi\in S_m\\ \pi'\in S_n}} \varphi(\eta_{\pi(1)},\ldots,\eta_{\pi(m)};(\xi_{\pi'(1)},s_{\pi'(1)}),\ldots,(\xi_{\pi'(n)},s_{\pi'(n)}))\,.\end{aligned}$$

(ii) Denote by $\mathcal{F}_m(n;\Sigma)$ the space of all translation invariant, complex valued functions

$$\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))$$

on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma)^n$ that are antisymmetric in their external ($= \eta$) variables and whose Fourier transform $\check{\varphi}(\check{\eta}_1, \dots, \check{\eta}_m; (\check{\xi}_1, s_1), \dots, (\check{\xi}_n, s_n))$ vanishes unless $k_i \in \tilde{s}_i$ for all $1 \le j \le n$. Here, $\check{\xi}_i = (k_i, \sigma_i, a_i)$.

(iii) Let $f \in \mathcal{F}_m(n)$ be translation invariant. A Σ-sectorized representative for f is a function $\varphi \in \mathcal{F}_m(n; \Sigma)$ obeying

$$\check{f}(\check{\eta}_1, \dots, \check{\eta}_m; \check{\xi}_1, \dots, \check{\xi}_n) = \sum_{\substack{s_i \in \Sigma \\ i=1,\dots,n}} \check{\varphi}(\check{\eta}_1, \dots, \check{\eta}_m; (\check{\xi}_1, s_1), \dots, (\check{\xi}_n, s_n))$$

for all $\check{\xi}_i = (k_i, \sigma_i, a_i)$ with k_i in the jth neighborhood.

(iv) Let $u((\xi, s), (\xi', s'))$ be a translation invariant, spin independent, particle number conserving function on $(\mathcal{B} \times \Sigma)^2$. We define $\check{u}(k)$ by

$$\delta_{\sigma,\sigma'} \check{u}(k) = \sum_{s,s' \in \Sigma} \check{u}((k, \sigma, 1, s), (k, \sigma', 0, s'))\,.$$

Example XII.5. Set

$$\varphi(\eta_1, \dots, \eta_m; (\xi_1, s_1), \dots, (\xi_n, s_n)) = \int \prod_{i=1}^{n} (d\xi_i' \hat{\chi}_{s_i}(\xi_i, \xi_i')) f(\eta_1, \dots, \eta_m; \xi_1', \dots, \xi_n')$$

where χ_s is the partition of unity of Lemma XII.3 and $\hat{\chi}_s$ was defined in Definition IX.4. Then φ is a Σ-sectorized representative for f.

Recall that we want to control the renormalization group map Ω_C on $\bigwedge_A V$, where A is the Grassmann algebra generated by the fields $\phi(\eta)$ and V is the vector space generated by the fields $\psi(\xi)$. We shall do this by controlling norms of sectorized representatives of the coefficient functions. In preparation, we consider a renormalization group map that is adjusted to the sectorization.

Definition XII.6. (i) V_Σ is the vector space generated by $\psi(\xi, s)$, $\xi \in \mathcal{B}$, $s \in \Sigma$. If $\varphi \in \mathcal{F}_m(n; \Sigma)$ we define the element $\mathrm{Tens}(\varphi)$ of $A_m \otimes V_\Sigma^{\otimes n}$ by

$$\mathrm{Tens}(\varphi) = \sum_{\substack{s_j \in \Sigma \\ j=1,\dots,n}} \int \prod_{i=1}^{m} d\eta_i \prod_{j=1}^{n} d\xi_j \varphi(\eta_1, \dots, \eta_m; (\xi_1, s_1), \dots, (\xi_n, s_n))$$
$$\cdot \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1, s_1) \otimes \cdots \otimes \psi(\xi_n, s_n)$$

and the element $Gr(\varphi)$ of $A_m \otimes \bigwedge^n V_\Sigma$ as

$$Gr(\varphi) = \sum_{\substack{s_j \in \Sigma \\ j=1,\dots,n}} \int \prod_{i=1}^{m} d\eta_i \prod_{j=1}^{n} d\xi_j \varphi(\eta_1, \dots, \eta_m; (\xi_1, s_1), \dots, (\xi_n, s_n))$$
$$\cdot \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1, s_1) \cdots \psi(\xi_n, s_n)\,.$$

Elements of $A \otimes \bigwedge V_\Sigma$ are called sectorized Grassmann functions.

(ii) Let

$$\mathcal{W} = \sum_{m,n\geq 0} \int \prod_{i=1}^{m} d\eta_i \prod_{j=1}^{n} d\xi_j W_{m,n}(\eta_1,\dots,\eta_m;\xi_1,\dots,\xi_n)$$
$$\times \phi(\eta_1)\cdots\phi(\eta_m)\psi(\xi_1)\cdots\psi(\xi_n)$$

be a Grassmann function with $W_{m,n} \in \mathcal{F}_m(n)$ antisymmetric in its internal ($=\psi$) variables. A sectorized representative for $\mathcal{W}$ is a sectorized Grassmann function of the form

$$w = \sum_{m,n\geq 0} Gr(w_{m,n})$$

where, for each m, n, $w_{m,n}$ is a sectorized representative for $W_{m,n}$ that is also antisymmetric in the variables $(\xi_1, s_1), \dots, (\xi_n, s_n)$.

Remark XII.7. Let I_O be the ideal in $\bigwedge_A V$ consisting of all

$$\mathcal{W} = \sum_{n>0} \int \prod_{j=1}^{n} d\xi_j W_n(\xi_1,\dots,\xi_n)\psi(\xi_1)\cdots\psi(\xi_n)$$

obeying

$$\check{W}_n((k_1,\sigma_1,a_1),\dots,(k_n,\sigma_n,a_n)) = 0 \quad \text{for all } k_1,\dots,k_n \text{ in } j\text{th neighborhood}$$

and let $V_\Sigma^{\rm eff}$ be the linear subspace of V_Σ consisting of all

$$\mathcal{V} = \sum_{s\in\Sigma} \int d\xi \varphi((\xi,s))\psi(\xi,s)$$

obeying

$$\check{\varphi}((k,\sigma,a,s)) = 0 \quad \text{unless } k \in \tilde{s}\,.$$

Furthermore let $\pi : V_\Sigma \to V$ be the linear map that sends $\psi(\xi,s) \in V_\Sigma$ to $\psi(\xi) \in V$. It induces an algebra homomorphism from $\bigwedge_A V_\Sigma$ to $\bigwedge_A V$, which we again denote by π. Then the sectorized Grassmann function w is a sectorized representative for the Grassmann function $\mathcal{W}$ if and only if $w \in \bigwedge_A V_\Sigma^{\rm eff}$ and $\pi(w) - \mathcal{W} \in I_O$.

Proposition XII.8 (Functoriality). Let $C(\xi,\xi')$ be a skew symmetric function on $\mathcal{B}\times\mathcal{B}$. Assume that there is an antisymmetric function $c((\xi,s),(\xi',s')) \in \mathcal{F}_0(2;\Sigma)$ such that

$$C(\xi,\xi') = \sum_{s,s'\in\Sigma} c((\xi,s),(\xi',s'))$$

and

$$\check{c}((k,\sigma,a,s),(k',\sigma',a',s')) = 0$$

unless[a] $k \in s$, $k' \in s'$. Define a covariance on V_Σ by

$$C_\Sigma(\psi(\xi,s),\psi(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} c((\xi,t),(\xi',t'))\,.$$

(i) If $\varphi \in \mathcal{F}_0(n;\Sigma)$ then

$$\sum_{s_1,\ldots,s_n\in\Sigma} \iint d\xi_1\cdots d\xi_n \varphi((\xi_1,s_1),\ldots,(\xi_n,s_n))\psi(\xi_1,s_1)\cdots\psi(\xi_n,s_n)d\mu_{C_\Sigma}(\psi)$$

$$= \iint d\xi_1\cdots d\xi_n \left[\sum_{s_1,\ldots,s_n\in\Sigma}\varphi((\xi_1,s_1),\ldots,(\xi_n,s_n))\right]$$

$$\times\,\psi(\xi_1)\cdots\psi(\xi_n)d\mu_C(\psi)\,.$$

(ii) Let $\mathcal{W}(\phi;\psi)$ be an even Grassmann function and w a sectorized representative for $\mathcal{W}$. Then $\Omega_{C_\Sigma}(w)$ is a sectorized representative for $\Omega_C(\mathcal{W})$.

For any $\zeta(\xi)$, set, with some abuse of notation,

$$(C\zeta)(\xi,s) = \int d\xi' C(\xi,\xi')\zeta(\xi')\,.$$

Then $\frac{1}{2}\phi JCJ\phi + \Omega_{C_\Sigma}(w)(\phi,\psi+CJ\phi)$ is a sectorized representative for $\tilde\Omega_C(\mathcal{W})$.

(iii) Let $\mathcal{W}(\phi;\psi)$ be a Grassmann function and w a sectorized representative for $\mathcal{W}$. Then $:w:_{C_\Sigma}$ is a sectorized representative for $:\mathcal{W}:_C$.

Proof. (i) First consider $n=2$. Then

$$\sum_{s_1,s_2\in\Sigma}\iint d\xi_1 d\xi_2 \varphi((\xi_1,s_1),(\xi_2,s_2))\psi(\xi_1,s_1)\psi(\xi_2,s_2)d\mu_{C_\Sigma}(\psi)$$

$$= \sum_{s_1,s_2\in\Sigma}\int d\xi_1 d\xi_2\varphi((\xi_1,s_1),(\xi_2,s_2))C_\Sigma(\psi(\xi_1,s_1),\psi(\xi_2,s_2))$$

$$= \sum_{\substack{s_1,s_2,t_1,t_2\in\Sigma\\ t_1\cap s_1\neq\emptyset\\ t_2\cap s_2\neq\emptyset}}\int d\xi_1 d\xi_2\varphi((\xi_1,s_1),(\xi_2,s_2))c((\xi_1,t_1),(\xi_2,t_2))$$

$$= \sum_{s_1,s_2,t_1,t_2\in\Sigma}\int d\xi_1 d\xi_2\varphi((\xi_1,s_1),(\xi_2,s_2))c((\xi_1,t_1),(\xi_2,t_2))$$

[a] The hypothesis $c((\xi,s),(\xi',s')) \in \mathcal{F}_0(2;\Sigma)$ implies that $\check c((k,\cdot,s),(k',\cdot,s'))$ vanishes unless $k\in\tilde s$, $k'\in\tilde s'$. Here we further require that $\check c((k,\cdot,s),(k',\cdot,s'))$ vanish unless k and k' are in the jth neighborhood.

$$= \int d\xi_1 d\xi_2 \left[\sum_{s_1, s_2 \in \Sigma} \varphi((\xi_1, s_1), (\xi_2, s_2)) \right] C(\xi_1, \xi_2)$$

$$= \iint d\xi_1 d\xi_2 \left[\sum_{s_1, s_2 \in \Sigma} \varphi((\xi_1, s_1), (\xi_2, s_2)) \right] \psi(\xi_1)\psi(\xi_2) d\mu_C(\psi) .$$

In the third equality, we used conservation of momentum to imply that

$$\int d\xi_1 d\xi_2 \varphi((\xi_1, s_1), (\xi_2, s_2)) c((\xi_1, t_1), (\xi_2, t_2)) = 0$$

unless $\tilde{s}_1 \cap t_1 \neq \emptyset$ and $\tilde{s}_2 \cap t_2 \neq \emptyset$ and hence unless $s_1 \cap t_1 \neq \emptyset$ and $s_2 \cap t_2 \neq \emptyset$.

The claim for general n is now proven by induction on n using integration by parts (see, for example, [1, Sec. II.2]).

(ii) Set $\mathcal{W}' = \mathcal{W} - \pi(w) \in I_O$. By assumption, $\check{C}((k, \sigma, a), (k', \sigma', a')) = 0$ unless k and k' both lie in the jth neighborhood. Therefore $\int f(\psi + \zeta) d\mu_C(\zeta) \in I_O$ for all $f(\zeta) \in I_O$. Consequently

$$\int e^{\pi(w)(\phi, \psi+\zeta)} d\mu_C(\zeta) - \int e^{\mathcal{W}(\phi, \psi+\zeta)} d\mu_C(\zeta)$$

$$= \int e^{\pi(w)(\phi, \psi+\zeta)} [1 - e^{\mathcal{W}'(\phi, \psi+\zeta)}] d\mu_C(\zeta) \in I_O$$

since $1 - e^{\mathcal{W}'(\phi, \psi)} \in I_O$ and I_O is an ideal. In particular

$$Z(\pi(w)) = \int e^{\pi(w)(0, \zeta)} d\mu_C(\zeta) = \int e^{\mathcal{W}(0, \zeta)} d\mu_C(\zeta) = Z(\mathcal{W})$$

so that

$$\frac{1}{Z(\pi(w))} \int e^{\pi(w)(\phi, \psi+\zeta)} d\mu_C(\zeta) - \frac{1}{Z(\mathcal{W})} \int e^{\mathcal{W}(\phi, \psi+\zeta)} d\mu_C(\zeta) \in I_O .$$

Expanding the power series for $\log(1 + x)$, one sees that

$$\Omega_C(\pi(w)) - \Omega_C(\mathcal{W}) \in I_O .$$

As $C_\Sigma(v, v') = C(\pi(v), \pi(v'))$, for all $v, v' \in V_\Sigma^{\mathrm{eff}}$, functoriality of the renormalization group map ([1, Remarks III.3 and III.1(i)]) implies that $\Omega_C(\pi(w)) = \pi(\Omega_{C_\Sigma}(w))$. So $\Omega_C(\mathcal{W}) - \pi(\Omega_{C_\Sigma}(w)) \in I_O$. Also, by construction, $\Omega_{C_\Sigma}(w) \in \bigwedge_A V_\Sigma^{\mathrm{eff}}$. Hence, by Remark XII.7, $\Omega_{C_\Sigma}(w)$ is a sectorized representative for $\Omega_C(\mathcal{W})$.

If $v(\phi, \psi)$ is a sectorized representative for $V(\phi, \psi)$, then $v(\phi, \psi + CJ\phi)$ is a sectorized representative for $V(\phi, \psi + CJ\phi)$. Therefore the second claim follows from the first and Lemma VII.3.

(iii) Part (iii) is an immediate consequence of part (i) and [1, Proposition A.2(i)]. □

Definition XII.9 (Norms for sectorized functions). Let Σ be a sectorization at scale $j \geq 2$ and let $m, n \geq 0$ and $p > 0$ be integers.

(i) For a function φ on $\mathcal{B}^m \times (\mathcal{B} \times \Sigma)^n$ we define the seminorm $|\varphi|_{p,\Sigma}$ to be zero if $m \geq 1$, $p \geq 2$ or if $m = 0$, $p > n$.

In the case $m \geq 1$, $p = 1$ we set

$$|\varphi|_{p,\Sigma} = \sum_{s_i \in \Sigma} \|\varphi(\eta_1, \ldots, \eta_m; (\xi_1, s_1), \ldots, (\xi_n, s_n))\|_{1,\infty} .$$

In the case $m = 0$, $p \leq n$ we set

$$|\varphi|_{p,\Sigma} = \max_{1 \leq i_1 < \cdots < i_p \leq n} \max_{s_{i_1}, \ldots, s_{i_p} \in \Sigma} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \neq i_1, \ldots, i_p}} \|\varphi((\xi_1, s_1), \ldots, (\xi_n, s_n))\|_{1,\infty} .$$

In both cases, the $\|\cdot\|_{1,\infty}$ norm (defined in Example II.6) applies to all the position space variables. Furthermore, maxima acting on a formal power series $\sum_\delta a_\delta t^\delta$ are to be applied separately to each coefficient a_δ.

(ii) Let $f \in A_m \otimes (V_\Sigma^{\mathrm{eff}})^{\otimes n}$ be a sectorized Grassmann function. Then there is a unique $\varphi \in \mathcal{F}_m(n; \Sigma)$ such that $f = \mathrm{Tens}(\varphi)$. By definition

$$|f|_{p,\Sigma} = \begin{cases} |\varphi|_{p,\Sigma} & \text{if } \varphi \text{ is translation invariant, conserves particle numbers} \\ & \qquad \text{and is spin independent} \\ \infty & \text{otherwise} . \end{cases}$$

Example XII.10. Let $f \in \mathcal{F}_m(n)$ with $m \geq 1$. Then f has a sectorized representative φ fulfilling

$$|\varphi|_{1,\Sigma} \leq \left(\frac{const}{\mathfrak{l}}\right)^n \mathfrak{c}_j^n \|f\|_{1,\infty} .$$

Proof. Select a sectorized representative φ for f as in Example XII.5. For each choice of sectors $s_1, \ldots, s_n$ of sectors in Σ

$$\begin{aligned} &\|\varphi(\eta_1, \ldots, \eta_m; (\xi_1, s_1), \ldots, (\xi_n, s_n))\|_{1,\infty} \\ &\quad = \left\| \int \prod_{i=1}^n (d\xi_i' \hat{\chi}_{s_i}(\xi_i, \xi_i')) f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n) \right\|_{1,\infty} \\ &\quad \leq \|f\|_{1,\infty} \prod_{i=1}^n \|\hat{\chi}_{s_i}\|_{1,\infty} \\ &\quad \leq const^n \mathfrak{c}_{j-1}^n \|f\|_{1,\infty} \end{aligned}$$

by Lemma II.7 (n times) and Lemma XII.3(iii). As there are $\frac{const}{\mathfrak{l}}$ sectors, the claim follows from the definition. □

Remark XII.11. Let $\mathcal{D}$ be a decay operator and φ a function on $(\mathcal{B} \times \Sigma)^n$. Then

$$|\mathcal{D}\varphi|_{p,\Sigma} \leq \frac{\partial^{|\delta(\mathcal{D})|}}{\partial t^{\delta(\mathcal{D})}} |\varphi|_{p,\Sigma} .$$

Lemma XII.12. *Let $\varphi \in \mathcal{F}_0(n;\Sigma)$ be a sectorized representative for a translation invariant $f \in \mathcal{F}_0(n)$. Then, for $p \le n$,*

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!}\left[\max_{\substack{\mathrm{D}\ \text{differential operator}\\ \text{with}\ \delta(\mathrm{D})=\delta}} \sup_{\substack{\check\eta_1,\ldots,\check\eta_n\\ \check\eta_1+\cdots+\check\eta_n=0\\ k_1,\ldots,k_n\ \text{in}\ j\text{th neighborhood}}} |\mathrm{D}\check f(\check\eta_1,\ldots,\check\eta_n)|\right] t^\delta$$

$$\le 2^p\,|\varphi|_{p,\Sigma}\,.$$

Here $\check\eta_i = (k_i,\sigma_i,b_i)$ and the differential operators D *are defined in Definition* X.2(iii).

Proof. Fix a differential operator $\mathrm{D} = \mathrm{D}^{\delta^{(1)}}_{u_1;v_1}\cdots \mathrm{D}^{\delta^{(q)}}_{u_q;v_q}$ and let $\mathcal{D} = \mathcal{D}^{\delta^{(1)}}_{u_1,v_1}\cdots \mathcal{D}^{\delta^{(q)}}_{u_q,v_q}$ be the corresponding decay operator as in Definition II.3. Fix any $\check\eta_i = (k_i,\sigma_i,b_i)$, $1\le i\le n$ with $k_1,\ldots,k_n$ in the jth neighborhood and $\sum_{i=1}^n(-1)^{b_i}$ $k_i = 0$. Then

$$\check f(\check\eta_1,\ldots,\check\eta_n) = \sum_{\substack{s_i\ni k_i\\ 1\le i\le n}} \check\varphi((k_1,\sigma_1,b_1,s_1),\ldots,(k_n,\sigma_n,b_n,s_n))$$

so that

$$\begin{aligned}|\mathrm{D}\check f(\check\eta_1,\ldots,\check\eta_n)| &\le \sum_{\substack{s_i\ni k_i\\ 1\le i\le p}} \sum_{\substack{s_i\in\Sigma\\ p+1\le i\le n}} \int \prod_{\ell\ne n} d\xi_\ell \\ &\qquad \cdot |\mathcal{D}\varphi((\xi_1,s_1),\ldots,(\xi_{n-1},s_{n-1}),(0,\sigma_n,b_n,s_n))| \\ &\le 2^p \max_{s_1,\ldots,s_p} \sum_{\substack{s_i\in\Sigma\\ p+1\le i\le n}} |||\mathcal{D}\varphi((\cdot,s_1),\ldots,(\cdot,s_n))|||_{1,\infty}\end{aligned}$$

since each k_i can be contained in at most two sectors. □

Remark XII.13. We will use the norms of Definition XII in a multi scale analysis to prove the existence of Green's functions in the position space L_∞-norm. This is the reason why we take the suprema over the external variables $\eta_1,\ldots,\eta_m$, and why we do not "sectorize" these variables.

In Sec. XVI, we will introduce another set of norms, designed to study the smoothness properties of the amputated two and four-point functions in momentum space.

Lemma XII.14. *Let Σ be a sectorization and let φ be a function on $\mathcal{B}^m\times(\mathcal{B}\times\Sigma)^n$, φ' be a function on $\mathcal{B}^{m'}\times(\mathcal{B}\times\Sigma)^{n'}$ and $1\le i\le n$, $1\le i'\le n'$. Define the function γ on $\mathcal{B}^{m+m'}\times(\mathcal{B}\times\Sigma)^{n+n'-2}$ by*

$$\begin{aligned}\gamma(\eta_1,\ldots,\eta_{m+m'};(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),(\xi_{i+1},s_{i+1}),\ldots,(\xi_{n+i'-1},s_{n+i'-1})\,,\\ (\xi_{n+i'+1},s_{n+i'+1}),\ldots,(\xi_{n+n'},s_{n+n'}))\end{aligned}$$

$$= \sum_{\substack{s,s'\in\Sigma \\ s\cap s'\neq\emptyset}} \int_{\mathcal{B}} d\zeta\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),(\zeta,s),(\xi_{i+1},s_{i+1}),\ldots,$$

$$(\xi_n,s_n))\varphi'(\eta_{m+1},\ldots,\eta_{m+m'};(\xi_{n+1},s_{n+1}),\ldots,(\xi_{n+i'-1},s_{n+i'-1}),(\zeta,s'),$$

$$(\xi_{n+i'+1},s_{n+i'+1}),\ldots,(\xi_{n+n'},s_{n+n'}))\,.$$

If $m=0$ *or* $m'=0$,

$$|\gamma|_{p,\Sigma}\le 3\max_{\substack{p_1+p_2=p+1 \\ p_1,p_2\ \mathrm{odd}}}|\varphi|_{p_1,\Sigma}|\varphi'|_{p_2,\Sigma}$$

for all odd natural numbers p.

Proof. The variable indices for γ lie in the set $I\cup I'$, where

$$I=\{1,\ldots,i-1,\ i+1,\ldots,n\}$$

$$I'=\{n+1,\ldots,n+i'-1,\ n+i'+1,\ldots,n+n'\}\,.$$

Fix $u_1,\ldots,u_q\in I$, $u_{q+1},\ldots,u_p\in I'$ and fix sectors $s_{u_1},\ldots,s_{u_p}\in\Sigma$.

First assume that q is odd. Then $p-q$ is even. By Lemma II.7, for each choice of sectors s_ν, $\nu\in I\cup I'\backslash\{u_1,\ldots,u_p\}$ one has

$$\|\gamma(\eta_1,\ldots,\eta_{m+m'};(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),(\xi_{i+1},s_{i+1}),\ldots,$$

$$(\xi_{n+i'+1},s_{n+i'+1}),\ldots,(\xi_{n+n'},s_{n+n'}))\|_{1,\infty}$$

$$\le \sum_{\substack{s,s'\in\Sigma \\ s\cap s'\neq\emptyset}}\|\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),(\zeta,s),$$

$$(\xi_{i+1},s_{i+1}),\ldots,(\xi_n,s_n))\|_{1,\infty}\|\varphi'(\eta_{m+1},\ldots,\eta_{m+m'};$$

$$(\xi_{n+1},s_{n+1}),\ldots,(\xi_{n+i'-1},s_{n+i'-1}),(\zeta',s'),\ldots,(\xi_{n+n'},s_{n+n'}))\|_{1,\infty}\,.$$

Observe that for every $s\in\Sigma$ there are at most three sectors s' such that $s'\cap s\neq\emptyset$. Consequently

$$\sum_{\substack{s_\nu\in\Sigma \\ \nu\in I\cup I'\backslash\{u_1,\ldots,u_p\}}}\|\gamma(\eta_1,\ldots,\eta_{m+m'};(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),$$

$$(\xi_{i+1},s_{i+1}),\ldots,(\xi_{n+i'+1},s_{n+i'+1}),\ldots,(\xi_{n+n'},s_{n+n'}))\|_{1,\infty}$$

$$\le 3\sum_{\substack{s_\nu\in\Sigma \\ \nu\in I\backslash\{u_1,\ldots,u_q\}}}\ \sum_{s\in\Sigma}\|\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),(\zeta,s),$$

$$(\xi_{i+1},s_{i+1}),\ldots,(\xi_n,s_n))\|_{1,\infty}\max_{s'\in\Sigma}\sum_{\substack{s'_\mu\in\Sigma\\ \mu\in I'\setminus\{u_{q+1},\ldots,u_p\}}}\|\varphi'(\eta_{m+1},\ldots,\eta_{m+m'};$$

$$(\xi_{n+1},s_{n+1}),\ldots,(\xi_{n+i'-1},s_{n+i'-1}),(\zeta',s'),\ldots,(\xi_{n+n'},s_{n+n'}))\|_{1,\infty}$$

$$\le 3|\varphi|_{q,\Sigma}|\varphi'|_{p-q+1,\Sigma}\,.$$

The case that q is even follows as in the case discussed above by interchanging the roles of φ and φ'. □

We define "contraction" for sectorized functions as the obvious generalization of Definition III.1.

Definition XII.15. Let $c((\xi,s),(\xi',s'))$ be any skew symmetric function on $(\mathcal{B}\times\Sigma)^2$. Let $m,n\ge 0$ and $1\le i<j\le n$. For $\varphi\in\mathcal{F}_m(n;\Sigma)$ the contraction $\underset{i\to j}{\mathcal{C}\mathrm{on}_c}\,\varphi\in\mathcal{F}_m(n-2;\Sigma)$ is defined as

$$\underset{i\to j}{\mathcal{C}\mathrm{on}_c}\,\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),$$

$$(\xi_{i+1},s_{i+1}),\ldots,(\xi_{j-1},s_{j-1}),(\xi_{j+1},s_{j+1}),\ldots,(\xi_n,s_n))$$

$$=(-1)^{j-i+1}\sum_{\substack{s_i,s_j,t_i,t_j\in\Sigma\\ t_i\cap s_i\neq\emptyset\\ t_j\cap s_j\neq\emptyset}}\int d\xi_i d\xi_j\, c((\xi_i,t_i),(\xi_j,t_j))$$

$$\cdot\,\varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))\,.$$

Proposition XII.16. *Let Σ be a sectorization of length $\mathfrak{l}$ at scale $j\ge 2$ and let $c((\xi,s),(\xi',s'))\in\mathcal{F}_0(2;\Sigma)$ be an antisymmetric function.*

(i) *Let p be an odd integer, $m,m'\ge 0$, $n,n'\ge 1$ and $\varphi\in\mathcal{F}_m(n;\Sigma)$, $\varphi'\in\mathcal{F}_{m'}(n',\Sigma)$. If $m=0$ or $m'=0$*

$$\left|\underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\,\mathrm{Ant}_{\mathrm{ext}}(\varphi\otimes\varphi')\right|_{p,\Sigma}\le 9|c|_{1,\Sigma}\max_{\substack{p_1+p_2=p+1\\ p_1,p_2\ \mathrm{odd}}}|\varphi|_{p_1,\Sigma}|\varphi'|_{p_2,\Sigma}\,.$$

If $m\neq 0$ and $m'\neq 0$

$$\left|\underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\,\mathrm{Ant}_{\mathrm{ext}}(\varphi\otimes\varphi')\right|_{1,\Sigma}\le 9\left(\sup_{\xi,\xi',s,s'}|c((\xi,s),(\xi',s'))|\right)|\varphi|_{1,\Sigma}|\varphi'|_{1,\Sigma}\,.$$

(ii) *Assume that there is a function $C(k)$ that is supported in the jth neighborhood, such that $c((\cdot,s),(\cdot,s'))$ is the Fourier transform of $\chi_s(k)C(k)\chi_{s'}(k)$ in the sense of Definition* IX.3 *and that $|C(k)|\le\frac{\varepsilon}{|\imath k_0-e(\mathbf{k})|}$ for some $\varepsilon\ge 0$.*

Let $\varphi \in \mathcal{F}_m(n;\Sigma)$, $n' \le n$ *and set, as in Definition* III.5 *of integral bound,*

$$\varphi'(\eta_1,\dots,\eta_m;(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))$$
$$= \sum_{\substack{s_i\in\Sigma\\ i=1,\dots,n'}} \iint d\xi_1\cdots d\xi_{n'}\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,$$
$$(\xi_{n'},s_{n'}),\dots,(\xi_n,s_n))\psi(\xi_1,s_1)\cdots\psi(\xi_{n'},s_{n'})d\mu_{C_\Sigma}(\psi)$$

where

$$C_\Sigma(\psi(\xi,s),\psi(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} c((\xi,t),(\xi',t'))\,.$$

Then for all p

$$|\varphi'|_{p,\Sigma} \le \left(\varepsilon \mathrm{B}_1 \frac{\mathfrak{l}}{M^j}\right)^{n'/2} |\varphi|_{p,\Sigma}$$

with a constant B_1 *that is independent of* j *and* Σ.

Proof. (i) Set

$$\gamma(\eta_1,\dots,\eta_{m+m'};(\xi_2,s_2),\dots,(\xi_n,s_n),(\xi_{n+2},s_{n+2}),\dots,(\xi_{n+n'},s_{n+n'}))$$
$$= \sum_{\substack{s,s',t,t'\in\Sigma\\ t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} \int d\zeta d\eta \varphi(\eta_1,\dots,\eta_m;(\zeta,s),(\xi_2,s_2),\dots,(\xi_n,s_n))c((\zeta,t),(\eta,t'))$$
$$\times \varphi'(\eta_{m+1},\dots,\eta_{m+m'};(\eta,s')(\xi_{n+2},s_{n+2}),\dots,(\xi_{n+n'},s_{n+n'}))\,.$$

Then $(-1)^{n+1}\,\mathrm{Ant}_{\mathrm{ext}}\gamma = \underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\mathrm{Ant}_{\mathrm{ext}}(\varphi\otimes\varphi')$. If $m\neq 0$ and $m'\neq 0$ then

$$\left|\underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\mathrm{Ant}_{\mathrm{ext}}(\varphi\otimes\varphi')\right|_{1,\Sigma} = |\mathrm{Ant}_{\mathrm{ext}}\gamma|_{1,\Sigma}$$
$$\le 9\left(\sup_{\xi,\xi',t,t'}|c((\xi,t),(\xi',t'))|\right)|\varphi|_{1,\Sigma}|\varphi'|_{1,\Sigma}\,.$$

If $m=0$ or $m'=0$, by iterated application of Lemma XII.14

$$|\gamma|_{p,\Sigma} \le 3 \max_{\substack{p_1+p_2=p+1\\ p_1,p_2 \text{ odd}}} \left| \sum_{\substack{s,t\in\Sigma\\ s\cap t\neq\emptyset}} \int d\zeta\varphi(\eta_1,\dots,\eta_m;\right.$$
$$\left.(\zeta,s),(\xi_2,s_2),\dots,(\xi_n,s_n))c((\zeta,t),(\eta,t'))\right|_{p_1,\Sigma} |\varphi'|_{p_2,\Sigma}$$
$$\le 9 \max_{\substack{p_1+p_2=p+1\\ p_1,p_2 \text{ odd}}} |\varphi|_{p_1,\Sigma}|c|_{1,\Sigma}|\varphi'|_{p_2,\Sigma}\,.$$

(ii) Define the covariance $C(\xi,\xi')$ to be the Fourier transform of $C(k)$. By part (i) of Proposition XII,

$$\varphi'(\eta_1,\dots,\eta_m;(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))$$

$$= \iint d\xi_1\cdots d\xi_{n'}\left[\sum_{s_1,\dots,s_{n'}\in\Sigma}\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_{n'},s_{n'}),\dots,(\xi_n,s_n))\right]$$

$$\cdot\,\psi(\xi_1)\cdots\psi(\xi_{n'})d\mu_C(\psi)\,.$$

Since $\varphi\in\mathcal{F}_m(n;\Sigma)$

$$\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_{n'},s_{n'}),(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))$$

$$= \int d\xi'_1\cdots d\xi'_{n'}\varphi(\eta_1,\dots,\eta_m;(\xi'_1,s_1),\dots,(\xi'_{n'},s_{n'})\,,$$

$$(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))\prod_{i=1}^{n'}\hat{\tilde{\chi}}_{s_i}(\xi'_i,\xi_i)\,.$$

Consequently

$$\varphi'(\eta_1,\dots,\eta_m;(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))$$

$$= \sum_{s_1,\dots,s_{n'}\in\Sigma}\iint d\xi'_1\cdots d\xi'_{n'}\varphi(\eta_1,\dots,\eta_m;(\xi'_1,s_1),\dots,$$

$$(\xi'_{n'},s_{n'}),\dots,(\xi_n,s_n))\psi_{s_1}(\xi'_1)\cdots\psi_{s_{n'}}(\xi'_{n'})d\mu_C(\psi)$$

where

$$\psi_s(\xi') = \int d\xi\,\hat{\tilde{\chi}}_s(\xi',\xi)\psi(\xi)\,.$$

Therefore, by Proposition IV.3(ii)

$$|\varphi'(\eta_1,\dots,\eta_m;(\xi_{n'+1},s_{n'+1}),\dots,(\xi_n,s_n))|$$

$$\le \sum_{s_1,\dots,s_{n'}\in\Sigma}\int d\xi_1\cdots d\xi_{n'}|\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_n,s_n))|$$

$$\cdot\left|\int\psi_{s_1}(\xi_1)\cdots\psi_{s_{n'}}(\xi_{n'})d\mu_C(\psi)\right|$$

$$\le G^{n'/2}\sum_{s_1,\dots,s_{n'}\in\Sigma}\int d\xi_1\cdots d\xi_{n'}|\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_n,s_n))|$$

with

$$G = \int\frac{d^{d+1}k}{(2\pi)^{d+1}}\tilde{\chi}_s(k)^2|C(k)| \le \varepsilon\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{\tilde{\chi}_s(k)^2}{|\imath k_0 - e(\mathbf{k})|} \le \text{const}\,\varepsilon\frac{\mathfrak{l}}{M^j}\,. \qquad \square$$

Remark XII.17. If C fulfills the hypothesis of part (i) of the proposition, then

$$\mathfrak{c} = 9 \max \left\{ |c|_{1,\Sigma},\ \sup_{\xi,\xi',s,s'} |c((\xi, s), (\xi', s'))| \right\}$$

is a contraction bound for the system $|\cdot|_{1,\Sigma}$ of seminorms. We shall show in Proposition XIII.5, that for $C^{(j)}(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k})}$ and $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$, the constant coefficient $\mathfrak{c}_{\mathbf{0}}$ of $\mathfrak{c}$ is bounded by const M^j. On the other hand, part (ii) of Proposition XII.16 shows that $\mathrm{b} = \mathrm{const}\sqrt{\frac{\mathfrak{l}}{M^j}}$ is an integral bound for $C_\Sigma^{(j)}$ with respect to this system of seminorms. Thus, if $\mathcal{W}(\psi)$ is an even Grassmann function with sectorized representative

$$w(\psi) = \sum_{n=0}^{\infty} \sum_{s_1,\ldots,s_{2n}\in\Sigma} \int d\xi_1 \cdots d\xi_{2n} w_{2n}((\xi_1, s_1), \ldots, (\xi_{2n}, s_{2n})) \cdot \psi(\xi_1, s_1) \cdots \psi(\xi_{2n}, s_{2n})$$

the quantity $N(\mathcal{W}; \mathfrak{c}, \mathrm{b}, \alpha)$ of [1, Definition II.23] (with V replaced by V_Σ) has

$$N(w; \mathfrak{c}, \mathrm{b}, \alpha)_{\mathbf{0}} \leq \mathrm{const} \left\{ \alpha^2 M^j |w_2|_{1,\Sigma} + \alpha^4 \mathfrak{l} |w_4|_{1,\Sigma} + \frac{M^{2j}}{\mathfrak{l}} \sum_{n\geq 3} \left(\mathrm{const}\, \frac{\alpha^2 \mathfrak{l}}{M^j} \right)^n |w_{2n}|_{1,\Sigma} \right\}.$$

In contrast to the situation of Remark VIII.8, this norm is of order one if $|w_4|_{1,\Sigma}$ is of order $\frac{1}{\mathfrak{l}}$, which is approximately the number of sectors. As $d = 2$, this is a realistic estimate for the original interaction $\mathcal{V}$, with all momenta restricted to the jth shell. Observe, however, that for $d \geq 3$ this estimate is not expected to hold. See [2, Sec. II, Subsec. 8].

For more precise control of W_4 one also uses the norm $|w_4|_{3,\Sigma}$.

To prepare for the application of [3, Theorem VI.6] about overlapping loops, we state

Proposition XII.18. *Let Σ be a sectorization of length $\mathfrak{l}$ at scale $j \geq 2$ and let $c((\xi, s), (\xi', s')) \in \mathcal{F}_0(2; \Sigma)$ be an antisymmetric function. Let $D(k)$, $D'(k)$ be functions obeying $|D(k)|$, $|D'(k)| \leq \frac{2}{|\imath k_0 - e(\mathbf{k})|}$ and let $d((\cdot, s), (\cdot, s'))$ resp. $d'((\cdot, s), (\cdot, s'))$ be the Fourier transform of $\chi_s(k)D(k)\chi_{s'}(k)$ resp. $\chi_s(k)D'(k)\chi_{s'}(k)$ in the sense of Definition* IX.3.

Let $1 \leq i_1, i_2, i_3 \leq n$, $1 \leq i'_1, i'_2, i'_3 \leq n'$ with $i_1 \neq i_2 \neq i_3 \neq i_1$, $i'_1 \neq i'_2 \neq i'_3 \neq i'_1$, and let p be an odd natural number. Then for $\varphi \in \mathcal{F}_0(n; \Sigma)$, $\varphi' \in \mathcal{F}_0(n', \Sigma)$

$$\left| \underset{i_1\to n+i'_1}{\mathcal{C}\mathrm{on}_c}\ \underset{i_2\to n+i'_2}{\mathcal{C}\mathrm{on}_d}\ \underset{i_3\to n+i'_3}{\mathcal{C}\mathrm{on}_{d'}} (\varphi \otimes \varphi') \right|_{p,\Sigma} \leq \left(\mathrm{B}_2 \frac{\mathfrak{l}}{M^j} \right)^2 |c|_{1,\Sigma} \max_{\substack{p_1+p_2=p+3\\ p_1,p_2\ \mathrm{odd}}} |\varphi|_{p_1,\Sigma} |\varphi'|_{p_2,\Sigma}$$

with a constant B_2 *that is independent of j and Σ.*

Proof. By the symmetry of the norms we may assume that $i_1 = i'_1 = 1$, $i_2 = i'_2 = 2$, $i_3 = i'_3 = 3$. Set

$$\begin{aligned}
&\gamma((\xi_4, s_4), \ldots, (\xi_n, s_n), (\xi_{n+4}, s_{n+4}), \ldots, (\xi_{n+n'}, s_{n+n'})) \\
&\quad = \sum_{\substack{s_i, t_i, t'_i, s'_i \in \Sigma \\ i=1,2,3}} \int \prod_{i=1}^{3} (d\zeta_i d\eta_i) \varphi((\zeta_1, s_1), (\zeta_2, s_2), (\zeta_3, s_3), (\xi_4, s_4), \ldots, (\xi_n, s_n)) \\
&\qquad \cdot c((\zeta_1, t_1), (\eta_1, t'_1)) d((\zeta_2, t_2), (\eta_2, t'_2)) d'((\zeta_3, t_3), (\eta_3, t'_3)) \\
&\qquad \cdot \varphi'((\eta_1, s'_1), (\eta_2, s'_2), (\eta_3, s'_3), (\xi_{n+4}, s_{n+4}), \ldots, (\xi_{n+n'}, s_{n+n'}))\,.
\end{aligned}$$

Then

$$(-1)^{3(n+1)} \operatorname{Ant}_{\text{ext}} \gamma = \underset{i_1 \to n+i'_1}{\mathcal{C}\text{on}_c} \; \underset{i_2 \to n+i'_2}{\mathcal{C}\text{on}_d} \; \underset{i_3 \to n+i'_3}{\mathcal{C}\text{on}_{d'}} (\varphi \otimes \varphi')\,.$$

Observe that $d((\zeta, t), (\eta, t')) = 0$ if $t \cap t' = \emptyset$ and that

$$\begin{aligned}
\sup_{\zeta, \eta; t, t'} |d((\zeta, t), (\eta, t')| &\le \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \chi_t(k) |D(k)| \chi_{t'}(k) \\
&\le \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{2\chi_t(k)\chi_{t'}(k)}{|\imath k_0 - e(\mathbf{k})|} \\
&\le \text{const}\, \frac{\mathfrak{l}}{M^j}\,. \qquad \text{(XII.1)}
\end{aligned}$$

The same properties hold for d'.

Set

$$\begin{aligned}
&\varphi''((\zeta_1, t_1), (\xi_2, s_2), \ldots, (\xi_{n'}, s_{n'})) \\
&\quad = \sum_{t'_1, s'_1 \in \Sigma} \int d\eta_1 c((\zeta_1, t_1), (\eta_1, t'_1)) \varphi'((\eta_1, s'_1), (\xi_2, s_2), \ldots, (\xi_{n'}, s_{n'}))\,.
\end{aligned}$$

By Lemma XII.14

$$|\varphi''|_{p,\Sigma} \le 3|c|_{1,\Sigma} |\varphi'|_{p,\Sigma}\,.$$

Furthermore $\check{\varphi}''((k_1, \sigma_1, a_1, t_1), (\check{\xi}_2, s_2), \ldots, (\check{\xi}_{n'}, s_{n'})) = 0$ unless $k_1 \in \tilde{t}_1$. Set

$$\begin{aligned}
&\tilde{\gamma}((\xi_4, s_4), \ldots, (\xi_n, s_n), (\xi_{n+4}, s_{n+4}), \ldots, (\xi_{n+n'}, s_{n+n'})\,; \\
&\qquad s_1, t_1, s_2, t_2, t'_2, s'_2, s_3, t_3, t'_3, s'_3) \\
&\quad = \int d\zeta_1 d\zeta_2 d\eta_2 d\zeta_3 d\eta_3 \varphi((\zeta_1, s_1), (\zeta_2, s_2), (\zeta_3, s_3), (\xi_4, s_4), \ldots, (\xi_n, s_n)) \\
&\qquad \cdot d((\zeta_2, t_2), (\eta_2, t'_2)) d'((\zeta_3, t_3), (\eta_3, t'_3)) \varphi''((\zeta_1, t_1), (\eta_2, s'_2), (\eta_3, s'_3)\,, \\
&\qquad (\xi_{n+4}, s_{n+4}), \ldots, (\xi_{n+n'}, s_{n+n'}))\,.
\end{aligned}$$

Then

$$\gamma((\xi_4,s_4),\dots,(\xi_n,s_n),(\xi_{n+4},s_{n+4}),\dots,(\xi_{n+n'},s_{n+n'}))$$
$$= \sum_{s_1,t_1\in\Sigma} \sum_{\substack{s_i,t_i,t'_i,s'_i\in\Sigma \\ i=2,3}} \tilde\gamma((\xi_4,s_4),\dots,(\xi_n,s_n),(\xi_{n+4},s_{n+4}),\dots,$$
$$(\xi_{n+n'},s_{n+n'}); s_1,t_1,s_2,t_2,t'_2,s'_2,s_3,t_3,t'_3,s'_3)$$

and $\tilde\gamma(\cdot;s_1,t_1,s_2,t_2,t'_2,s'_2,s_3,t_3,t'_3,s'_3) = 0$ unless $s_1\cap t_1 \neq \emptyset$ and $s_i\cap t_i\cap t'_i\cap s'_i \neq \emptyset$ for $i = 2,3$. By Corollary II.8, for all choices of sectors,

$$\|\tilde\gamma((\cdot,s_4),\dots,(\cdot,s_n),\cdot,s_{n+4}),\dots,(\cdot,s_{n+n'});s_1,t_1,s_2,t_2,t'_2,s'_2,s_3,t_3,t'_3,s'_3)\|_{1,\infty}$$
$$\le \sup|d|\sup|d'|\|\varphi((\cdot,s_1),(\cdot,s_2),(\cdot,s_3),(\cdot,s_4),\dots,(\cdot,s_n))\|_{1,\infty}$$
$$\cdot\|\varphi''((\cdot,t_1),(\cdot,s'_2),(\cdot,s'_3),(\cdot,s_{n+4}),\dots,(\cdot,s_{n+n'}))\|_{1,\infty}\,.$$

The variable indices for γ lie in the set $I\cup I'$, where

$$I = \{4,\dots,n\}\,,\quad I' = \{n+4,\dots,n+n'\}\,.$$

Fix $u_1,\dots,u_q\in I$, $u_{q+1},\dots,u_p\in I'$ and fix sectors $s_{u_1},\dots,s_{u_p}\in\Sigma$. First assume that q is odd so that $p-q$ is even. Then, by (XII.1) and the estimate on $\tilde\gamma$ above

$$\sum_{\substack{s_i\in\Sigma\\ i\in I\cup I'\setminus\{u_1,\dots,u_p\}}} \|\gamma((\cdot,s_4),\dots,(\cdot,s_n),(\cdot,s_{n+4}),\dots,(\cdot,s_{n+n'}))\|_{1,\infty}$$
$$\le \mathrm{const}\,\frac{\mathfrak{l}^2}{M^{2j}} \sum_{\substack{s_i\in\Sigma\\ i\in\{1,\dots,n\}\\ i\neq u_1,\dots,u_q}} \|\varphi((\cdot,s_1),(\cdot,s_2),(\cdot,s_3),(\cdot,s_4),\dots,(\cdot,s_n))\|_{1,\infty}$$
$$\cdot \max_{t_1,s'_2,s'_3\in\Sigma} \sum_{\substack{s_i\in\Sigma\\ i\in I'\setminus\{u_{q+1},\dots,u_p\}}}$$
$$\cdot\|\varphi''((\cdot,t_1),(\cdot,s'_2),(\cdot,s'_3),(\cdot,s_{n+4}),\dots,(\cdot,s_{n+n'}))\|_{1,\infty}$$
$$\le \mathrm{const}\,\frac{\mathfrak{l}^2}{M^{2j}}|\varphi|_{q,\Sigma}|\varphi''|_{p-q+3,\Sigma}$$
$$\le \mathrm{const}\,\frac{\mathfrak{l}^2}{M^{2j}}|c|_{1,\Sigma}|\varphi|_{q,\Sigma}|\varphi'|_{p-q+3,\Sigma}\,.$$

The case that q is even is similar. □

To treat source terms, we state, motivated by Definition VII.4,

Lemma XII.19 (External improving). Let $C(k)$ be a function obeying $|C(k)| \le \frac{2}{|\imath k_0 - e(\mathbf{k})|}$ and $c((\cdot,s),(\cdot,s'))$ be the Fourier transform of $\chi_s(k)C(k)\chi_{s'}(k)$ in the sense of Definition IX.3. Let $\varphi \in \mathcal{F}_m(n;\Sigma)$, $1 \le i \le n$ and set

$$\begin{aligned}
&\varphi'(\eta_1,\dots,\eta_{m+1};(\xi_1,s_1),\dots,(\xi_{n-1},s_{n-1})) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s,t,t'\in\Sigma} \int d\zeta d\zeta' \varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\xi_{i-1},s_{i-1}), \\
&\qquad (\zeta',t),(\xi_i,s_i),\dots,(\xi_{n-1},s_{n-1}))c((\zeta',t',),(\zeta,s))J(\zeta,\eta_{m+1})\,.
\end{aligned}$$

Then

$$|\varphi'|_{1,\Sigma} \le \mathrm{const}\,|\varphi|_{1,\Sigma} \begin{cases} \frac{1}{\mathfrak{l}}|c|_{1,\Sigma} & \text{if } m=0 \\ \frac{\mathfrak{l}}{M^j} & \text{if } m\ne 0 \end{cases}$$

with a constant that is independent of j and Σ.

Proof. First consider the case $m=0$. Define the function φ'' on $(\mathcal{B}\times\Sigma)\times(\mathcal{B}\times\Sigma)^{n-1}$ by

$$\begin{aligned}
&\varphi''((\eta,s);(\xi_1,s_1),\dots,(\xi_{n-1},s_{n-1})) \\
&\quad = \sum_{t,t'\in\Sigma} \int d\zeta d\zeta' \varphi((\xi_1,s_1),\dots,(\xi_{i-1},s_{i-1}), \\
&\qquad (\zeta',t),(\xi_i,s_i),\dots,(\xi_{n-1},s_{n-1}))c((\zeta',t'),(\zeta,s))J(\zeta,\eta)\,.
\end{aligned}$$

Then

$$\varphi'(\eta;(\xi_1,s_1),\dots,(\xi_{n-1},s_{n-1})) = \sum_{s\in\Sigma} \varphi''((\eta,s);(\xi_1,s_1),\dots,(\xi_{n-1},s_{n-1}))\,.$$

Hence, by Lemma XII.14,

$$|\varphi'|_{1,\Sigma} \le |\Sigma|\,|\varphi''|_{1,\Sigma} \le \frac{const}{\mathfrak{l}}|\varphi|_{1,\Sigma}|cJ|_{1,\Sigma} \le \frac{const}{\mathfrak{l}}|c|_{1,\Sigma}|\varphi|_{1,\Sigma}\,.$$

Now suppose that $m\ne 0$. Then

$$\begin{aligned}
&|\varphi'(\eta_1,\dots,\eta_{m+1};(\xi_1,s_1),\dots,(\xi_{n-1},s_{n-1}))|_{1,\Sigma} \\
&\quad \le \sum_{\substack{s_1,\dots,s_{n-1}\\ s,t,t'}} \sup_{\eta_1,\dots,\eta_{m+1}} \int d\zeta d\zeta' d\xi_1,\dots,d\xi_{n-1} |\varphi(\eta_1,\dots,\eta_m; \\
&\qquad (\xi_1,s_1),\dots,(\zeta',t),\dots,(\xi_{n-1},s_{n-1}))c((\zeta',t'),(\zeta,s))J(\zeta,\eta_{m+1})|
\end{aligned}$$

$$\leq \sum_{s_1,\dots,s_{n-1},t} \sum_{\substack{s,t' \\ s\cap t\cap t'\neq\emptyset}} \sup_{\eta_{m+1},\zeta'} |c((\zeta',t'),(\eta_{m+1},s))|$$

$$\cdot \|\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\zeta',t),\dots,(\xi_{n-1},s_{n-1}))\|_{1,\infty}$$

$$\leq 9 \sup|c| \sum_{s_1,\dots,s_{n-1},t} \|\varphi(\eta_1,\dots,\eta_m;(\xi_1,s_1),\dots,(\zeta',t),\dots,(\xi_{n-1},s_{n-1}))\|_{1,\infty}$$

$$\leq \text{const}\, \frac{\mathfrak{l}}{M^j} |\varphi|_{1,\Sigma}$$

by (XII.1). □

XIII. Bounds for Sectorized Propagators

In this section we prove the existence of the partitions of unity, $\{\chi_s(k)\}$, and enveloping functions satisfying Lemma XII.3. We derive bounds on $|c|_{1,\Sigma}$, for various sectorized covariances c whose Fourier transforms are related to $\chi_s(k)\frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k})}\chi_{s'}(k)$, that, together with Proposition XII.16, give good contraction bounds.

The reason it is not easy to get good L_1–L_∞-bounds on the propagators in position space is that integration by parts in Cartesian coordinates is not well suited to the curvature of the Fermi surface and the shells around it. This is why we introduce sectorization. If the sectors are not too long (more precisely, at most of order $\frac{1}{M^{j/2}}$), the curvature of the sector has little effect. The first step in deriving L_1–L_∞-bounds using sectorization is

Proposition XIII.1. *Let $j \geq 2$. Let I be an interval of the Fermi curve with length $\mathfrak{l}$ and let $f(k)$ be a function that is supported on $\{k \in \mathbb{R}^3 \,|\, |\imath k_0 - e(\mathbf{k})| \leq \frac{2}{M^j}$, $\pi_F(k) \in I\}$. Set, as in Lemma* IX.6

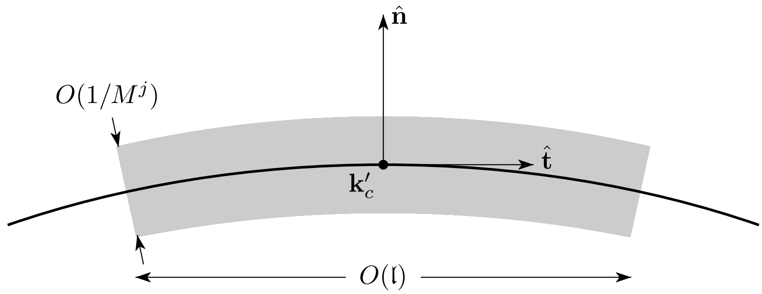
$$f'(x) = \int e^{\imath\langle k,x\rangle_-} f(k) \frac{d^3k}{(2\pi)^3} \,.$$

Fix any point $\mathbf{k}_c' \in I$, let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors to the Fermi curve at $\mathbf{k}_c'$ and let $\mathbf{x}_\parallel$ be the component of $\mathbf{x}$ parallel to $\hat{\mathbf{t}}$ and $\mathbf{x}_\perp$ the component parallel to $\hat{\mathbf{n}}$. There is a constant, const, *depending on $e(\mathbf{k})$, but independent of M, f, j and x such that*

(i) *For all multiindices $\gamma \in \mathbb{N}_0 \times \mathbb{N}_0^2$*

$$\left|\left(\frac{x_0}{M^j}\right)^{\gamma_0} \left(\frac{\mathbf{x}_\perp}{M^j}\right)^{\gamma_1} (\mathfrak{l}\,\mathbf{x}_\parallel)^{\gamma_2} f'(x)\right|$$

$$\leq \mathit{const}\, \frac{\mathfrak{l}}{M^{2j}} \sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{j(\gamma_0+\gamma_1)}} |\partial_{k_0}^{\gamma_0} (\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1} (\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2} f(k)| \,.$$



(ii)

$$\sup_x |f'(x)| \le {\scriptstyle const}\, \frac{\mathfrak{l}}{M^{2j}} \sup_k |f(k)|\,.$$

(iii) *If* $\mathfrak{l} \ge \frac{1}{M^j}$, *then, for all multiindices* δ

$$\int dx |x^\delta f'(x)| \le {\scriptstyle const}\, 2^{\delta_1+\delta_2} M^{|\delta| j} \max_{\substack{\gamma_0 \le \delta_0+2\\ \gamma_1+\gamma_2 \le \delta_1+\delta_2+3}} \sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{j(\gamma_0+\gamma_1)}}$$

$$\cdot |\partial_{k_0}^{\gamma_0} (\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}})^{\gamma_1} (\hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}})^{\gamma_2} f(k)|\,.$$

Proof. (i) By integration by parts

$$\left| \left(\frac{x_0}{M^j}\right)^{\gamma_0} \left(\frac{\mathbf{x}_\perp}{M^j}\right)^{\gamma_1} (\mathfrak{l}\, \mathbf{x}_\parallel)^{\gamma_2} f'(x) \right|$$

$$= \left| \int \frac{d^3k}{(2\pi)^3} e^{\imath \langle k,x\rangle_-} \left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} \left(\frac{1}{M^j}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}}\right)^{\gamma_1} (\mathfrak{l}\, \hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}})^{\gamma_2} f(k) \right|.$$

Use S to denote the support of $f(k)$. Observe that S has volume at most ${\scriptstyle const}$ $M^{-2j}\mathfrak{l}$, since k_0 is supported in an interval of length ${\scriptstyle const}\, M^{-j}$, the distance of $\mathbf{k}$ from F is bounded by ${\scriptstyle const}\, M^{-j}$ and the $\pi_F(\mathbf{k})$ runs over an interval of length $\mathfrak{l}$. Hence

$$\left| \left(\frac{x_0}{M^j}\right)^{\gamma_0} \left(\frac{\mathbf{x}_\perp}{M^j}\right)^{\gamma_1} (\mathfrak{l}\, \mathbf{x}_\parallel)^{\gamma_2} f'(x) \right|$$

$$\le \mathrm{vol}(S) \sup_k \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} \left(\frac{1}{M^j}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}}\right)^{\gamma_1} (\mathfrak{l}\, \hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}})^{\gamma_2} f(k) \right|$$

$$\le {\scriptstyle const}\, \frac{\mathfrak{l}}{M^{2j}} \sup_k \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} \left(\frac{1}{M^j}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}}\right)^{\gamma_1} (\mathfrak{l}\, \hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}})^{\gamma_2} f(k) \right|.$$

(ii) This is simply a restatement of part (i) with $\gamma = \mathbf{0}$.

(iii) Set

$$\rho(x) = [1 + M^{-j}|x_0|]^2 [1 + M^{-j}|\mathbf{x}_\perp| + \mathfrak{l}|\mathbf{x}_\parallel|]^3\,.$$

By part (i)

$$\rho(x)|x^\delta f'(x)| \le {\scriptstyle const}\, 2^{\delta_1+\delta_2} M^{j|\delta|} \frac{\mathfrak{l}}{M^{2j}} \max_{\substack{\gamma_0 \le \delta_0+2 \\ \gamma_1+\gamma_2 \le \delta_1+\delta_2+3}}$$

$$\cdot \sup_k \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} \left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\gamma_1} (\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2} f(k)\right|$$

since

$$|x^\delta| \le M^{|\delta|j}[M^{-j}|x_0|]^{\delta_0}[M^{-j}|\mathbf{x}_\perp| + \mathfrak{l}|\mathbf{x}_\parallel|]^{\delta_1+\delta_2}\,.$$

The desired bound now follows from

$$\int dx \frac{1}{\rho(x)} \le {\scriptstyle const}\, M^{2j}/\mathfrak{l}\,.$$

To see this, just make the change of variables $x_0 = M^j z_0$, $\mathbf{x}_\perp = M^j z_1$, $\mathbf{x}_\parallel = z_2/\mathfrak{l}$. □

We parametrize the Fermi curve F by arc length, using a real variable $\mathbf{k}'$ for the parametrization. To simplify notation, set $\mathbf{k}'(k) = \pi_F(k)$, the projection on the Fermi surface.

Lemma XIII.2. *Let $j > 0$. Let I be an interval of the Fermi curve with length $\mathfrak{l} \in [\frac{1}{M^j}, \frac{1}{M^{j/2}}]$. Let $\chi(k) = R(k_0, e(\mathbf{k}))\Theta(\mathbf{k}'(k))$ with $R(x,y)$ vanishing unless $|y| \le \sqrt{2}\mathfrak{l}$ and Θ supported in I. Fix any point $\mathbf{k}'_c \in I$ and let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors to the Fermi curve at $\mathbf{k}'_c$. There is a constant, ${\scriptstyle const}$, depending on r_0, r and $e(\mathbf{k})$, but independent of M, χ and j such that, for all $\gamma \in \mathbb{N}_0 \times \mathbb{N}_0^2$ with $\gamma_0 \le r_0 + 2$, $\gamma_1 + \gamma_2 \le r + 3$,*

$$\sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{j(\gamma_0+\gamma_1)}} |\partial_{k_0}^{\gamma_0}(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1}(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2}\chi(k)|$$

$$\le {\scriptstyle const} \max_{m+n \le \gamma_1+\gamma_2} \sup_{\mathbf{k}'} \mathfrak{l}^m |\partial_{\mathbf{k}'}^m \Theta(\mathbf{k}')| \sup_{x,y} \frac{1}{M^{j(\gamma_0+n)}} \left|\frac{\partial^{\gamma_0+n} R}{\partial x^{\gamma_0}\partial^n y}(x,y)\right|.$$

Proof. Since $\mathfrak{l} \ge \frac{1}{M^j}$ and all derivatives of $\mathbf{k}'(k)$ to order $\gamma_1+\gamma_2$ are bounded,

$$\sup_k \left| \left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\gamma_1-\beta_1} \left(\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}}\right)^{\gamma_2-\beta_2} \Theta(\mathbf{k}'(k))\right| \le {\scriptstyle const} \max_{\substack{m \le \gamma_1+\gamma_2 \\ -\beta_1-\beta_2}} \sup_{\mathbf{k}'} \mathfrak{l}^m |\partial_{\mathbf{k}'}^m \Theta(\mathbf{k}')|$$

for all $\beta_1 \le \gamma_1$ and $\beta_2 \le \gamma_2$. So, by the product rule, it suffices to prove

$$\sup_{k \in S} \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} \left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1} (\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\beta_2} R(k_0, e(\mathbf{k}))\right|$$

$$\le {\scriptstyle const} \max_{n \le \beta_1+\beta_2} \sup_{x,y} \frac{1}{M^{j(\gamma_0+n)}} \left|\frac{\partial^{\gamma_0+n} R}{\partial x^{\gamma_0}\partial^n y}(x,y)\right|$$

where S is the support of $\chi(k)$. Set $\pi = \{1, \ldots, \beta_1 + \beta_2\}$,

$$d_i = \begin{cases} \frac{1}{M^j}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}} & \text{if } 1 \le i \le \beta_1 \\ \mathfrak{l}\,\hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}} & \text{if } \beta_1 + 1 \le i \le \beta_1 + \beta_2 \end{cases}$$

and, for each $\pi' \subset \pi$, $d^{\pi'} = \prod_{i\in\pi'} d_i$. By the product and chain rules

$$\left(\frac{1}{M^j}\partial_{k_0}\right)^{\gamma_0} d^{\pi} R(k_0, e(\mathbf{k}))$$

$$= \sum_{n=1}^{\beta_1+\beta_2} \sum_{(\pi_1,\ldots,\pi_n)\in\mathcal{P}_n} \frac{1}{M^{j(\gamma_0+n)}} \frac{\partial^{\gamma_0+n} R}{\partial x^{\gamma_0}\partial^n y}(k_0, e(\mathbf{k})) \prod_{i=1}^{n} M^j d^{\pi_i} e(\mathbf{k})$$

where $\mathcal{P}_n$ is the set of all partitions of π into n nonempty subsets $\pi_1, \ldots, \pi_n$ with, for all $i < i'$, the smallest element of π_i smaller than the smallest element of $\pi_{i'}$. So to prove the lemma, it suffices to prove that

$$\max_{1\le\beta_1+\beta_2\le\gamma_1+\gamma_2} \sup_{k\in S} \left| M^j \left(\frac{1}{M^j}\hat{\mathbf{n}} \cdot \nabla_{\mathbf{k}}\right)^{\beta_1} (\mathfrak{l}\,\hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}})^{\beta_2} e(\mathbf{k}) \right| \le \mathit{const}\,. \tag{XIII.1}$$

If $\beta_1 \ge 1$ or $\beta_2 \ge 2$, this follows from $\frac{M^j \mathfrak{l}^{\beta_2}}{M^{\beta_1 j}} \le 1$. (Recall that $\mathfrak{l} \le \frac{1}{M^{j/2}}$.) The only remaining possibility is $\beta_1 = 0$, $\beta_2 = 1$.

If $\hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}} e(\mathbf{k})$ is evaluated at $\mathbf{k} = \mathbf{k}'_c$, it vanishes, since $\nabla_{\mathbf{k}} e(\mathbf{k}'_c)$ is parallel to $\hat{\mathbf{n}}$. The second derivative of e is bounded so that,

$$M^j \mathfrak{l} \sup_{k\in S} |\hat{\mathbf{t}} \cdot \nabla_{\mathbf{k}} e(\mathbf{k})| \le \mathit{const}\, M^j \mathfrak{l} \sup_{k\in S} |\mathbf{k} - \mathbf{k}'_c| \le \mathit{const}\, M^j \mathfrak{l}^2 \le \mathit{const}$$

since $\mathfrak{l} \le \frac{1}{M^{j/2}}$. □

For the rest of this section, we fix a sectorization Σ of scale $j \ge 2$ and length $\frac{1}{M^{j-3/2}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$. We choose a smooth partition of unity $\Theta_s(\mathbf{k}')$, $s \in \Sigma$ of the Fermi curve F subordinate to the sets $s \cap F$, such that $|\partial^m_{\mathbf{k}'}\Theta_s(\mathbf{k}')| \le \mathit{const}\frac{1}{\mathfrak{l}^m}$ for $m = 0, 1, \ldots, r+3$. Furthermore choose enveloping functions $\tilde{\Theta}_s(\mathbf{k}')$, $s \in \Sigma$ that are identically one on $s \cap F$ and obey $|\partial^m_{\mathbf{k}'}\tilde{\Theta}_s(\mathbf{k}')| \le \mathit{const}\frac{1}{\mathfrak{l}^m}$ for $m = 0, 1, \ldots, r+3$ and $\tilde{\Theta}_s\tilde{\Theta}_{s'} = 0$ if $s \cap s' = \emptyset$. Set

$$\begin{aligned} \chi_s(k) &= \tilde{\nu}^{(\ge j)}(k)\Theta_s(\mathbf{k}'(k)) = \varphi(M^{2j-2}(k_0^2 + e(\mathbf{k})^2))\Theta_s(\mathbf{k}'(k)) \\ \tilde{\chi}_s(k) &= \bar{\nu}^{(\ge j)}(k)\tilde{\Theta}_s(\mathbf{k}'(k)) = \varphi(M^{2j-3}(k_0^2 + e(\mathbf{k})^2))\Theta_s(\mathbf{k}'(k)) \end{aligned} \tag{XIII.2}$$

where $\tilde{\nu}^{(\ge j)}$, $\bar{\nu}^{(\ge j)}$ are the functions of Definition VIII.4.

Lemma XIII.3. *Let $s \in \Sigma$. Set for $x = (x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^2$*

$$\chi'_s(x) = \int e^{\imath\langle k,x\rangle_-} \chi_s(k) \frac{d^3k}{(2\pi)^3}\,, \quad \tilde{\chi}'_s(x) = \int e^{\imath\langle k,x\rangle_-} \tilde{\chi}_s(k) \frac{d^3k}{(2\pi)^3}$$

$$\chi^0_s(\mathbf{x}) = \int e^{\imath\mathbf{k}\cdot\mathbf{x}} \chi_s((0,\mathbf{k})) \frac{d^2\mathbf{k}}{(2\pi)^2}\,.$$

Then

$$\|\chi'_s\|_{L^1} \le const\, \mathfrak{c}_{j-1}\,, \qquad \left\|\frac{\partial}{\partial x_0}\chi'_s\right\|_{L^1} \le const\, \frac{1}{M^{j-1}}\mathfrak{c}_{j-1}\,,$$

$$\left\|x_0\frac{\partial}{\partial x_0}\chi'_s\right\|_{L^1} \le const\, \mathfrak{c}_{j-1}\,, \qquad \|\tilde\chi'_s\|_{L^1} \le const\, \mathfrak{c}_{j-\frac{3}{2}}\,,$$

$$\left\|\frac{\partial}{\partial x_0}\tilde\chi'_s\right\|_{L^1} \le const\, \frac{1}{M^{j-3/2}}\mathfrak{c}_{j-\frac{3}{2}}\,, \qquad \left\|x_0\frac{\partial}{\partial x_0}\tilde\chi'_s\right\|_{L^1} \le const\, \mathfrak{c}_{j-\frac{3}{2}}$$

and

$$\|\chi^0_s\|_{L^1} \le const\, \mathfrak{c}_{j-1}\,.$$

Here, for a function $f(x)$ *on* $\mathbb{R}\times\mathbb{R}^2$,

$$\|f\|_{L^1} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\left[\int |x^\delta f(x)|dx\right] t^\delta \in \mathfrak{N}_{d+1}$$

is the norm defined before Lemma IX.6, *and for a function* $g(\mathbf{x})$ *on* $\mathbb{R}^2$ *we set*

$$\|g\|_{L^1} = \sum_{\substack{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \delta_0=0}} \frac{1}{\delta!}\left[\int |\mathbf{x}^\delta f(\mathbf{x})|d\mathbf{x}\right] t^\delta\,.$$

Proof. Fix a point $\mathbf{k}'_c \in s\cap F$ and let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors to F at $\mathbf{k}'_c$. By Lemma XIII.2, with j replaced by $j-1$ and $R(x,y) = \varphi(M^{2j-2}(x^2+y^2))$,

$$\max_{\substack{\gamma_0\le\delta_0+2\\ \gamma_1+\gamma_2\le\delta_1+\delta_2+3}} \sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{(j-1)(\gamma_0+\gamma_1)}}\left|\partial_{k_0}^{\gamma_0}(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1}(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2}\chi_s(k)\right| \le const \qquad \text{(XIII.3)}$$

for every multiindex $\delta\in\mathbb{N}_0\times\mathbb{N}_0^2$ with $\delta_0\le r_0$ and $\delta_1+\delta_2\le r$. Here, we have used that $M^{j-1}|x|, M^{j-1}|y|\le\sqrt{2}$ on the support of $R(x,y)$. Therefore, by Proposition XIII.1(iii),

$$\frac{1}{M^{(j-1)|\delta|}}\int dx|x^\delta\chi'(x)| \le const\,.$$

By definition of $\|\cdot\|_{L^1}$, this implies that $\|\chi'_s\|_{L^1}\le const\,\mathfrak{c}_{j-1}$.

By Lemma XIII.2 and the product rule

$$\max_{\substack{\gamma_0\le\delta_0+2\\ \gamma_1+\gamma_2\le\delta_1+\delta_2+3}} \sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{(j-1)(\gamma_0+\gamma_1)}}\left|\partial_{k_0}^{\gamma_0}(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1}(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2}(k_0\chi_s(k))\right| \le const\,\frac{1}{M^{j-1}}$$

and as above it follows that $\|\frac{\partial}{\partial x_0}\chi'_s\|_{L^1}\le const\,\frac{1}{M^{j-1}}\mathfrak{c}_{j-1}$.

Again, by Lemma XIII.2

$$\max_{\substack{\gamma_0\le\delta_0+2\\ \gamma_1+\gamma_2\le\delta_1+\delta_2+3}} \sup_k \frac{\mathfrak{l}^{\gamma_2}}{M^{(j-1)(\gamma_0+\gamma_1)}}\left|\partial_{k_0}^{\gamma_0}(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1}(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2}\left(\frac{\partial}{\partial k_0}k_0\chi_s(k)\right)\right| \le const$$

and the proof that $\|x_0\frac{\partial}{\partial x_0}\chi'_s\|_{L^1}\le const\,\mathfrak{c}_{j-1}$ is as before.

The bounds on $\tilde{\chi}'_s$ are obtained in the same way, with $j-1$ replaced by $j-\frac{3}{2}$. The estimate on $\|\chi^0_s\|_{L^1}$ follows from the fact that

$$\chi^0_s(\mathbf{x}) = \int dx_0 \chi'_s(x_0, \mathbf{x})\,. \qquad \square$$

Proof of Lemma XII.3. Parts (i) and (ii) of the lemma are trivial. To prove part (iii) observe that

$$\hat{\chi}_s(\xi, \xi') = \delta_{\sigma,\sigma'}\delta_{a,a'}\chi'_s((-1)^a(x-x'))$$

for $\xi = (x, \sigma, a)$, $\xi' = (x', \sigma', a') \in \mathcal{B}$. Therefore, by Lemma XIII.3

$$\|\hat{\chi}_s\|_{1,\infty} \le \mathit{const}\, \|\chi'_s\|_{L^1} \le \mathit{const}\, \mathfrak{c}_{j-1}\,.$$

The estimate for $\|\hat{\tilde{\chi}}_s\|_{1,\infty}$ is obtained in the same way. $\square$

From now on, we fix for each sectorization Σ, a partition of unity χ_s, $s \in \Sigma$ and a system of functions $\tilde{\chi}_s$, $s \in \Sigma$ that fulfill the conclusions of Lemmas XII.3 and XIII.3.

Recall from Definition XII.2 that

$$\mathfrak{c}_j = \sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} M^{j|\delta|}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|> r\\ \text{or } |\delta_0|> r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}\,.$$

Observe that by Corollary A.5, there is a constant *const* that is independent of M such that for $2 \le i \le j$

$$\mathfrak{c}_i\mathfrak{c}_j \le \mathit{const}\, \mathfrak{c}_j\,. \tag{XIII.4}$$

Lemma XIII.4. *Set*

$$C^{(j)}(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k})}\,, \quad \tilde{C}^{(j)}(k) = \frac{\tilde{\nu}^{(j)}(k)}{\imath k_0 - e(\mathbf{k})}$$

and, for $s, s' \in \Sigma$, let $c^{(j)}((\xi, s), (\xi', s'))$ resp. $\tilde{c}^{(j)}((\xi, s), (\xi', s'))$ be the Fourier transforms (as in Definition IX.3*) of $\chi_s(k)C^{(j)}(k)\chi_{s'}(k)$ resp. $\chi_s(k)\tilde{C}^{(j)}(k)\chi_{s'}(k)$. Then*

(i) $c^{(j)}((\cdot, s), (\cdot, s')) = \tilde{c}^{(j)}((\cdot, s), (\cdot, s')) = 0$ *if* $s \cap s' = \emptyset$.

(ii)

$$|c^{(j)}|_{1,\Sigma}\,, \quad |\tilde{c}^{(j)}|_{1,\Sigma} \le \mathrm{const}\, M^j \mathfrak{c}_j\,.$$

(iii) *Let $c^{(j)}_0((\xi, s), (\xi', s'))$ the Fourier transform (as in Definition* IX.3*) of $\chi_s(k)k_0C^{(j)}(k)\chi_{s'}(k)$ or $\chi_s(k)e(\mathbf{k})C^{(j)}(k)\chi_{s'}(k)$ and let $\tilde{c}^{(j)}_0((\xi, s), (\xi', s'))$ be the Fourier transform of either $\chi_s(k)k_0\tilde{C}^{(j)}(k)\chi_{s'}(k)$ or $\chi_s(k)e(\mathbf{k})\tilde{C}^{(j)}(k)$ $\chi_{s'}(k)$. Then*

$$|c^{(j)}_0|_{1,\Sigma}\,, \quad |\tilde{c}^{(j)}_0|_{1,\Sigma} \le \mathrm{const}\, \mathfrak{c}_j\,.$$

(iv)

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{s,s'\in\Sigma}\sup_{\xi,\xi'\in\mathcal{B}}|\mathcal{D}^\delta_{1,2}c^{(j)}((\xi,s),(\xi',s'))|t^\delta\le \text{const}\,\frac{\mathfrak{l}}{M^j}\mathfrak{c}_j\,.$$

Proof. Part (i) is obvious. To prove part (ii) fix sectors $s,s'\in\Sigma$, with $s\cap s'\neq\emptyset$ and a point $\mathbf{k}_c'\in s\cap F$. Let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors to F at $\mathbf{k}_c'$. First we claim that for all k in the intersection of s with the jth shell

$$\max_{\substack{\beta_0\le r_0+2\\ \beta_1+\beta_2\le r+3}}\left|\left(\frac{1}{M^j}\partial_{k_0}\right)^{\beta_0}\left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1}(\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\beta_2}\frac{1}{ik_0-e(\mathbf{k})}\right|$$
$$\le\text{const}\,M^j\,. \tag{XIII.5}$$

To see this, set $\pi=\{1,\ldots,|\beta|\}$,

$$d_i=\begin{cases}\dfrac{1}{M^j}\partial_{k_0} & \text{if } 1\le i\le\beta_0\\[2mm] \dfrac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}} & \text{if } \beta_0+1\le i\le\beta_0+\beta_1\\[2mm] \mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}} & \text{if } \beta_0+\beta_1+1\le i\le|\beta|\end{cases}$$

and, for each $\pi'\subset\pi$, $d^{\pi'}=\prod_{i\in\pi'}d_i$. By the product and chain rules

$$d^\pi\frac{1}{ik_0-e(\mathbf{k})}=M^j\sum_{n=1}^{|\beta|}(-1)^n n!\sum_{(\pi_1,\ldots,\pi_n)\in\mathcal{P}_n}\left(\frac{1/M^j}{ik_0-e(\mathbf{k})}\right)^{n+1}$$
$$\cdot\prod_{i=1}^n M^j d^{\pi_i}(ik_0-e(\mathbf{k}))\,.$$

In the sector s, $|ik_0-e(\mathbf{k})|\ge\text{const}\,\frac{1}{M^j}$ so that $(\frac{1/M^j}{ik_0-e(\mathbf{k})})^{n+1}$ is bounded uniformly in j. That $M^jd^{\pi_i}(ik_0-e(\mathbf{k}))$ is bounded uniformly in j follows immediately from (XIII.1) and the fact that $|k_0|\le\frac{\text{const}}{M^j}$ on the jth shell. This proves (XIII.5).

As in (XIII.3), for all k in the intersection of s with the jth shell,

$$\max_{\substack{\beta_0\le r_0+2\\ \beta_1+\beta_2\le r+3}}\left|\left(\frac{1}{M^j}\partial_{k_0}\right)^{\beta_0}\left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1}(\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\beta_2}\nu^{(j)}(k)\right|\le\text{const}\,.$$

By Leibniz's rule it follows from this inequality and the inequalities (XIII.3) and (XIII.5) that

$$\max_{\substack{\gamma_0\le r_0+2\\ \gamma_1+\gamma_2\le r+3}}\sup_k\frac{\mathfrak{l}^{\gamma_2}}{M^{j(\gamma_0+\gamma_1)}}|\partial_{k_0}^{\gamma_0}(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}})^{\gamma_1}(\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\gamma_2}\chi_s(k)C^{(j)}(k)\chi_{s'}(k)|\le\text{const}\,M^j\,.$$

Hence, by Proposition XIII.1

$$|c^{(j)}|_{1,\Sigma}\le\text{const}\,M^j\,\mathfrak{c}_j\,.$$

The proof for $|\tilde{c}^{(j)}|_{1,\Sigma}$ is analogous.

The proof of part (iii) is the same as the proof of part (ii) with (XIII.5) replaced by

$$\max_{\substack{\beta_0\le r_0+2\\ \beta_1+\beta_2\le r+3}} \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\beta_0} \left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1} (\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\beta_2} \frac{k_0}{ik_0-e(\mathbf{k})}\right| \le \mathrm{const}$$

$$\max_{\substack{\beta_0\le r_0+2\\ \beta_1+\beta_2\le r+3}} \left| \left(\frac{1}{M^j}\partial_{k_0}\right)^{\beta_0} \left(\frac{1}{M^j}\hat{\mathbf{n}}\cdot\nabla_{\mathbf{k}}\right)^{\beta_1} (\mathfrak{l}\,\hat{\mathbf{t}}\cdot\nabla_{\mathbf{k}})^{\beta_2} \frac{e(\mathbf{k})}{ik_0-e(\mathbf{k})}\right| \le \mathrm{const}\,.$$

To prove part (iv), observe that, by (XIII.5) and the fact that $\chi_s(k)C^{(j)}(k)\chi_{s'}(k)$ is supported in a region of volume const $\frac{\mathfrak{l}}{M^{2j}}$,

$$\max_{s,s'\in\Sigma}\sup_{\xi,\xi'\in\mathcal{B}} |\mathcal{D}^{\delta}_{1,2}c^{(j)}((\xi,s),(\xi',s'))| \le \mathrm{const}\,\frac{\mathfrak{l}}{M^{2j}}M^{j(1+|\delta|)}$$

for all $\delta_0\le r_0$ and $|\boldsymbol{\delta}|\le r$. □

Proposition XIII.5. *There are constants* τ_1, const *that depend on* $e(\mathbf{k})$ *and* M, *but not on* j *or* Σ *with the following property*:

Let $u((\xi,s),(\xi',s'))\in\mathcal{F}_0(2;\Sigma)$ *be antisymmetric, spin independent and particle number conserving and obey* $|u|_{1,\Sigma}\le\frac{\tau_1}{M^j}+\sum_{\delta\ne\mathbf{0}}\infty t^{\delta}$. *Let*

$$C(k)=\frac{\nu^{(j)}(k)}{\imath k_0-e(\mathbf{k})-\check{u}(k)}$$

$c((\xi,s),(\xi',s'))$ *be the Fourier transform* (*as in Definition* IX.3) *of* $\chi_s(k)C(k)\chi_{s'}(k)$ *and* $c_0((\xi,s),(\xi',s'))$ *be the Fourier transform of* $\chi_s(k)k_0C(k)\chi_{s'}(k)$ *or* $\chi_s(k)e(\mathbf{k})C(k)\chi_{s'}(k)$. *Then*

(i) $c((\cdot,s),(\cdot,s'))=0$ *if* $s\cap s'=\emptyset$.
(ii) $|c|_{1,\Sigma}\le\mathrm{const}\,\frac{M^j\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}}$.
(iii) $\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\sup_{\xi,\xi',s,s'}|\mathcal{D}^{\delta}_{1,2}c((\xi,s),(\xi',s'))|t^{\delta}\le\mathrm{const}\,\frac{\mathfrak{l}}{M^j}\frac{\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}}$.
(iv) $|c_0|_{1,\Sigma}\le\mathrm{const}\,\frac{\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}}$.

Proof. Again part (i) is trivial. To prove part (ii), observe that

$$C(k)=\frac{C^{(j)}(k)}{1-\frac{\check{u}(k)}{\imath k_0-e(\mathbf{k})}}=\frac{C^{(j)}(k)}{1-\frac{\check{u}(k)\tilde{\nu}^{(j)}(k)}{\imath k_0-e(\mathbf{k})}}=\frac{C^{(j)}(k)}{1-\check{u}(k)\tilde{C}^{(j)}(k)}$$

$$=C^{(j)}(k)\sum_{n=0}^{\infty}(\check{u}(k)\tilde{C}^{(j)}(k))^n\,.$$

Introducing the local notation

$$C^{(j)}(k;s,s')=\chi_s(k)C^{(j)}(k)\chi_{s'}(k)$$

$$\tilde{C}^{(j)}(k;s,s')=\chi_s(k)\tilde{C}^{(j)}(k)\chi_{s'}(k)$$

we have

$$\chi_s(k)C(k)\chi_{s'}(k) = C^{(j)}(k;s,s') + \sum_{n=1}^{\infty}\sum_{s''\in\Sigma}\sum_{\substack{t_i,t_i'\in\Sigma\\ \text{for } i=1,\dots,n\\ \text{with } t_n'=s'}} C^{(j)}(k;s,s'')$$

$$\cdot\prod_{i=1}^{n}\check{u}(k)\tilde{C}^{(j)}(k;t_i,t_i')\,.$$

Define the operator $*$-product of the sectorized functions $A((\xi,s),(\xi',s'))$ and $B((\xi',s'),(\xi'',s''))$ by

$$(A*B)((\xi,s),(\xi'',s'')) = \sum_{\substack{s',t'\in\Sigma\\ s'\cap t'\neq\emptyset}}\int d\xi' A((\xi,s),(\xi',s'))B((\xi',t'),(\xi'',s''))\,.$$

Then, by part (i) of Lemma XIII.4

$$c = c^{(j)}\sum_{n=0}^{\infty}(*\,u*\tilde{c}^{(j)})^n \tag{XIII.6}$$

so that by iterated application of Lemmas XII.14 and XIII.4(ii)

$$\begin{aligned}|c|_{1,\Sigma} &\le |c^{(j)}|_{1,\Sigma}\sum_{n=0}^{\infty}(9|u|_{1,\Sigma}|\tilde{c}^{(j)}|_{1,\Sigma})^n\\ &\le \text{const}\,M^j\mathfrak{c}_j\sum_{n=0}^{\infty}(\text{const}'M^j\mathfrak{c}_j|u|_{1,\Sigma})^n\\ &= \text{const}\,\frac{M^j\mathfrak{c}_j}{1-\text{const}'M^j\mathfrak{c}_j|u|_{1,\Sigma}}\,.\end{aligned}$$

If $\tau_1 < \min\{\frac{1}{2\,\text{const}'},1\}$, then, by Corollary A.5(i), with $X = M^j|u|_{1,\Sigma}$, $\Lambda = M^j$ and $\mu = \text{const}'$,

$$|c|_{1,\Sigma} \le \text{const}\,\frac{M^j\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}}\,.$$

(iii) The bound

$$\begin{aligned}&\sum_{\delta}\frac{1}{\delta!}\sup_{\xi,\xi',s,s'}|\mathcal{D}_{1,2}^{\delta}(A*B)((\xi,s),(\xi',s'))|t^{\delta}\\ &\quad\le 3\left\{\sum_{\delta}\frac{1}{\delta!}\sup_{\xi,\xi',s,s'}|\mathcal{D}_{1,2}^{\delta}A((\xi,s),(\xi',s'))|t^{\delta}\right\}|B|_{1,\Sigma}\end{aligned}$$

is proven in much the same way as Lemma II.7, but uses

$$\sup_{\xi,\xi'}\left|\int d\zeta A(\xi,\zeta)B(\zeta,\xi')\right| \le \sup_{\xi,\zeta}|A(\xi,\zeta)|\sup_{\zeta}\int d\xi'|B(\zeta,\xi')|$$

in place of

$$\sup_{\xi}\int d\xi' \left|\int d\zeta A(\xi,\zeta)B(\zeta,\xi')\right| \le \sup_{\xi}\int d\zeta |A(\xi,\zeta)| \sup_{\zeta}\int d\xi' |B(\zeta,\xi')| \,.$$

Repeatedly applying this bound to (XIII.6) and using Lemma XIII.4(iv) yields

$$\begin{aligned}
&\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \sup_{\xi,\xi',s,s'} |\mathcal{D}_{1,2}^{\delta} c((\xi,s),(\xi',s'))| t^{\delta} \\
&\quad \le \left\{ \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \sup_{\xi,\xi',s,s'} |\mathcal{D}_{1,2}^{\delta} c^{(j)}((\xi,s),(\xi',s'))| t^{\delta} \right\} \sum_{n=0}^{\infty} (9|u|_{1,\Sigma} |\tilde{c}^{(j)}|_{1,\Sigma})^n \\
&\quad \le \text{const}\, \frac{\mathfrak{l}}{M^j} \mathfrak{c}_j \sum_{n=0}^{\infty} (\text{const}' M^j \mathfrak{c}_j |u|_{1,\Sigma})^n \\
&\quad = \text{const}\, \frac{\mathfrak{l}}{M^j} \frac{\mathfrak{c}_j}{1-\text{const}' M^j \mathfrak{c}_j |u|_{1,\Sigma}} \\
&\quad \le \text{const}\, \frac{\mathfrak{l}}{M^j} \frac{\mathfrak{c}_j}{1- M^j |u|_{1,\Sigma}} \,.
\end{aligned}$$

(iv) Repeat the proof of (ii) with (XIII.6) replaced by

$$c_0 = c_0^{(j)} \sum_{n=0}^{\infty} (* \, u * \tilde{c}^{(j)})^n$$

and using Lemma XIII.4(iii). □

Lemma XIII.6. *There are constants* τ_2, const *that depend on* $e(\mathbf{k})$ *and* M, *but not on* j *or* Σ *with the following property*:

Let, for κ *in a neighborhood of zero,* $u_\kappa \in \mathcal{F}_0(2;\Sigma)$ *be antisymmetric, spin independent and particle number conserving and obey* $|u_0|_{1,\Sigma} \le \frac{\tau_2}{M^j} + \sum_{\delta\ne\mathbf{0}} \infty t^{\delta}$. *Let*

$$C_\kappa(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}_\kappa(k)}$$

and $c_\kappa((\xi,s),(\xi',s'))$ *be the Fourier transform of* $\chi_s(k) C_\kappa(k) \chi_{s'}(k)$. *Let* $c_{\kappa,0}((\xi,s),(\xi',s'))$ *be the Fourier transform of* $\chi_s(k) k_0 C_\kappa(k) \chi_{s'}(k)$ *or* $\chi_s(k) e(\mathbf{k}) C_\kappa(k) \chi_{s'}(k)$. *Then*

(i) $|\frac{d}{d\kappa} c_\kappa|_{\kappa=0}|_{1,\Sigma} \le \text{const}\, M^j \mathfrak{c}_j \frac{M^j |\frac{d}{d\kappa} u_\kappa|_{\kappa=0}|_{1,\Sigma}}{1-M^j |u_0|_{1,\Sigma}}$.
(ii) $\sup_{\xi,\xi',s,s'} |\frac{d}{d\kappa} c_\kappa((\xi,s),(\xi',s'))|_{\kappa=0}| \le \text{const}\, \mathfrak{l} |\frac{d}{d\kappa} u_\kappa|_{\kappa=0}|_{1,\Sigma}$.
(iii) $|\frac{d}{d\kappa} c_{\kappa,0}|_{\kappa=0}|_{1,\Sigma} \le \text{const}\, \mathfrak{c}_j \frac{M^j |\frac{d}{d\kappa} u_\kappa|_{\kappa=0}|_{1,\Sigma}}{1-M^j |u_0|_{1,\Sigma}}$.

Proof. The proof is similar to that of Proposition XIII.5, using

$$\begin{aligned}\frac{d}{d\kappa}C_\kappa(k) &= \frac{d}{d\kappa}\frac{C^{(j)}(k)}{1-\check{u}_\kappa(k)\tilde{C}^{(j)}(k)} \\ &= C^{(j)}(k)\frac{1}{1-\check{u}_\kappa(k)\tilde{C}^{(j)}(k)}\left[\frac{d}{d\kappa}\check{u}_\kappa(k)\tilde{C}^{(j)}(k)\right]\frac{1}{1-\check{u}_\kappa(k)\tilde{C}^{(j)}(k)} \\ &= C^{(j)}(k)\sum_{m,n=0}^{\infty}(\check{u}_\kappa(k)\tilde{C}^{(j)}(k))^m\left[\frac{d}{d\kappa}\check{u}_\kappa(k)\tilde{C}^{(j)}(k)\right](\check{u}_\kappa(k)\tilde{C}^{(j)}(k))^n\end{aligned}$$

and [6, Corollary A.5], which implies that $(\frac{\mathfrak{c}_j}{1-M^j|u_0|_{1,\Sigma}})^2 \le \mathit{const}\,\frac{\mathfrak{c}_j}{1-M^j|u_0|_{1,\Sigma}}$. □

Lemma XIII.7. *Let $u((\xi,s),(\xi',s'))$ be a translation invariant function on $(\mathcal{B}\times\Sigma)^2$ with the property that $\check{u}((k,\sigma,a,s),(k',\sigma',a',s'))$ vanishes unless $\pi_F(k)\in\pi_F(s)$ and $\pi_F(k')\in\pi_F(s')$. Let $\mu(t)$ be a C_0^∞ function on $\mathbb{R}$ and set, for each $\Lambda>0$*

$$\mu_\Lambda(k) = \mu(\Lambda^2[k_0^2+e(\mathbf{k})^2])$$

$$(u*\hat{\mu}_\Lambda)((\xi,s),(\xi',s')) = \int_{\mathcal{B}} d\zeta\, u((\xi,s),(\zeta,s'))\hat{\mu}_\Lambda(\zeta,\xi')$$

$$(\hat{\mu}_\Lambda*u)((\xi,s),(\xi',s')) = \int_{\mathcal{B}} d\zeta\, u((\zeta,s),(\xi',s'))\hat{\mu}_\Lambda(\zeta,\xi)$$

where $\hat{\mu}_\Lambda$ was defined in Definition IX.4. *Denote $j(\Lambda)=\min\{i\in\mathbb{N}|M^i\ge\Lambda\}$. Then, there is a constant const, depending on μ, but not on M, j or Λ, such that*

$$|u*\hat{\mu}_\Lambda|_{1,\Sigma}\,,\quad |\hat{\mu}_\Lambda*u|_{1,\Sigma}\le \mathit{const}\,\mathfrak{c}_{j(\Lambda)}|u|_{1,\Sigma}\,.$$

Proof. Let $\{\Theta_s|s\in\Sigma\}$ be the smooth partition of unity of the Fermi curve F that was chosen just before (XIII.2) and set

$$\chi_{\Lambda,s}(k)=\mu_\Lambda(k)\Theta_s(\pi_F(k))\,.$$

Then, by Lemma XIII.2 and Proposition XIII.1(iii), as in Lemma XIII.3,

$$\|\hat{\chi}_{\Lambda,s}\|_{1,\infty}\le \mathit{const}\,\mathfrak{c}_{j(\Lambda)}\,.$$

We treat $u*\hat{\mu}_\Lambda$. The other case is similar. As

$$u*\hat{\mu}_\Lambda((\xi,s),(\xi',s')) = \sum_{\substack{s''\in\Sigma\\ s''\cap s'\ne\emptyset}}\int_{\mathcal{B}} d\zeta\, u((\xi,s),(\zeta,s'))\hat{\chi}_{\Lambda,s''}(\zeta,\xi')\,.$$

Lemma II.7 implies that

$$\begin{aligned}\|u*\hat{\mu}_\Lambda((\cdot,s),(\cdot,s'))\|_{1,\infty} &\le \sum_{\substack{s''\in\Sigma\\ s''\cap s'\ne\emptyset}} \mathit{const}\,\mathfrak{c}_{j(\Lambda)}\|u((\cdot,s),(\cdot,s'))\|_{1,\infty} \\ &\le \mathit{const}\,\mathfrak{c}_{j(\Lambda)}\|u((\cdot,s),(\cdot,s'))\|_{1,\infty}\end{aligned}$$

since, for each $s' \in \Sigma$, there are only three $s'' \in \Sigma$ with $s'' \cap s' \neq \emptyset$. The lemma follows. □

Remark XIII.8. In the notation of Lemma XIII.7,

$$(u * \hat{\mu}_\Lambda)\check{}(k) = (\hat{\mu}_\Lambda * u)\check{}(k) = \check{u}(k)\mu_\Lambda(k)\,.$$

XIV. Ladders

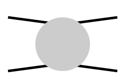
In Sec. XV, we will apply [3, Theorem VI.6] to estimate the renormalization group map $\tilde{\Omega}$ of Definition VII.1, with respect to the sectorized norms of Definition XII. It will give "improved power counting" for two-legged contributions and "improved power counting" for those four-legged contributions that are not ladders. A similar result using the norms of Sec. X, will be derived in Sec. XVII. Depending on the geometry of the Fermi curve, ladders have different behavior. We shall investigate ladders in Sec. XXII, [4, Sec. VII] and the paper [5]. In this section, we introduce notation for ladders that will be useful in all these investigations.

In this section, the internal lines of ladders will be functions with arguments running over an arbitrary measure space $\mathfrak{X}$. We think of $\mathfrak{X}$ as $\mathcal{B} = \mathbb{R} \times \mathbb{R}^2 \times \{\uparrow,\downarrow\} \times \{0,1\}$ or $\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow,\downarrow\}$ or $\mathbb{R} \times \mathbb{R}^2$ or $\mathcal{B} \times \Sigma$, where Σ is a sectorization.

Definition XIV.1. (i) A complex valued function on $\mathfrak{X} \times \mathfrak{X}$ is called a propagator over $\mathfrak{X}$.

(ii) A four-legged kernel over $\mathfrak{X}$ is a complex valued function on $\mathfrak{X}^2 \times \mathfrak{X}^2$. We sometimes consider it as a bubble propagator over $\mathfrak{X}$, graphically depicted by

or as a rung over $\mathfrak{X}$, graphically depicted by



(iii) If A and B are propagators over $\mathfrak{X}$ then the tensor product

$$A \otimes B(x_1, x_2, x_3, x_4) = A(x_1, x_3)B(x_2, x_4)$$

is a bubble propagator over $\mathfrak{X}$. We set

$$\mathcal{C}(A, B) = A \otimes A + A \otimes B + B \otimes A\,.$$

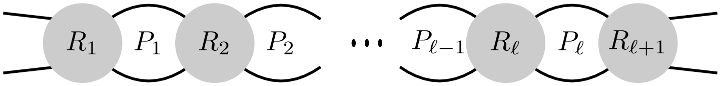
(iv) Let F, F' be four-legged kernels over $\mathfrak{X}$. We define the four-legged kernel $F \circ F'$ as

$$(F \circ F')(x_1, x_2; x_3, x_4) = \int dx_1' dx_2' F(x_1, x_2; x_1', x_2') F'(x_1', x_2'; x_3, x_4)$$

whenever the integral is well-defined.

(v) Let $\ell \geq 1$. The ladder with rungs $R_1, \ldots, R_{\ell+1}$ and bubble propagators $P_1, \ldots, P_\ell$ is defined to be

$$R_1 \circ P_1 \circ R_2 \circ P_2 \circ \cdots \circ P_{\ell-1} \circ R_\ell \circ P_\ell \circ R_{\ell+1}$$



If R is a rung and A, B are propagators we define $L_\ell(R; A, B)$ as the ladder with $\ell+1$ rungs R and ℓ bubble propagators $\mathcal{C}(A, B)$.

The ladders contribute to the four-point function, which is antisymmetric. So the ladders must be antisymmetrized.

Definition XIV.2. Let F be a four-legged kernel. The antisymmetrization of F is the four-legged kernel

$$(\mathrm{Ant}\, F)(x_1, x_2, x_3, x_4) = \frac{1}{4!} \sum_{\pi \in S_4} \mathrm{sign}(\pi) F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)})$$

F is called antisymmetric if $F = \mathrm{Ant}\, F$.

In the direct application of [3, Theorem VI.6], we will consider ladders with internal lines taking values in the measure space $\mathfrak{X} = \mathcal{B} \times \Sigma$, where Σ is a sectorization. However, the propagators are not naturally sectorized, and in [4, Sec. VII] we will combine bubble propagators of different scales. This motivates the following variant of the previous definitions.

Definition XIV.3. Let S be a finite set.[b] It is endowed with the counting measure. Then $\mathfrak{X} \times S$ is also a measure space.

(i) Let P be a propagator over $\mathfrak{X}$, f a four-legged kernel over $\mathfrak{X} \times S$ and F a function on $(\mathfrak{X} \times S)^2 \times \mathfrak{X}^2$. We define

$$(f \bullet P)((x_1, s_1), (x_2, s_2); x_3, x_4)$$

$$= \sum_{s_1', s_2' \in S} \int dx_1' dx_2' f((x_1, s_1), (x_2, s_2), (x_1', s_1'), (x_2', s_2')) P(x_1', x_2'; x_3, x_4)$$

$$(F \bullet f)((x_1, s_1), \ldots, (x_4, s_4))$$

$$= \sum_{s_1', s_2' \in S} \int dx_1' dx_2' F((x_1, s_1), (x_2, s_2); x_1', x_2')$$

$$\cdot f((x_1', s_1'), (x_2', s_2'), (x_3, s_3), (x_4, s_4))$$

whenever the integrals are well-defined. Observe that $(f \bullet P)$ is a function on $(\mathfrak{X} \times S)^2 \times \mathfrak{X}^2$ and $F \bullet f$ is a four-legged kernel over $\mathfrak{X} \times S$.

[b]In practice, S will be a set of sectors and $\mathfrak{X}$ will be $\mathcal{B}$ or $\mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\}$ or $\mathbb{R} \times \mathbb{R}^2$.

(ii) Let $\ell \geq 1, r_1, \ldots, r_{\ell+1}$ be rungs over $\mathfrak{X} \times S$ and $P_1, \ldots, P_\ell$ be bubble propagators over $\mathfrak{X}$. The ladder with rungs $r_1, \ldots, r_{\ell+1}$ and bubble propagators $P_1, \ldots, P_\ell$ is defined to be

$$r_1 \bullet P_1 \bullet r_2 \bullet P_2 \bullet \cdots \bullet r_\ell \bullet P_\ell \bullet r_{\ell+1}\,.$$

If r is a rung over $\mathfrak{X} \times S$ and A, B are propagators over $\mathfrak{X}$, we define $L_\ell(r; A, B)$ as the ladder with $\ell + 1$ rungs r and ℓ bubble propagators $\mathcal{C}(A, B)$.

Lemma XIV.4. *Let c and d be propagators over $\mathfrak{X} \times S$ and r a rung over $\mathfrak{X} \times S$. Define the propagators C and D over $\mathfrak{X}$ by*

$$C(x_1, x_2) = \sum_{t_1, t_2 \in S} c((x_1, t_1), (x_2, t_2)) \qquad D(x_1, x_2) = \sum_{t_1, t_2 \in S} d((x_1, t_1), (x_2, t_2))$$

and new propagators $\tilde{c}$ and $\tilde{d}$ over $\mathfrak{X} \times S$ by

$$\tilde{c}((x_1, s_1), (x_2, s_2)) = C(x_1, x_2) \qquad \tilde{d}((x_1, s_1), (x_2, s_2)) = D(x_1, x_2)\,.$$

Then

$$L_\ell(r; C, D) = L_\ell(r; \tilde{c}, \tilde{d})$$

for all $\ell \geq 1$. Here, the ladder $L_\ell(r; \tilde{c}, \tilde{d})$, of the right-hand side, is defined over the measure space $\mathfrak{X} \times S$ and uses the $\circ$ product while the ladder on the left-hand side is as in Definition XIV.3(ii).

Proof. For any rungs r', r'' over $\mathfrak{X} \times S$,

$$r' \bullet \mathcal{C}(C, D) \bullet r'' = r' \circ \mathcal{C}(\tilde{c}, \tilde{d}) \circ r''\,.$$

The lemma now follows by induction on ℓ. □

Lemma XIV.5. *Let Σ be a sectorization at scale j and $\varphi \in \mathcal{F}_0(4, \Sigma)$. Let $C(k)$ and $D(k)$ be functions on $\mathbb{R} \times \mathbb{R}^2$, that are supported in the jth neighborhood, and $C(\xi, \xi')$, $D(\xi, \xi')$ their Fourier transforms as in Definition* IX.3. *Furthermore, let $c((\cdot, s), (\cdot, s'))$ and $d((\cdot, s), (\cdot, s'))$ be the Fourier transforms of $\chi_s(k)C(k)\chi_{s'}(k)$ and $\chi_s(k)D(k)\chi_{s'}(k)$. Define propagators over $\mathcal{B} \times \Sigma$ by*

$$c_\Sigma((\xi, s), (\xi', s')) = \sum_{\substack{t \cap s \neq \emptyset \\ t' \cap s' \neq \emptyset}} c((\xi, t), (\xi', t'))$$

$$d_\Sigma((\xi, s), (\xi', s')) = \sum_{\substack{t \cap s \neq \emptyset \\ t' \cap s' \neq \emptyset}} d((\xi, t), (\xi', t'))\,.$$

Then

$$L_\ell(\varphi; C, D) = L_\ell(\varphi; c_\Sigma, d_\Sigma)$$

for all $\ell \ge 1$. *Here, the ladder on the right-hand side is defined over the measure space* $\mathcal{B} \times \Sigma$ *and uses the* $\circ$ *product while the ladders on the left-hand side are as in Definition* XIV.3(ii).

Proof. Since $\sum_{s\in\Sigma} \chi_s(k)$ is identically one on the support of $C(k)$ and $D(k)$,

$$C(\xi, \xi_2) = \sum_{t,t'\in\Sigma} c((\xi, t), (\xi', t')) \qquad D(\xi_1, \xi_2) = \sum_{t,t'\in\Sigma} d((\xi, t), (\xi', t'))\,.$$

As in Lemma XIV.4, set

$$\tilde{c}((\xi_1, s_1), (\xi_2, s_2)) = C(\xi_1, \xi_2) \qquad \tilde{d}((\xi_1, s_1), (\xi_2, s_2)) = D(\xi_1, \xi_2)\,.$$

Denote

$$p = \mathcal{C}(c, d) \qquad p_\Sigma = \mathcal{C}(c_\Sigma, d_\Sigma) \qquad \tilde{p} = \mathcal{C}(\tilde{c}, \tilde{d})\,.$$

Then, p is a Σ-sectorized bubble propagator and

$$p_\Sigma((\xi_1, s_1), (\xi_2, s_2), (\xi_3, s_3), (\xi_4, s_4)) = \sum_{\substack{t_i\cap s_i\neq\emptyset\\ 1\le i\le 4}} p((\xi_1, t_1), (\xi_2, t_2), (\xi_3, t_3), (\xi_4, t_4))$$

$$\tilde{p}((\xi_1, s_1), (\xi_2, s_2), (\xi_3, s_3), (\xi_4, s_4)) = \sum_{\substack{t_i\in\Sigma\\ 1\le i\le 4}} p((\xi_1, t_1), (\xi_2, t_2), (\xi_3, t_3), (\xi_4, t_4))\,.$$

For any $w \in \mathcal{F}_0(4, \Sigma)$,

$$\begin{aligned}
w \circ p_\Sigma \circ \varphi &= \sum_{\substack{s_i'\in\Sigma\\ 1\le i\le 4}} \int d\xi_1' \cdots d\xi_4' w(\cdot, \cdot; (\xi_1', s_1'), (\xi_2', s_2'))\\
&\qquad \cdot p_\Sigma((\xi_1', s_1'), \ldots, (\xi_4', s_4'))\varphi((\xi_3', s_3'), (\xi_4', s_4'), \cdot, \cdot)\\
&= \sum_{\substack{s_i', t_i\in\Sigma\\ s_i'\cap t_i\neq\emptyset\\ 1\le i\le 4}} \int d\xi_1' \cdots d\xi_4' w(\cdot, \cdot; (\xi_1', s_1'), (\xi_2', s_2'))\\
&\qquad \cdot p((\xi_1', t_1), \ldots, (\xi_4', t_4))\varphi((\xi_3', s_3'), (\xi_4', s_4'), \cdot, \cdot)\\
&= \sum_{\substack{s_i', t_i\in\Sigma\\ 1\le i\le 4}} \int d\xi_1' \cdots d\xi_4' w(\cdot, \cdot; (\xi_1', s_1'), (\xi_2', s_2'))\\
&\qquad \cdot p((\xi_1', t_1), \ldots, (\xi_4', t_4))\varphi((\xi_3', s_3'), (\xi_4', s_4'), \cdot, \cdot)\\
&= w \circ \tilde{p} \circ \varphi
\end{aligned} \tag{XIV.1}$$

because

$$\int d\xi_1' \cdots d\xi_4' w(\cdot, \cdot; (\xi_1', s_1'), (\xi_2', s_2'))p((\xi_1', t_1), \ldots, (\xi_4', t_4))\varphi((\xi_3', s_3'), (\xi_4', s_4'), \cdot, \cdot)$$

vanishes unless $s_i' \cap t_i \neq \emptyset$ for all $1 \leq i \leq 4$. Observe that $\tilde{w} \circ \tilde{p} \circ \varphi$ is again in $\mathcal{F}_0(4, \Sigma)$.

It follows by induction from (XIV.1) that

$$L_\ell(\varphi; c_\Sigma, d_\Sigma) = L_\ell(\varphi; \tilde{c}, \tilde{d}) .$$

The lemma follows by Lemma XIV.4. □

XV. Norm Estimates on the Renormalization Group Map

Again, let $j \geq 2$ and Σ be a sectorization of scale j and length $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$. Fix a system $\boldsymbol{\rho} = (\rho_{m;n})$ of positive real numbers such that

$$\begin{aligned} \rho_{m;n} &\leq \rho_{m;n'} && \text{if } n \leq n' \\ \rho_{m+m';n+n'-2} &\leq \rho_{m;n}\rho_{m';n'} && \\ \rho_{m+1;n-1} &\leq \rho_{m;n} && \text{if } m \geq 1 \\ \rho_{1;n-1} &\leq \sqrt{\mathfrak{l} M^j}\rho_{0;n} . && \end{aligned} \tag{XV.1}$$

Definition XV.1. (i) For $\varphi \in \mathcal{F}_m(n; \Sigma)$ set

$$|\varphi|_\Sigma = \rho_{m;n} \begin{cases} |\varphi|_{1,\Sigma} + \frac{1}{\mathfrak{l}}|\varphi|_{3,\Sigma} + \frac{1}{\mathfrak{l}^2}|\varphi|_{5,\Sigma} & \text{if } m = 0 \\ \frac{\mathfrak{l}}{M^{2j}}|\varphi|_{1,\Sigma} & \text{if } m \neq 0 \end{cases} .$$

(ii) We set, for $X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} X_\delta t^\delta \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \frac{1}{M^j}$,

$$\mathfrak{e}_j(X) = \frac{\mathfrak{c}_j}{1 - M^j X} .$$

(iii) A sectorized Grassmann function w can be uniquely written in the form

$$\begin{aligned} w(\phi, \psi) = \sum_{m,n} \sum_{s_1,\ldots,s_n \in \Sigma} \int & d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n \\ & \cdot w_{m,n}(\eta_1, \ldots, \eta_m(\xi_1, s_1), \ldots, (\xi_n, s_n)) \\ & \cdot \phi(\eta_1) \cdots \phi(\eta_m)\psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n)) \end{aligned}$$

with $w_{m,n}$ antisymmetric separately in the η and in the ξ variables. Set, in analogy with Theorem VIII.6, for $\alpha > 0$ and $X \in \mathfrak{N}_{d+1}$,

$$N_j(w; \alpha; X, \Sigma, \boldsymbol{\rho}) = \frac{M^{2j}}{\mathfrak{l}} \mathfrak{e}_j(X) \sum_{m,n \geq 0} \alpha^n \left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2} |w_{m,n}|_\Sigma .$$

The constant B will be chosen in Definition XVII.1(iii). It will obey $\mathrm{B} > 4\max\{8\mathrm{B}_1, \mathrm{B}_2\}$ with $\mathrm{B}_1, \mathrm{B}_2$ being the constants of Propositions XII.16 and XII.18.

Remark XV.2. (i) By definition, for even w

$$N_j(w;\Lambda,\alpha;X,\Sigma,\boldsymbol{\rho}) = \mathfrak{e}_j(X)\sum_{n\geq 1}\mathrm{B}^n\alpha^{2n}\frac{\mathfrak{l}^{n-1}}{M^{j(n-2)}}\rho_{0;2n}|w_{0,2n}|_{1,\Sigma}$$
$$+\,\mathfrak{e}_j(X)\sum_{n\geq 2}\mathrm{B}^n\alpha^{2n}\frac{\mathfrak{l}^{n-2}}{M^{j(n-2)}}\rho_{0;2n}|w_{0,2n}|_{3,\Sigma}$$
$$+\,\mathfrak{e}_j(X)\sum_{n\geq 3}\mathrm{B}^n\alpha^{2n}\frac{\mathfrak{l}^{n-3}}{M^{j(n-2)}}\rho_{0;2n}|w_{0,2n}|_{5,\Sigma}$$
$$+\,\mathfrak{e}_j(X)\sum_{m\geq 1}\sum_{n\geq 0}\alpha^n\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2}\rho_{m;n}|w_{m,n}|_{1,\Sigma}\,.$$

If, in a renormalization group analysis, $\rho_{0;2n}$ is independent of the scale number, j, then boundedness of the norms N_j imply that

$$|w_{0,2}|_{1,\Sigma} = O\left(\frac{1}{M^j}\right) \qquad |w_{0,4}|_{3,\Sigma} = O(1)$$

modulo t.

(ii) If $X \leq \frac{1}{2M^j}\mathfrak{c}_j$ then $\mathfrak{e}_j(X) \leq {\scriptstyle const}\,\mathfrak{c}_j$.

(iii) $\frac{\partial}{\partial t_0}\mathfrak{c}_j \leq {\scriptstyle const}\, M^j\mathfrak{c}_j + \sum_{\delta_0=r_0}\infty t^\delta$.

(iv) If X is independent of t_0, then $\frac{\partial}{\partial t_0}\mathfrak{e}_j(X) \leq {\scriptstyle const}\, M^j\mathfrak{e}_j(X) + \sum_{\delta_0=r_0}\infty t^\delta$.

(v) The j-dependent factors in the definition of N_j were largely motivated by the discussion in [2, Sec. II, Subsec. 8] and [4, Remark VI.8].

The main result of this paper is, that the norms of Definition XV.1 are not changed very much by the renormalization group map $\tilde\Omega_C$ of Definition VII.1, and that there is volume improvement for the two-point function and all contributions to the four-point function with the exception of ladders.

Theorem XV.3. *There are constants* const, const$_0$, α_0 *and* τ_0 *that are independent of* j, Σ, $\boldsymbol{\rho}$ *such that for all* $\alpha \geq \alpha_0$ *the following estimates hold*:

Let $u((\xi,s),(\xi',s')), v((\xi,s),(\xi',s')) \in \mathcal{F}_0(2;\Sigma)$ *be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms obey* $|\check u(k)|$, $|\check v(k)| \leq \frac{1}{2}|\imath k_0 - e(k)|$. *Furthermore, let* $X \in \mathfrak{N}_{d+1}$, $\mu,\Lambda > 0$ *and assume that* $|u|_{1,\Sigma} \leq \mu(\Lambda + X)\mathfrak{e}_j(X)$ *and* $(1+\mu)(\Lambda + X_{\mathbf{0}}) \leq \frac{\tau_0}{M^j}$. *Set*

$$C(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u(k)}\,, \quad D(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check v(k)}$$

and let $C(\xi,\xi')$, $D(\xi,\xi')$ *be the Fourier transforms of* $C(k)$, $D(k)$ *as in Definition* IX.3. *Let* $\mathcal{W}(\phi,\psi)$ *be a Grassmann function and set*[c]

$$:\mathcal{W}'(\phi,\psi):_{\psi,D} = \tilde\Omega_C(:\mathcal{W}(\phi,\psi):_{\psi,C+D})\,.$$

[c]The definition of $\mathcal{W}'$ as an analytic function, rather than merely a formal Taylor series will be explained in Remark XV.11.

Assume that $\mathcal{W}$ *has a sectorized representative* w *with* $w_{0,2} = 0$ *and*

$$N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho}) \le \mathrm{const}_0\, \alpha + \sum_{\delta \ne 0} \infty t^{\delta}\,.$$

Then $\mathcal{W}'$ *has a sectorized representative* w' *such that*

$$N_j\left(w' - \frac{1}{2}\phi J C J \phi - w; \alpha; X, \Sigma, \boldsymbol{\rho}\right) \le \frac{\mathrm{const}}{\alpha} \frac{N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})}{1 - \frac{\mathrm{const}}{\alpha} N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})}\,.$$

Furthermore

$$|w'_{0,2}|_{1,\Sigma} \le \frac{\mathrm{const}}{\alpha^8 \rho_{0;2}} \frac{\mathfrak{c}}{M^j} \frac{N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})}$$

and

$$\left| w'_{0,4} - w_{0,4} - \frac{1}{4} \sum_{\ell=1}^{\infty} (-1)^{\ell} (12)^{\ell+1} \mathrm{Ant}\, L_{\ell}(w_{0,4}; C, D) \right|_{3,\Sigma}$$

$$\le \frac{\mathrm{const}}{\alpha^{10} \rho_{0;4}} \mathfrak{c} \frac{N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w; 64\alpha; X, \Sigma, \boldsymbol{\rho})}\,.$$

Remark XV.4. (i) When we use Theorem XV.3 in a renormalization group analysis, u will depend on counterterms that will ultimately be generated at scales $j' > j$. Then the derivatives of $\check{u}(k)$ can have a scaling behavior characteristic of scale j'. In this case $|u|_{1,\Sigma}$ will not be of order $\mathfrak{c}_j$. This is why we introduce the factor $\mathfrak{e}_j(X)$ in the definition of N_j.

(ii) The hypothesis that $w_{0,2} = 0$ is used, in conjunction with Wick ordering, to ensure that all non-ladder contributions to $w'_{0,2}$ and $w'_{0,4}$ contain overlapping loops. See [2, Sec. II, Subsecs. 4 and 9].

(iii) In Appendix D, we give naive power-counting bounds for ladders $L_\ell(w_{0,4}; C, D)$. These estimates are not good enough for a renormalization group analysis. They would lead to logarithmic divergences. Stronger estimates on the "particle–particle" part of the ladders are derived in Theorem XXII.7. The "particle–hole" parts of the ladders are treated in [5].

Most of the rest of this paper is devoted to the proof of Theorem XV.3. To simplify notation we write $N_j(w; \alpha)$ for $N_j(w; \alpha; X, \Sigma, \boldsymbol{\rho})$. We define a family of seminorms on the spaces $\mathcal{F}_m(n; \Sigma)$ by

$$|\varphi|_p = \rho_{m;n} \begin{cases} |\varphi|_{p,\Sigma} & \text{if } m = 0 \\ \frac{\mathfrak{c}}{M^{2j}} |\varphi|_{p,\Sigma} & \text{if } m \ne 0 \end{cases}$$

with $p = 1, 3, 5$. As in Definition XII.6, these norms induce a family of symmetric seminorms on the spaces $A_m \otimes V_\Sigma^{\otimes n}$. This family of seminorms will only appear in the proof of Theorem XV.3 and in the preliminary Lemma XV.5.

Let $c((\cdot,s),(\cdot,s'))$ and $d((\cdot,s),(\cdot,s'))$ be the Fourier transform of $\chi_s(k)C(k)\chi_{s'}(k)$ and $\chi_s(k)D(k)\chi_{s'}(k)$ in the sense of Definition IX.3. As in Lemma XIV.5, let

$$c_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset \\ t'\cap s'\neq\emptyset}} c((\xi,t),(\xi',t'))$$

$$d_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset \\ t'\cap s'\neq\emptyset}} d((\xi,t),(\xi',t'))\,.$$

As in Proposition XII,

$$C_\Sigma(\psi(\xi,s),\psi(\xi',s')) = c_\Sigma((\xi,s),(\xi',s'))$$
$$D_\Sigma(\psi(\xi,s),\psi(\xi',s')) = d_\Sigma((\xi,s),(\xi',s'))$$

are covariances on V_Σ.

Lemma XV.5. *Under the hypotheses of Theorem* XV.3, *there exists a constant* const_1 *that is independent of* j *and* Σ *such that the covariances* C_Σ, D_Σ *have integration constants*[d]

$$\mathfrak{c} = \mathrm{const}_1\, M^j \mathfrak{e}_j(X)\,, \quad \frac{1}{2}\mathrm{b} = \sqrt{\frac{\mathrm{B}\,\mathfrak{l}}{4M^j}}$$

(*in the sense of* [3, *Definition* VI.13]) *for the configuration* $|\cdot|_p$ *of seminorms.*

Proof. Clearly, the functions $C(k)$ and $D(k)$ are supported on the jth neighborhood, and $|C(k)|$, $|D(k)| \le \frac{2}{|\imath k_0 - e(\mathbf{k})|}$. By Proposition XII.16(ii) and the first condition of (XV.1), $\frac{1}{2}\mathrm{b}$ is an integral bound both for C_Σ and D_Σ.

We now verify the contraction estimates of [3, Definition VI.13]. Contraction by c for functions on $\mathcal{B}^m \times (\mathcal{B}\times\Sigma)^m$ as in Definition XII.15 corresponds to contraction by C_Σ in the Grassmann algebra over V_Σ as in [1, Definition II.5]. Set

$$\mathfrak{c}' = 9\max\left\{ |c|_{1,\Sigma},\ \frac{M^{2j}}{\mathfrak{l}} \sup_{\xi,\xi',s,s'} |c((\xi,s),(\xi',s'))| \right\}.$$

It follows from Proposition XII.16(i), combined with the second property of $\boldsymbol{\rho}$, and Proposition XII.18, combined with the first two properties of $\boldsymbol{\rho}$, that C_Σ, D_Σ have integration constants $\mathfrak{c}'$, $\frac{1}{2}\mathrm{b}$. By Proposition XIII.5, if τ_0 is small enough,

$$\mathfrak{c}' \le \mathrm{const}\max\left\{ \frac{M^j\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}},\ M^j \right\} \le \frac{\mathrm{const}\, M^j\mathfrak{c}_j}{1-M^j|u|_{1,\Sigma}}\,.$$

[d]We shall, in the proof of Theorem XV.7, apply [1, Theorem IV.4], which requires integral bounds $\frac{1}{2}\mathrm{b}$. Of course, then b is also an integral bound, as is required in the proof of the current Theorem XV.3.

Therefore, by the hypotheses on u,

$$\mathfrak{c}' \le \frac{\mathrm{const}\, M^j \mathfrak{c}_j}{1-\mu M^j(\Lambda+X)\frac{\mathfrak{c}_j}{1-M^jX}} \le \frac{\mathrm{const}\, M^j \mathfrak{c}_j}{1-\mu\frac{M^j\mathfrak{c}_j(\Lambda+X)}{1-M^j\mathfrak{c}_j(\Lambda+X)}} = \mathrm{const}\, M^j \mathfrak{c}_j f(Y)$$

where $Y = M^j \mathfrak{c}_j(\Lambda+X)$ and

$$f(z) = \frac{1}{1-\mu\frac{z}{1-z}} = \frac{1-z}{1-(1+\mu)z}\,.$$

By Lemma A.7, $f(Y) \le {\scriptstyle const}\, \frac{1}{1-Y}$ so that

$$\begin{aligned}\mathfrak{c}' &\le \frac{\mathrm{const}\, M^j \mathfrak{c}_j}{1-M^j\mathfrak{c}_j\Lambda - M^j\mathfrak{c}_jX} \le \mathrm{const}\, M^j \frac{\mathfrak{c}_j}{1-M^j\mathfrak{c}_j\Lambda}\frac{1}{1-M^j\mathfrak{c}_jX}\\ &\le \mathrm{const}\, M^j \frac{\mathfrak{c}_j}{1-\mathfrak{c}_j/3}\frac{1}{1-M^j\mathfrak{c}_jX}\,.\end{aligned}$$

In the second inequality we used Lemma A.4(ii). As, by Corollary A.5(i), $\frac{\mathfrak{c}_j}{1-\mathfrak{c}_j/3} \le {\scriptstyle const}\, \mathfrak{c}_j$ and $\frac{\mathfrak{c}_j}{1-M^j\mathfrak{c}_jX} \le {\scriptstyle const}\, \frac{\mathfrak{c}_j}{1-M^jX}$, we have

$$\mathfrak{c}' \le \mathrm{const}\, M^j \mathfrak{e}_j(X)\,. \tag{XV.2}$$

□

Lemma XV.6. *Let $g(\phi,\psi)$ be a sectorized Grassmann function. Let $C(k)$ be a function obeying $|C(k)| \le \frac{2}{|\imath k_0 - e(\mathbf{k})|}$ and $C(\xi,\xi')$, resp. $c((\xi,s),(\xi',s'))$, be the Fourier transforms of $C(k)$, resp. $\chi_s(k)C(k)\chi_{s'}(k)$, in the sense of Definition* IX.3. *Set*

$$g'(\phi,\psi) = g(\phi,\psi + CJ\phi)\,.$$

If $|c|_{1,\Sigma} \le \mathrm{const}\, M^j + \sum_{\delta\ne 0}\infty t^\delta$, then

$$N_j(g'-g;\alpha) \le \frac{\mathrm{const}}{\alpha}N_j(g;2\alpha)\,.$$

In particular, this bound is true under the hypotheses of Theorem XV.3.

Proof. Let $\varphi \in \mathcal{F}_m(n;\Sigma)$, $1 \le i \le n$ and set

$$\begin{aligned}&\varphi'(\eta_1,\ldots,\eta_{m+1};(\xi_1,s_1),\ldots,(\xi_{n-1},s_{n-1}))\\ &\quad= \mathrm{Ant}_{\mathrm{ext}} \sum_{s,t,t'\in\Sigma}\int d\zeta d\zeta' \varphi(\eta_1,\ldots,\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1}),\\ &\qquad (\zeta',t),(\xi_i,s_i),\ldots,(\xi_{n-1},s_{n-1}))c((\zeta',t',),(\zeta,s))J(\zeta,\eta_{m+1})\,.\end{aligned}$$

Under the hypotheses of Theorem XV.3, $|c|_{1,\Sigma} \le \mathrm{const}\, M^j + \sum_{\delta\ne 0}\infty t^\delta$ by Proposition XIII.5(ii). Hence, by Lemma XII,

$$|\varphi'|_{1,\Sigma} \le \mathrm{const}\, |\varphi|_{1,\Sigma}\begin{cases}\dfrac{M^j}{\mathfrak{l}} & \text{if } m=0\\[2ex] \dfrac{\mathfrak{l}}{M^j} & \text{if } m\ne 0\end{cases}\,.$$

Here we have used that the coefficient of t^δ in $|\varphi'|_{1,\Sigma}$ vanishes for $\delta \neq 0$ so that in Lemma XII we may replace $|c|_{1,\Sigma}$ by its value at $t=0$. Hence, for $m=0$,

$$|\varphi'|_\Sigma = \rho_{1;n-1}\frac{\mathfrak{l}}{M^{2j}}|\varphi'|_{1,\Sigma} \le \text{const}\,\rho_{1;n-1}\frac{1}{M^j}|\varphi|_{1,\Sigma}$$
$$\le \text{const}\,\frac{\rho_{1;n-1}}{\rho_{0;n}}\frac{1}{M^j}|\varphi|_\Sigma \le \text{const}\,\sqrt{\mathfrak{l}\,M^j}\,\frac{1}{M^j}|\varphi|_\Sigma \le \text{const}\,\mathrm{b}|\varphi|_\Sigma$$

and, for $m \neq 0$,

$$|\varphi'|_\Sigma = \rho_{m+1;n-1}\frac{\mathfrak{l}}{M^{2j}}|\varphi'|_{1,\Sigma} \le \text{const}\,\rho_{m+1;n-1}\frac{\mathfrak{l}^2}{M^{3j}}|\varphi|_{1,\Sigma}$$
$$\le \text{const}\,\frac{\rho_{m+1;n-1}}{\rho_{m;n}}\frac{\mathfrak{l}}{M^j}|\varphi|_\Sigma \le \text{const}\,\frac{\mathfrak{l}}{M^j}|\varphi|_\Sigma \le \text{const}\,\mathrm{b}|\varphi|_\Sigma\,.$$

The lemma now follows from $|\varphi'|_\Sigma \le \text{const}\,\mathrm{b}|\varphi|_\Sigma$ as Proposition VII.6 follows from the bound of Definition VII.4. □

Proof of Theorem XV.3. For $\varphi \in \mathcal{F}_m(n;\Sigma)$ set

$$|\varphi|_{\mathrm{impr},\Sigma} = \rho_{m;n}\begin{cases} |\varphi|_{1,\Sigma} + \frac{1}{\mathfrak{l}}|\varphi|_{3,\Sigma} & \text{if } m=0 \\ 0 & \text{if } m\neq 0 \end{cases}.$$

This family of seminorms will only appear in this proof. By Lemma XV.5 and [3, Lemma VI.15], with $q=5$, $J=\mathfrak{l}$ and $\|\cdot\|_p = |\cdot|_p$, the covariances (C_Σ, D_Σ) have improved integration constants $\mathfrak{c}, \mathrm{b}, \mathfrak{l}$ for the families $|\cdot|_\Sigma$ and $|\cdot|_{\mathrm{impr},\Sigma}$ of seminorms (in the sense of [3, Definition VI.1]). For a sectorized Grassmann function $v = \sum_{m,n} v_{m,n}$ with $v_{m,n} \in A_m \otimes \bigwedge^n V_\Sigma$ let

$$N(v;\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n}\alpha^n \mathrm{b}^n |v_{m,n}|_\Sigma$$

$$N_{\mathrm{impr}}(v;\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{n}\alpha^n \mathrm{b}^n |v_{0,n}|_{\mathrm{impr},\Sigma}$$

be the quantities introduced in [1, Definition II.23] and just after [3, Lemma VI.2]. Then

$$N(v;\alpha) = \frac{\text{const}_1}{\mathrm{B}} N_j(v;\alpha;X,\Sigma,\boldsymbol{\rho})$$

where const_1 is the constant of Lemma XV.5.

Set $:w'':_{\psi,D_\Sigma} = \Omega_{C_\Sigma}(:w:_{\psi,C_\Sigma+D_\Sigma})$ and

$$w' = \frac{1}{2}\phi JCJ\phi + w''(\phi,\psi + CJ\phi)\,.$$

By parts (ii) and (iii) of Proposition XII, $:w':_{\psi,D_\Sigma}$ is a sectorized representative for $:\mathcal{W}'(\phi,\psi):_{\psi,D}$. Hence, by Proposition XII(i) and [1, Proposition A.2(ii)], w' is a sectorized representative for $\mathcal{W}'$. We apply [3, Theorem VI.6] to get estimates on w''. With $\mathrm{const}_0 = \frac{\mathrm{B}}{8\,\mathrm{const}_1}$ the hypotheses of this theorem are fulfilled. Consequently

$$N(w'' - w; \alpha) \le \frac{1}{2\alpha^2} \frac{N(w; 32\alpha)^2}{1 - \frac{1}{\alpha^2} N(w; 32\alpha)} \tag{XV.3}$$

and

$$\alpha^2 \mathfrak{c} |w''_{0,2}|_{\mathrm{impr},\Sigma} \le \frac{2^{10}\mathfrak{l}}{\alpha^6} \frac{N(w; 64\alpha)^2}{1 - \frac{8}{\alpha} N(w; 64\alpha)}$$

$$\alpha^4 \mathrm{b}^2 \mathfrak{c} \left| w''_{0,4} - w_{0,4} - \frac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, L_\ell(w_{0,4}; c_\Sigma, d_\Sigma) \right|_{\mathrm{impr},\Sigma}$$

$$\le \frac{2^{10}\mathfrak{l}}{\alpha^6} \frac{N(w; 64\alpha)^2}{1 - \frac{8}{\alpha} N(w; 64\alpha)}\,.$$

For the last estimate, we also used the description of ladders in terms of kernels of [3, Proposition C.4]. As $w'_{0,2} = w''_{0,2}$ and $w'_{0,4} = w''_{0,4}$ this implies that

$$\mathfrak{e}_j(X) |w'_{0,2}|_{1,\Sigma} \le \frac{\mathrm{const}}{\alpha^8 \rho_{0;2}} \frac{\mathfrak{l}}{M^j} \frac{N_j(w; 64\alpha)^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w; 64\alpha)}$$

and, using Lemma XIV.5

$$\mathfrak{e}_j(X) \left| w'_{0,4} - w_{0,4} - \frac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, L_\ell(w_{0,4}; C, D) \right|_{3,\Sigma}$$

$$\le \frac{1}{\rho_{0;4}} \mathfrak{l}\mathfrak{e}_j(X) \left| w''_{0,4} - w_{0,4} - \frac{1}{4} \sum_{\ell=1}^{\infty} (-1)^\ell (12)^{\ell+1} \mathrm{Ant}\, L_\ell(w_{0,4}; c_\Sigma, d_\Sigma) \right|_{\mathrm{impr},\Sigma}$$

$$\le \frac{\mathrm{const}}{\alpha^{10} \rho_{0;4}} \mathfrak{l} \frac{N_j(w; 64\alpha)^2}{1 - \frac{\mathrm{const}}{\alpha} N_j(w; 64\alpha)}\,.$$

By Lemma XV.6,

$$N_j\left(w' - \frac{1}{2}\phi JCJ\phi - w; \alpha\right)$$

$$= N_j(w''(\phi, \psi + CJ\phi) - w(\phi, \psi); \alpha)$$

$$\le N_j(w''(\phi, \psi + CJ\phi) - w''(\phi, \psi); \alpha) + N_j(w''(\phi, \psi) - w(\phi, \psi); \alpha)$$

$$\le \frac{\mathrm{const}}{\alpha} N_j(w''; 2\alpha) + N_j(w'' - w; \alpha)$$

$$\le \frac{\mathrm{const}}{\alpha} N_j(w; 2\alpha) + \left(1 + \frac{\mathrm{const}}{\alpha}\right) \frac{\mathrm{B}}{\mathrm{const}_1} N(w'' - w; 2\alpha)$$

$$\leq \frac{\text{const}}{\alpha} N_j(w;2\alpha) + \left(1 + \frac{\text{const}}{\alpha}\right) \frac{\text{B}}{\text{const}_1} \frac{1}{8\alpha^2} \frac{N(w;64\alpha)^2}{1 - \frac{1}{4\alpha^2} N(w;64\alpha)}$$

$$\leq \frac{\text{const}}{\alpha} N_j(w;2\alpha) + \frac{\text{const}}{\alpha^2} \frac{N_j(w;64\alpha)^2}{1 - \frac{\text{const}}{\alpha^2} N_j(w;64\alpha)}$$

$$\leq \frac{\text{const}}{\alpha} N_j(w;64\alpha) + \frac{\text{const}}{\alpha^2} \frac{N_j(w;64\alpha)^2}{1 - \frac{\text{const}}{\alpha} N_j(w;64\alpha)}$$

$$\leq \frac{\text{const}}{\alpha} \frac{N_j(w;64\alpha)}{1 - \frac{\text{const}}{\alpha} N_j(w;64\alpha)} . \qquad \square$$

We also wish to allow the functions u and v of Theorem XV.7 to depend on a parameter κ.

Theorem XV.7. *There are constants* const, const_0, α_0, τ_0 *that are independent of* j, Σ, $\boldsymbol{\rho}$ *such that for all* $\varepsilon > 0$ *and* $\alpha \geq \alpha_0$ *the following estimates hold*:

Let, for κ *in a neighborhood of zero,* $u_\kappa, v_\kappa \in \mathcal{F}_0(2;\Sigma)$ *be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms satisfy* $|\check{u}_0(k)|, |\check{v}_0(k)| \leq \frac{1}{2}|\imath k_0 - e(k)|$ *and* $|\frac{d}{d\kappa}\check{v}_\kappa(k)|_{\kappa=0}| \leq \varepsilon|\imath k_0 - e(k)|$. *Furthermore, let* $X, Y \in \mathfrak{N}_{d+1}$, $\mu, \Lambda > 0$ *and assume that*

$$|u_0|_{1,\Sigma} \leq \mu(\Lambda + X)\mathfrak{e}_j(X) \qquad \left|\frac{d}{d\kappa} u_\kappa\Big|_{\kappa=0}\right|_{1,\Sigma} \leq \mathfrak{e}_j(X) Y$$

and $(1+\mu)(\Lambda + X_{\mathbf{0}}) \leq \frac{\tau_0}{M^j}$. *Set*

$$C_\kappa(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}_\kappa(k)} , \quad D_\kappa(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}_\kappa(k)}$$

and let $C_\kappa(\xi,\xi')$, $D_\kappa(\xi,\xi')$ *be the Fourier transforms of* $C_\kappa(k)$, $D_\kappa(k)$. *Let, for* κ *in a neighborhood of zero,* $\mathcal{W}_\kappa(\phi,\psi)$ *be an even Grassmann function and set*

$$:\mathcal{W}'_\kappa(\psi):_{\psi,D_\kappa} = \tilde{\Omega}_{C_\kappa}(:\mathcal{W}_\kappa:_{\psi,C_\kappa+D_\kappa}) .$$

Assume that $\mathcal{W}_\kappa$ *has a sectorized representative* w_κ *with*

$$\mathfrak{n} \equiv N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho}) \leq \text{const}_0\, \alpha + \sum_{\delta \neq 0} \infty t^\delta .$$

Then $\mathcal{W}'_\kappa$ *has a sectorized representative* w'_κ *such that*

$$N_j\left(\frac{d}{d\kappa}\left[w'_\kappa - \frac{1}{2}\phi J C_\kappa J \phi - w_\kappa\right]_{\kappa=0};\alpha;X,\Sigma,\boldsymbol{\rho}\right)$$

$$\leq \text{const}\left\{\frac{1}{\alpha} + \frac{1}{\alpha^2}\frac{\mathfrak{n}}{1 - \frac{\text{const}}{\alpha^2}\mathfrak{n}}\right\} N_j\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha;X,\Sigma,\boldsymbol{\rho}\right)$$

$$+ \text{const}\,\frac{\mathfrak{n}}{1 - \frac{\text{const}}{\alpha^2}\mathfrak{n}}\left\{\left(\frac{1}{\alpha} + \frac{\mathfrak{n}}{\alpha^2}\right) M^j Y + \frac{\varepsilon}{\alpha^2}\mathfrak{n}\right\} .$$

Lemma XV.8. *Under the hypotheses of Theorem* XV.7, *there exists a constant* const_2 *that is independent of* j *and* Σ *such that* $C_{0,\Sigma}$ *has contraction bound* $\mathfrak{c}$, $C_{0,\Sigma}$ *and* $D_{0,\Sigma}$ *have integral bound* $\frac{1}{2}\mathrm{b}$ *and*

$$\frac{d}{d\kappa}C_{\kappa,\Sigma}\Big|_{\kappa=0} \quad \textit{has contraction bound } \mathfrak{c}' = \mathrm{const}_2\, M^{2j}\mathfrak{e}_j(X)Y$$

$$\frac{d}{d\kappa}D_{\kappa,\Sigma}\Big|_{\kappa=0} \quad \textit{has integral bound } \frac{1}{2}\mathrm{b}' = \sqrt{\varepsilon}\mathrm{b}$$

for the family $|\cdot|_\Sigma$ *of symmetric seminorms.*

Proof. The contraction and integral bounds on $C_{0,\Sigma}$ and $D_{0,\Sigma}$ were proven in Lemma XV.5. Clearly, the function

$$\frac{d}{d\kappa}D_\kappa(k) = \frac{d}{d\kappa}\frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}_\kappa(k)} = \frac{\nu^{(\ge j+1)}(k)}{[\imath k_0 - e(\mathbf{k}) - \check{v}_\kappa(k)]^2}\frac{d}{d\kappa}\check{v}_\kappa(k)$$

is supported on the jth neighborhood and obeys $|\frac{d}{d\kappa}D_\kappa(k)|_{\kappa=0}| \le \frac{4\varepsilon}{|\imath k_0 - e(\mathbf{k})|}$. By Proposition XII.16(ii), $2\sqrt{4\mathrm{B}_1\varepsilon\frac{\mathfrak{l}}{M^j}} \le \sqrt{\varepsilon}\mathrm{b}$ is an integral bound for $\frac{d}{d\kappa}D_{\kappa,\Sigma}|_{\kappa=0}$.

Set

$$\mathfrak{c}'' = 9\max\left\{\left|\frac{d}{d\kappa}c_\kappa\Big|_{\kappa=0}\right|_{1,\Sigma},\ \frac{M^{2j}}{\mathfrak{l}}\sup_{\xi,\xi',s,s'}\left|\frac{d}{d\kappa}c_\kappa((\xi,s),(\xi',s'))\Big|_{\kappa=0}\right|\right\}.$$

By Proposition XII.16(i) and the second property of $\boldsymbol{\rho}$, $(\frac{d}{d\kappa}c_\kappa|_{\kappa=0})_\Sigma$ has contraction bound $\mathfrak{c}''$. By Lemma XIII.6

$$\begin{aligned}
\mathfrak{c}'' &\le \mathrm{const}\max\left\{M^j\mathfrak{c}_j\frac{M^j|\frac{d}{d\kappa}u_\kappa|_{\kappa=0}|_{1,\Sigma}}{1-M^j|u_0|_{1,\Sigma}},\ M^{2j}\left|\frac{d}{d\kappa}u_\kappa\Big|_{\kappa=0}\right|_{1,\Sigma}\right\}\\
&\le \mathrm{const}\,M^j\mathfrak{c}_j\frac{M^j\mathfrak{e}_j(X)}{1-M^j\mu(\Lambda+X)\mathfrak{e}_j(X)}Y\\
&\le \mathrm{const}\,M^{2j}\mathfrak{c}_j\frac{\frac{1}{1-M^jX}}{1-M^j\mu(\Lambda+X)\frac{\mathfrak{c}_j}{1-M^jX}}Y\\
&\le \mathrm{const}\,M^{2j}\mathfrak{c}_jY\frac{\frac{1}{1-M^j\mathfrak{c}_j(\Lambda+X)}}{1-\mu\frac{M^j\mathfrak{c}_j(\Lambda+X)}{1-M^j\mathfrak{c}_j(\Lambda+X)}}\\
&= \mathrm{const}\,M^{2j}\mathfrak{c}_jYf(Z)
\end{aligned} \tag{XV.4}$$

where $Z = M^j\mathfrak{c}_j(\Lambda+X)$ and

$$f(z) = \frac{\frac{1}{1-z}}{1-\mu\frac{z}{1-z}} = \frac{1}{1-(1+\mu)z}.$$

By Lemma A.7, $f(Z) \le \mathit{const}\,\frac{1}{1-Z}$ so that

$$\mathfrak{c}'' \le \mathit{const}\,M^{2j}\mathfrak{e}_j(X)Y$$

as in Lemma XV.5. □

Lemma XV.9. *Let $g(\phi, \psi)$ be a sectorized Grassmann function and set*

$$g'_\kappa(\phi,\psi) = g(\phi, \psi + C_\kappa J\phi)\,.$$

Under the hypotheses of Theorem XV.7,

$$N_j\left(\frac{d}{d\kappa} g'_\kappa\bigg|_{\kappa=0}; \alpha;\ X, \Sigma, \boldsymbol{\rho}\right) \le \frac{\text{const}}{\alpha} M^j Y_{\mathbf{0}} N_j(g; 2\alpha; X, \Sigma, \boldsymbol{\rho})\,.$$

Proof. Define

$$\tilde{c}_z = c_0 + z\frac{d}{d\kappa}c_\kappa\bigg|_{\kappa=0}$$

and

$$\tilde{g}_z(\phi,\psi) = g\left(\phi, \psi + \left[C_0 + z\frac{d}{d\kappa}C_\kappa\right]_{\kappa=0} J\phi\right).$$

Let $\varphi \in \mathcal{F}_m(n;\Sigma)$, $1 \le i \le n$ and set

$$\begin{aligned}
&\varphi'_z(\eta_1, \ldots, \eta_{m+1}; (\xi_1, s_1), \ldots, (\xi_{n-1}, s_{n-1}))\\
&\quad = \text{Ant}_{\text{ext}} \sum_{s,t,t'\in\Sigma} \int d\zeta d\zeta' \varphi(\eta_1, \ldots, \eta_m; (\xi_1, s_1), \ldots, (\xi_{i-1}, s_{i-1})\,,\\
&\qquad (\zeta', t), (\xi_i, s_i), \ldots, (\xi_{n-1}, s_{n-1}))\tilde{c}_z((\zeta', t',), (\zeta, s))J(\zeta, \eta_{m+1})\,.
\end{aligned}$$

By Lemma XIII.6(i) and (XV.4)

$$\left|\frac{d}{d\kappa}c_\kappa\right|_{1,\Sigma}\Bigg|_{\substack{\kappa=0\\ t=0}} \le \text{const}\, M^{2j} Y_{\mathbf{0}}$$

so that, using the bound on $|c_0|_{1,\Sigma}$ that was derived in Lemma XV.5,

$$|\tilde{c}_z|_{1,\Sigma}|_{t=0} \le \text{const}\, M^j$$

for all $|z| \le \frac{1}{M^j Y_{\mathbf{0}}}$. Consequently, for all $|z| \le \frac{1}{M^j Y_{\mathbf{0}}}$, Lemma XII yields

$$|\varphi'_z|_{1,\Sigma} \le \text{const}\, |\varphi|_{1,\Sigma} \begin{cases} \dfrac{M^j}{\mathfrak{l}} & \text{if } m = 0\\[2ex] \dfrac{\mathfrak{l}}{M^j} & \text{if } m \ne 0 \end{cases}$$

so that, for $m = 0$,

$$\begin{aligned}
|\varphi'_z|_\Sigma &= \rho_{1;n-1}\frac{\mathfrak{l}}{M^{2j}}|\varphi'|_{1,\Sigma} \le \text{const}\, \rho_{1;n-1}\frac{1}{M^j}|\varphi|_{1,\Sigma}\\
&\le \text{const}\, \frac{\rho_{1;n-1}}{\rho_{0;n}}\frac{1}{M^j}|\varphi|_\Sigma \le \text{const}\, \sqrt{\mathfrak{l} M^j}\frac{1}{M^j}|\varphi|_\Sigma \le \text{const}\, \mathrm{b}|\varphi|_\Sigma
\end{aligned}$$

and, for $m \neq 0$,

$$|\varphi'_z|_\Sigma = \rho_{m+1;n-1}\frac{\mathfrak{l}}{M^{2j}}|\varphi'|_{1,\Sigma} \le \mathrm{const}\,\rho_{m+1;n-1}\frac{\mathfrak{l}^2}{M^{3j}}|\varphi|_{1,\Sigma}$$

$$\le \mathrm{const}\,\frac{\rho_{m+1;n-1}}{\rho_{m;n}}\frac{\mathfrak{l}}{M^j}|\varphi|_\Sigma \le \mathrm{const}\,\frac{\mathfrak{l}}{M^j}|\varphi|_\Sigma \le \mathrm{const}\,\mathrm{b}|\varphi|_\Sigma\,.$$

Hence, as in Lemma XV.6,

$$N_j(\tilde g_z - g;\alpha;X,\Sigma,\boldsymbol{\rho}) \le \frac{\mathrm{const}}{\alpha}N_j(g,2\alpha;X,\Sigma,\boldsymbol{\rho})$$

for all $|z| \le \frac{1}{M^j Y_{\mathbf{0}}}$ and, by the Cauchy integral theorem,

$$N_j\left(\frac{d}{d\kappa}g'_\kappa\bigg|_{\kappa=0};\alpha;X,\Sigma,\boldsymbol{\rho}\right) = N_j\left(\frac{d}{dz}[\tilde g_z - g]\bigg|_{z=0};\alpha;X,\Sigma,\boldsymbol{\rho}\right)$$

$$\le \frac{\mathrm{const}}{\alpha}M^j Y_{\mathbf{0}} N_j(g,2\alpha;X,\Sigma,\boldsymbol{\rho})\,. \qquad \square$$

Proof of Theorem XV.7. As in the proof of Theorem XV.3, let, for a sectorized Grassmann function $v = \sum_{m,n} v_{m,n}$ with $v_{m,n} \in A_m \otimes \bigwedge^n V_\Sigma$,

$$N(v;\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n}\alpha^n \mathrm{b}^n |v_{m,n}|_\Sigma = \frac{\mathrm{const}_1}{\mathrm{B}}N_j(v;\alpha;X,\Sigma,\boldsymbol{\rho})$$

and

$$:w''_\kappa:_{\psi,D_{\kappa,\Sigma}} = \Omega_{C_{\kappa,\Sigma}}(:w_\kappa:_{\psi,C_{\kappa,\Sigma}+D_{\kappa,\Sigma}})\,.$$

By Proposition XII, parts (ii) and (iii), and [1, Proposition A.2(ii)],

$$w'_\kappa = \frac{1}{2}\phi J C_\kappa J\phi + w''_\kappa(\phi,\psi + C_\kappa J\phi)$$

is a sectorized representative for $\mathcal{W}'_\kappa$. By the chain rule and the triangle inequality

$$N\left(\frac{d}{d\kappa}\left[w'_\kappa - \frac{1}{2}\phi J C_\kappa J\phi - w_\kappa\right]_{\kappa=0};\alpha\right)$$

$$\le N\left(\frac{d}{d\kappa}w''_0(\phi,\psi + C_\kappa J\phi)\bigg|_{\kappa=0};\alpha\right)$$

$$+ N\left(\frac{d}{d\kappa}[w''_\kappa(\phi,\psi + C_0 J\phi) - w''_\kappa(\phi,\psi)]_{\kappa=0};\alpha\right)$$

$$+ N\left(\frac{d}{d\kappa}[w''_\kappa(\phi,\psi) - w_\kappa(\phi,\psi)]_{\kappa=0};\alpha\right). \qquad \text{(XV.5)}$$

By Lemma XV.9,

$$N\left(\frac{d}{d\kappa}w''_0(\phi,\psi + C_\kappa J\phi)\bigg|_{\kappa=0};\alpha\right) \le \frac{\mathrm{const}}{\alpha}M^j Y_{\mathbf{0}} N_j(w''_0;2\alpha;X,\Sigma,\boldsymbol{\rho})\,.$$

By (XV.3),

$$\begin{aligned} N_j(w_0'';2\alpha;X,\Sigma,\boldsymbol{\rho}) &\le N_j(w_0;2\alpha;X,\Sigma,\boldsymbol{\rho}) + N_j(w_0''-w_0;2\alpha;X,\Sigma,\boldsymbol{\rho}) \\ &\le N_j(w_0;2\alpha;X,\Sigma,\boldsymbol{\rho}) + \frac{\mathrm{B}}{\mathrm{const}_1}\frac{1}{8\alpha^2}\frac{N(w_0;64\alpha)^2}{1-\frac{1}{4\alpha^2}N(w_0;64\alpha)} \\ &\le N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho}) + \frac{\mathrm{const}}{\alpha^2}\frac{N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho})^2}{1-\frac{\mathrm{const}}{\alpha^2}N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho})} \\ &\le \mathrm{const}\,\frac{N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho})}{1-\frac{\mathrm{const}}{\alpha^2}N_j(w_0;64\alpha;X,\Sigma,\boldsymbol{\rho})} \end{aligned}$$

so that

$$N\left(\frac{d}{d\kappa}w_0''(\phi,\psi+C_\kappa J\phi)\Big|_{\kappa=0};\alpha\right) \le \frac{\mathrm{const}}{\alpha}\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}M^jY_{\mathbf{0}}\,. \tag{XV.6}$$

By Lemma XV.6, with $g = \frac{d}{d\kappa}w_\kappa''|_{\kappa=0}$,

$$\begin{aligned} &N\left(\frac{d}{d\kappa}[w_\kappa''(\phi,\psi+C_0J\phi)-w_\kappa''(\phi,\psi)]_{\kappa=0};\alpha\right) \\ &\quad\le \frac{\mathrm{const}}{\alpha}N_j\left(\frac{d}{d\kappa}w_\kappa''\Big|_{\kappa=0};2\alpha;X,\Sigma,\boldsymbol{\rho}\right) \\ &\quad\le \frac{\mathrm{const}}{\alpha}N_j\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};2\alpha;X,\Sigma,\boldsymbol{\rho}\right) \\ &\qquad+\frac{\mathrm{const}}{\alpha}N_j\left(\frac{d}{d\kappa}[w_\kappa''-w_\kappa]_{\kappa=0};2\alpha;X,\Sigma,\boldsymbol{\rho}\right)\,. \end{aligned} \tag{XV.7}$$

By [1, Theorem IV.4], with $\mu = \frac{1}{M^j}$,

$$\begin{aligned} &N\left(\frac{d}{d\kappa}[w_\kappa''-w_\kappa]_{\kappa=0};\alpha\right) \\ &\quad\le \frac{1}{2\alpha^2}\frac{N(w_0;32\alpha)}{1-\frac{1}{\alpha^2}N(w_0;32\alpha)}N\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};8\alpha\right) \\ &\qquad+\frac{1}{2\alpha^2}\frac{N(w_0;32\alpha)^2}{1-\frac{1}{\alpha^2}N(w_0;32\alpha)}\left\{\frac{1}{4M^j}\,\mathrm{const}_2\,M^{2j}\mathfrak{e}_j(X)Y+4\varepsilon\right\} \\ &\quad\le \frac{\mathrm{const}}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\left\{N\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};8\alpha\right)+M^jY\mathfrak{n}+4\varepsilon\mathfrak{n}\right\} \end{aligned} \tag{XV.8}$$

since $\mathfrak{e}_j(X)N(w_0;32\alpha) \le \mathrm{const}\,N(w_0;32\alpha)$. Also

$$\begin{aligned} &N_j\left(\frac{d}{d\kappa}[w_\kappa''-w_\kappa]_{\kappa=0};2\alpha;X,\Sigma,\boldsymbol{\rho}\right) \\ &\quad\le \frac{\mathrm{const}}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\left\{N\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha\right)+M^jY\mathfrak{n}+4\varepsilon\mathfrak{n}\right\}. \end{aligned}$$

Substituting (XV.6)–(XV.8) into (XV.5),

$$N\left(\frac{d}{d\kappa}\left[w'_\kappa - \frac{1}{2}\phi JC_\kappa J\phi - w_\kappa\right]_{\kappa=0};\alpha\right)$$

$$\leq \frac{\text{const}}{\alpha}\frac{\mathfrak{n}}{1-\frac{\text{const}}{\alpha^2}\mathfrak{n}}M^jY_{\mathbf{0}} + \frac{\text{const}}{\alpha}N\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};2\alpha\right)$$

$$+\frac{\text{const}}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\text{const}}{\alpha^2}\mathfrak{n}}\left\{N\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha\right)+M^jY\mathfrak{n}+4\varepsilon\mathfrak{n}\right\}$$

$$\leq \text{const}\left\{\frac{1}{\alpha}+\frac{1}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\text{const}}{\alpha^2}\mathfrak{n}}\right\}N_j\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha;X,\Sigma,\boldsymbol{\rho}\right)$$

$$+\,\text{const}\,\frac{\mathfrak{n}}{1-\frac{\text{const}}{\alpha^2}\mathfrak{n}}\left\{\left(\frac{1}{\alpha}+\frac{\mathfrak{n}}{\alpha^2}\right)M^jY+\frac{\varepsilon}{\alpha^2}\mathfrak{n}\right\}. \qquad \square$$

We also must control the pure ϕ contributions in a situation similar to that of Theorem XV.3.

Proposition XV.10. *There are constants α_0 and τ_0 that are independent of j, Σ, $\boldsymbol{\rho}$ such that for all $\alpha \geq \alpha_0$ the following estimates hold*:

Let $u((\xi,s),(\xi',s')), v((\xi,s),(\xi',s')) \in \mathcal{F}_0(2;\Sigma)$ be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms obey $|\check{u}(k)|$, $|\check{v}(k)| \leq \frac{1}{2}|\imath k_0 - e(k)|$. Furthermore, let $X \in \mathfrak{N}_{d+1}$ and assume that $X, |u|_{1,\Sigma} \leq \frac{\tau_0}{M^j} + \sum_{\delta\neq 0}\infty t^\delta$. Let $\jmath$ be a real number in $(j+1,j+2]$ and set

$$S(k) = \frac{\nu^{(\geq j+1)}(k) - \nu^{(\geq \jmath)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)}\,, \quad D(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}(k)}$$

and let $S(\xi,\xi')$, $D(\xi,\xi')$ be the Fourier transforms of $S(k)$, $D(k)$ as in Definition IX.3. *Let $\mathcal{W}(\phi,\psi)$ be a Grassmann function obeying $\mathcal{W}(\phi,0)=0$ and set*

$$\mathcal{G}(\phi) = \tilde{\Omega}_S(:\mathcal{W}(\phi,\psi):_{\psi,D})(\phi,0)\,.$$

Assume that $\mathcal{W}$ has a sectorized representative w with

$$N_j(w;\alpha;X,\Sigma,\boldsymbol{\rho}) \leq 2 + \sum_{\delta\neq 0}\infty t^\delta\,.$$

Write

$$\mathcal{G}(\phi) - \frac{1}{2}\phi JSJ\phi = \sum_m \int d\eta_1\cdots d\eta_m G_m(\eta_1,\ldots,\eta_m)\phi(\eta_1)\cdots\phi(\eta_m)$$

with G_m antisymmetric. Then

$$\sum_{m>0}\rho_{m;0}|||G_m|||_\infty \leq 10\,.$$

Proof. We use the notation of the proof of Theorem XV.3. As in Lemma XV.5, $\mathfrak{c} = \mathrm{const}_1\, M^j + \sum_{\delta\neq 0} \infty t^\delta$ is a contraction bound for S_Σ and $\mathrm{b} = \sqrt{\frac{\mathrm{B}\,\mathfrak{l}}{M^j}}$ is an integral bound for both S_Σ and D_Σ (in the sense of [1, Definition II.25]). Write

$$:w(\phi,\psi):_{\psi,D_\Sigma} = :\tilde{w}(\phi,\psi):_{\psi,S_\Sigma}$$
$$w''(\phi,\psi) = \Omega_{S_\Sigma}(:w(\phi,\psi):_{\psi,D_\Sigma})\,.$$

By Proposition XII

$$\mathcal{G}(\phi) = \frac{1}{2}\phi JSJ\phi + w''(\phi, SJ\phi)\,.$$

By [1, Corollary II.32(ii)]

$$N_j\left(\tilde{w}; \frac{\alpha}{2}; X, \Sigma, \boldsymbol{\rho}\right) \le 2N_j(w;\alpha;X,\Sigma,\boldsymbol{\rho})\,.$$

By [1, Theorem II.28], with α replaced by $\frac{\alpha}{16}$,

$$\begin{aligned} N_j\left(w''(\phi,\psi); \frac{\alpha}{16}; X,\Sigma,\boldsymbol{\rho}\right) &\le N_j\left(\tilde{w}; \frac{\alpha}{16}; X,\Sigma,\boldsymbol{\rho}\right) + \frac{2^9}{\alpha^2}\frac{N_j(\tilde{w};\frac{1}{2}\alpha;X,\Sigma,\boldsymbol{\rho})^2}{1-\frac{2^{10}}{\alpha^2}N_j(\tilde{w};\frac{1}{2}\alpha;X,\Sigma,\boldsymbol{\rho})} \\ &\le 5 + \sum_{\delta\neq 0}\infty t^\delta\,. \end{aligned}$$

By Lemma XV.6

$$\begin{aligned} N_j\left(w''(\phi,\psi+SJ\phi); \frac{\alpha}{32}; X,\Sigma,\boldsymbol{\rho}\right) &\le 2N_j\left(w''(\phi,\psi); \frac{\alpha}{16}; X,\Sigma,\boldsymbol{\rho}\right) \\ &\le 10 + \sum_{\delta\neq 0}\infty t^\delta \end{aligned}$$

so that

$$\begin{aligned} \mathfrak{e}_j(X)\sum_{m>0}\rho_{m;0}|||G_m|||_\infty &= N_j\left(\mathcal{G}(\phi) - \frac{1}{2}\phi JSJ\phi; \frac{\alpha}{32}; X,\Sigma,\boldsymbol{\rho}\right) \\ &\le N_j\left(w''(\phi,\psi+SJ\phi); \frac{\alpha}{32}; X,\Sigma,\boldsymbol{\rho}\right) \\ &\le 10 + \sum_{\delta\neq 0}\infty t^\delta\,. \end{aligned}$$

□

Remark XV.11. In Theorem XV.3, the sectorized representative w' of $\mathcal{W}'$ may be obtained from the sectorized representative w of $\mathcal{W}$ by

$$:w':_{\psi,D_\Sigma} = \frac{1}{2}\phi JCJ\phi + \Omega_{C_\Sigma}(:w:_{\psi,C_\Sigma+D_\Sigma})(\phi,\psi+CJ\phi)\,.$$

Again, w' is initially defined as a formal Taylor series in w. By [1, Remark IV.3] and the observation that, as in Proposition IV.11(i), C_Σ and D_Σ are analytic functions of u and v, respectively, this formal Taylor series converges to a function that is jointly analytic in w, u and v. By the functoriality Proposition XII.8, if w_1 and w_2 are two sectorized representatives of $\mathcal{W}$, then the corresponding w_1' and w_2' represent the same unsectorized Grassmann function $\mathcal{W}'$. In this way one sees that the formal Taylor series for $\mathcal{W}'$ converges.

The obvious analogs of these statements apply to Theorem XV.7 and Proposition XV.10.

XVI. Sectorized Momentum Space Norms

Again, let Σ be a sectorization of length $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$ at scale $j \geq 2$. In Sec. XV we described the renormalization group map $\tilde{\Omega}_C$ using the algebra $\bigwedge_A V_\Sigma$, where V_Σ is the vector space generated by $\psi(\xi, s)$, $\xi \in \mathcal{B}$, $s \in \Sigma$ (see Definition XII.6) and A is the Grassmann algebra in the external fields $\phi(\eta)$, $\eta \in \mathcal{B}$. To deal with amputated Green's functions in momentum space, we set for $\check{\eta} = (k, \sigma, a) \in \check{\mathcal{B}}$

$$\check{\phi}(\check{\eta}) = \int d^{d+1}x e^{-(-1)^a \imath \langle k,x\rangle_-} \phi(x, \sigma, a)$$

and denote by $V_{\rm ext}$ the vector space generated by $\check{\phi}(\check{\eta})$, $\check{\eta} \in \check{\mathcal{B}}$. Furthermore set

$$\tilde{V} = V_{\rm ext} \oplus V_\Sigma \,.$$

Then $\bigwedge_A V_\Sigma$ is canonically isomorphic to the Grassmann algebra $\bigwedge \tilde{V}$ over $\tilde{V}$ with complex coefficients. In terms of Grassmann functions, this isomorphism amounts to the following: a translation invariant sectorized Grassmann function w can be uniquely written in the form

$$\begin{aligned} w(\phi, \psi) = \sum_{m,n} \sum_{s_1,\ldots,s_n \in \Sigma} \int & d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n \\ & \cdot w_{m,n}(\eta_1, \ldots, \eta_m (\xi_1, s_1), \ldots, (\xi_n, s_n)) \\ & \cdot \phi(\eta_1) \cdots \phi(\eta_m) \psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n)) \end{aligned}$$

with $w_{m,n}$ antisymmetric separately in the η and in the ξ variables. As well,

$$\begin{aligned} w(\phi, \psi) = & \sum_m \int d\check{\eta}_1 \cdots d\check{\eta}_m w^{\sim}_{m,0}(\check{\eta}_1, \ldots, \check{\eta}_m)(2\pi)^{d+1} \delta(\check{\eta}_1 + \cdots + \check{\eta}_m) \check{\phi}(\check{\eta}_1) \cdots \check{\phi}(\check{\eta}_m) \\ & + \sum_{\substack{m,n \\ n \geq 1}} \sum_{s_1,\ldots,s_n \in \Sigma} \int d\check{\eta}_1 \cdots d\check{\eta}_m d\xi_1 \cdots d\xi_n w^{\sim}_{m,n} \\ & \cdot (\check{\eta}_1, \ldots, \check{\eta}_m (\xi_1, s_1), \ldots, (\xi_n, s_n)) \check{\phi}(\check{\eta}_1) \cdots \check{\phi}(\check{\eta}_m) \psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n)) \,. \end{aligned}$$

Here $w^{\sim}_{m,n}$ is the partial Fourier transform of Definition IX.1.

The basis elements of the vector space $\tilde{V} = V_{\rm ext} \oplus V_\Sigma$ are in one-to-one correspondence with the points of the disjoint union $\mathfrak{X}_\Sigma$ of $\check{\mathcal{B}}$ and $\mathcal{B} \times \Sigma$. To simplify notation, we make the

Definition XVI.1. For $x \in \mathfrak{X}_\Sigma = \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$ set

$$\Psi(x) = \begin{cases} \check{\phi}(\check{\eta}) & \text{if } x = \check{\eta} \in \check{B} \\ \psi(\xi, s) & \text{if } x = (\xi, s) \in \mathcal{B} \times \Sigma \end{cases} .$$

The purpose of this section is to define and analyze norms on functions on $\mathfrak{X}_\Sigma^n$ to which the results of [1, 3] can be applied. First, we look at the structure of $\mathfrak{X}_\Sigma^n$ more carefully.

Definition XVI.2. Set $\mathfrak{X}_0 = \check{\mathcal{B}}$ and $\mathfrak{X}_1 = \mathcal{B} \times \Sigma$. Let $\boldsymbol{\imath} = (i_1, \ldots, i_n) \in \{0, 1\}^n$.

(i) The inclusions of $\mathfrak{X}_{i_j}$, $j = 1, \ldots, n$, in $\mathfrak{X}_\Sigma$ induce an inclusion of $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$ in $\mathfrak{X}_\Sigma^n$. We identify $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$ with its image in $\mathfrak{X}_\Sigma^n$.

(ii) Set $m(\boldsymbol{\imath}) = n - (i_1 + \cdots + i_n)$. Clearly, $m(\boldsymbol{\imath})$ is the number of copies of $\check{\mathcal{B}}$ in $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$.

(iii) If f is a function on $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$, then $\operatorname{Ord} f$ is the function on $\check{\mathcal{B}}^{m(\boldsymbol{\imath})} \times (\mathcal{B} \times \Sigma)^{n-m(\boldsymbol{\imath})}$ obtained from f by shifting all of the $\check{B}$ arguments before all of the $\mathcal{B} \times \Sigma$ arguments, while preserving the relative order of the $\check{B}$ arguments and the relative order of the $\mathcal{B} \times \Sigma$ arguments and multiplying by the sign of the permutation that implements the reordering of the arguments. That is, $\operatorname{Ord} f(x_1, \ldots, x_n) = \operatorname{sgn} \pi f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ where the permutation $\pi \in S_n$ is determined by $\pi(j) < \pi(j')$ if $i_j < i_{j'}$ or $i_j = i_{j'}$ $j < j'$.

Remark XVI.3. Using the identification of Definition XVI.2(i),

$$\mathfrak{X}_\Sigma^n = \dot{\bigcup_{i_1, \ldots, i_n \in \{0,1\}}} \mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$$

where, on the right-hand side we have a disjoint union. If f is a function on $\mathfrak{X}_\Sigma^n$ and $\boldsymbol{\imath} = (i_1, \ldots, i_n) \in \{0, 1\}^n$, we denote by $f|_{\boldsymbol{\imath}}$ the restriction of f to $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$.

To define norms for functions on $\mathfrak{X}_\Sigma^n$ it thus suffices to define norms for functions on each of the spaces $\mathfrak{X}_{i_1} \times \cdots \times \mathfrak{X}_{i_n}$. As we want these norms to be invariant under permutations, it suffices, using the map Ord of Definition XVI.2(iii), to define norms for functions on the spaces $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^{n-m}$.

Definition XVI.4. Let p be a natural number.

(i) For a function f on $\check{\mathcal{B}}_m$ we define

$$|f|^{\tilde{}}_{p,\Sigma} = \begin{cases} \|f\|^{\tilde{}} & \text{if } p = m - 1,\ m = 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

with $\|\cdot\|^{\tilde{}}$ being the norm of Definition X.4.

(ii) For a translation invariant function f on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^n$ with $n \geq 1$, we set $|f\widetilde{|}_{p,\Sigma} = 0$ when $p > m+n$ or $p < m$, and

$$|f\widetilde{|}_{p,\Sigma} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \sup_{\substack{1 \leq i_1 < \cdots < i_{p-m} \leq n \\ s_{i_1}, \ldots, s_{i_{p-m}} \in \Sigma \\ \check{\eta}_1, \ldots, \check{\eta}_m \in \check{\mathcal{B}}}} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \neq i_1, \ldots, i_{p-m}}} \frac{1}{\delta!} \max_{\substack{\mathrm{D} \text{ dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \cdot |||\mathrm{D} f(\check{\eta}_1, \ldots, \check{\eta}_m; (\xi_1, s_1), \ldots, (\xi_n, s_n))|||_{1,\infty} t^{\delta}$$

when $m \leq p \leq m+n$. The norm $|||\cdot|||_{1,\infty}$ of Example II.6 refers to the variables $\xi_1, \ldots, \xi_n$.

Remark XVI.5. In the case $m = 0$ and p odd, the norm $|\cdot|_{p,\Sigma}$ of Definition XII and the norm $|\cdot\widetilde{|}_{p,\Sigma}$ of Definition XVI.4 agree.

Lemma XVI.6. *Let f be a translation invariant function on $\check{\mathcal{B}}^m \times (\mathcal{B} \times \Sigma)^n$, f' a translation invariant function on $\check{\mathcal{B}}^{m'} \times (\mathcal{B} \times \Sigma)^{n'}$ and $1 \leq i \leq n$, $1 \leq i' \leq n'$.*

If $n \geq 2$ or $n' \geq 2$ define the function g on $\check{\mathcal{B}}^{m+m'} \times (\mathcal{B} \times \Sigma)^{n+n'-2}$ by

$$\begin{aligned} g(&\check{\eta}_1, \ldots, \check{\eta}_{m+m'}; (\xi_1, s_1), \ldots, (\xi_{i-1}, s_{i-1}), (\xi_{i+1}, s_{i+1}), \ldots, \\ &(\xi_{n+i'-1}, s_{n+i'-1}), (\xi_{n+i'+1}, s_{n+i'+1}), \ldots, (\xi_{n+n'}, s_{n+n'})) \\ = &\sum_{\substack{s, s' \in \Sigma \\ s \cap s' \neq \emptyset}} \int_{\mathcal{B}} d\zeta\, f(\check{\eta}_1, \ldots, \check{\eta}_m; (\xi_1, s_1), \ldots, (\xi_{i-1}, s_{i-1}), (\zeta, s), \\ &(\xi_{i+1}, s_{i+1}), \ldots, (\xi_n, s_n)) f'(\check{\eta}_{m+1}, \ldots, \check{\eta}_{m+m'}; (\xi_{n+1}, s_{n+1}), \ldots, \\ &(\xi_{n+i'-1}, s_{n+i'-1}), (\zeta, s'), (\xi_{n+i'+1}, s_{n+i'+1}), \ldots, (\xi_{n+n'}, s_{n+n'}))\,. \end{aligned}$$

If $n = n' = 1$, define the function g on $\check{\mathcal{B}}_{m+m'}$ by

$$\begin{aligned} g(\check{\eta}_1, \ldots, \check{\eta}_{m+m'})&(2\pi)^{d+1} \delta(\check{\eta}_1 + \cdots + \check{\eta}_{m+m'}) \\ &= \sum_{\substack{s, s' \in \Sigma \\ s \cap s' \neq \emptyset}} \int_{\mathcal{B}} d\zeta\, f(\check{\eta}_1, \ldots, \check{\eta}_m; (\zeta, s)) f'(\check{\eta}_{m+1}, \ldots, \check{\eta}_{m+m'}; (\zeta, s'))\,. \end{aligned}$$

Then, for all natural numbers p,

$$|g\widetilde{|}_{p,\Sigma} \leq \begin{cases} 3 \max\limits_{\substack{p_1 + p_2 = p \\ m \leq p_1 < m+n \\ m' \leq p_2 < m'+n'}} \min\{|f\widetilde{|}_{p_1,\Sigma} |f'\widetilde{|}_{p_2+1,\Sigma}, |f\widetilde{|}_{p_1+1,\Sigma} |f'\widetilde{|}_{p_2,\Sigma}\} & \textit{if } (n, n') \neq (1,1) \\ 4 |f\widetilde{|}_{m,\Sigma} |f'\widetilde{|}_{m',\Sigma} & \textit{if } (n, n') = (1,1) \end{cases}.$$

Proof. *The case $n \ge 2$ or $n' \ge 2$.* The proof is analogous to that of Lemma XII.14. The $\mathcal{B} \times \Sigma$ indices for g lie in the set $I \cup I'$, where

$$I = \{1, \ldots, i-1,\ i+1, \ldots, n\}$$

$$I' = \{n+1, \ldots, n+i'-1,\ n+i'+1, \ldots, n+n'\}\,.$$

Let q obey $0 \le q-m \le n-1$ and $0 \le p-q-m' \le n'-1$ or equivalently $m \le q < n+m$ and $m' \le p-q < m'+n'$. Fix $u_1, \ldots, u_{q-m} \in I$, $u_{q-m+1}, \ldots, u_{p-m-m'} \in I'$ and fix sectors $s_{u_1}, \ldots, s_{u_{p-m-m'}} \in \Sigma$. Let

$$F(\check\eta_1, \ldots, \check\eta_m; s_1, \ldots, s_n) = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} |||\mathrm{D} f(\check\eta_1, \ldots, \check\eta_m; (\cdot, s_1), \ldots, (\cdot, s_n))|||_{1,\infty} t^\delta$$

so that

$$|f|^{\sim}_{p,\Sigma} = \sup_{\substack{1 \le i_1 < \cdots < i_{p-m} \le n \\ s_{i_1}, \ldots, s_{i_{p-m}} \in \Sigma \\ \check\eta_1, \ldots, \check\eta_m \in \check{\mathcal{B}}}} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \ne i_1, \ldots, i_{p-m}}} F(\check\eta_1, \ldots, \check\eta_m; s_1, \ldots, s_n)$$

and define $G(\check\eta_1, \ldots, \check\eta_{m+m'};\ s_1, \ldots, \not{s}_i, \ldots, \not{s}_{n+i'}, \ldots, s_{n+n'})$ and $F'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; s_{n+1}, \ldots, s_{n+n'})$ similarly. By Remark X.7, for each choice of sectors s_ν, $\nu \in I \cup I'$, one has

$$\begin{aligned} &G(\check\eta_1, \ldots, \check\eta_{m+m'}; s_1, \ldots, \not{s}_i, \ldots, \not{s}_{n+i'}, \ldots, s_{n+n'}) \\ &\quad \le \sum_{\substack{s, s' \in \Sigma \\ s \cap s' \ne \emptyset}} F(\check\eta_1, \ldots, \check\eta_m; s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \\ &\qquad \cdot F'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; s_{n+1}, \ldots, s_{n+i'-1}, s', \ldots, s_{n+n'})\,. \end{aligned}$$

Observe that for every $s \in \Sigma$ there are at most three sectors s' such that $s' \cap s \ne \emptyset$. Consequently

$$\begin{aligned} &\sum_{\substack{s_\nu \in \Sigma \\ \nu \in I \cup I' \setminus \{u_1, \ldots, u_{p-m-m'}\}}} G(\check\eta_1, \ldots, \check\eta_{m+m'}; s_1, \ldots, \not{s}_i, \ldots, \not{s}_{n+i'}, \ldots, s_{n+n'}) \\ &\quad \le 3 \sum_{\substack{s_\nu \in \Sigma \\ \nu \in I \setminus \{u_1, \ldots, u_{q-m}\}}} \sum_{s \in \Sigma} F(\check\eta_1, \ldots, \check\eta_m; s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \\ &\qquad \cdot \max_{s' \in \Sigma} \sum_{\substack{s'_\mu \in \Sigma \\ \mu \in I' \setminus \{u_{q-m+1}, \ldots, u_{p-m-m'}\}}} \\ &\qquad \cdot F'(\check\eta_{m+1}, \ldots, \check\eta_{m+m'}; s_{n+1}, \ldots, s_{n+i'-1}, s', \ldots, s_{n+n'}) \\ &\quad \le 3 |f|^{\sim}_{q,\Sigma} |f'|^{\sim}_{p-q+1,\Sigma}\,. \end{aligned}$$

Taking the supremum over the $\check\eta$'s and the remaining s_ν's gives

$$|g\tilde{|}_{p,\Sigma} \le 3|f\tilde{|}_{q,\Sigma}|f'\tilde{|}_{p-q+1,\Sigma}\,.$$

By interchanging the roles of (f, q) and $(f', p - q)$, we get the bound $3|f\tilde{|}_{q+1,\Sigma}$ $|f'\tilde{|}_{p-q,\Sigma}$.

The case $n = n' = 1$. In this case, the norm $|g\tilde{|}_{p,\Sigma}$ is defined in Definition XVI.4(i). We need only consider the case $p = m + m' - 1$.

By Remark X.3(iii)

$$\sum_{\substack{s,s'\in\Sigma\\ s\cap s'\neq\emptyset}} \int_{\mathcal B} d\xi\, f(\check\eta_1,\ldots,\check\eta_m;(\xi,s))f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(\xi,s'))$$

$$= \int dx_0 \int d\mathbf{x} \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} \sum_{s,s'\in\Sigma} f(\check\eta_1,\ldots,\check\eta_m;(0,\sigma,b,s))$$

$$\cdot\, e^{\imath\langle\check\eta_1+\cdots+\check\eta_{m+m'},(x_0,\mathbf{x})\rangle_-} f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b,s'))$$

$$= \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} \sum_{s,s'\in\Sigma} f(\check\eta_1,\ldots,\check\eta_m;(0,\sigma,b,s))$$

$$\cdot\, f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b,s'))(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_{m+m'})\,.$$

Consequently

$$g(\check\eta_1,\ldots,\check\eta_{m+m'})$$

$$= \sum_{\substack{\sigma\in\{\uparrow,\downarrow\}\\ b\in\{0,1\}}} \sum_{s,s'\in\Sigma} f(\check\eta_1,\ldots,\check\eta_m;(0,\sigma,b,s))f'(\check\eta_{m+1},\ldots,\check\eta_{m+m'};(0,\sigma,b,s'))\,.$$

The claim now follows, as in Lemma II.7, by iterated application of the product rule for derivatives and Remark X.3(iii). □

Definition XVI.7. Let $m, n \ge 0$.

(i) For $n \ge 1$, denote by $\check{\mathcal F}_m(n;\Sigma)$ the space of all translation invariant, complex valued functions $f(\check\eta_1,\ldots,\check\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))$ on $\check{\mathcal B}^m\times(\mathcal B\times\Sigma)^n$ whose Fourier transform $\check f(\check\eta_1,\ldots,\check\eta_m;(\check\xi_1,s_1),\ldots,(\check\xi_n,s_n))$ vanishes unless $k_i\in\tilde s_i$ for all $1\le j\le n$. Here, $\check\xi_i=(k_i,\sigma_i,a_i)$. Also, let $\check{\mathcal F}_m(0;\Sigma)$ be the space of all momentum conserving, complex valued functions $f(\check\eta_1,\ldots,\check\eta_m)$ on $\check{\mathcal B}^m$.

(ii) Let $c((\xi,s),(\xi',s'))$ be any skew symmetric function on $(\mathcal B\times\Sigma)^2$. Let $f\in\check{\mathcal F}_m(n;\Sigma)$ and $1\le i<j\le n$. We define "contraction", for $n\ge 2$, by

$$\mathop{\mathcal{C}\mathrm{on}_c}_{i\to j} f(\check\eta_1,\ldots,\check\eta_m;(\xi_1,s_1),\ldots,(\xi_{i-1},s_{i-1})\,,$$

$$(\xi_{i+1},s_{i+1}),\ldots,(\xi_{j-1},s_{j-1}),(\xi_{j+1},s_{j+1}),\ldots,(\xi_n,s_n))$$

$$= (-1)^{j-i+1} \sum_{\substack{s_i,s_j,t_i,t_j\in\Sigma \\ t_i\cap s_i\neq\emptyset \\ t_j\cap s_j\neq\emptyset}} \int d\xi_i d\xi_j c((\xi_i,t_i),(\xi_j,t_j))$$

$$\cdot f(\check{\eta}_1,\ldots,\check{\eta}_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))$$

and, for $n=2$, by

$$\underset{1\to 2}{\mathcal{C}\mathrm{on}_c}\, f(\check{\eta}_1,\ldots,\check{\eta}_m)(2\pi)^{d+1}\delta(\check{\eta}_1+\cdots+\check{\eta}_m)$$

$$= \sum_{\substack{s_1,s_2,t_1,t_2\in\Sigma \\ t_1\cap s_1\neq\emptyset \\ t_2\cap s_2\neq\emptyset}} \int d\xi_1 d\xi_2 c((\xi_1,t_1),(\xi_2,t_2)) f(\check{\eta}_1,\ldots,\check{\eta}_m;(\xi_1,s_1),(\xi_2,s_2))\,.$$

(iii) We denote by $\check{\mathcal{F}}_{n;\Sigma}$ the set of functions on $\mathfrak{X}_\Sigma^n$ with the property that for each $\boldsymbol{\imath}=(i_1,\ldots,i_n)\in\{0,1\}^n$ with $m(\boldsymbol{\imath})<n$

$$\mathrm{Ord}(f|_{\boldsymbol{\imath}})\in\check{\mathcal{F}}_{m(\boldsymbol{\imath})}(n-m(\boldsymbol{\imath});\Sigma)$$

and there is a function g on $\check{\mathcal{B}}_n$ such that

$$f|_{(0,\ldots,0)}(\check{\eta}_1,\ldots,\check{\eta}_n)=(2\pi)^3\delta(\check{\eta}_1+\cdots+\check{\eta}_n)g(\check{\eta}_1,\ldots,\check{\eta}_n)\,.$$

The map Ord was introduced in Definition XVI.2(iii) and the restriction $f|_{\boldsymbol{\imath}}$ was introduced in Remark XVI.3.

The partial Fourier transforms $\varphi^\sim$ (as in Definition IX.1(ii)) of functions $\varphi\in\mathcal{F}_m(n;\Sigma)$ as in Definition XII(ii) are the functions in $\check{\mathcal{F}}_m(n;\Sigma)$ that are antisymmetric in their external variables. Also, $\underset{i\to j}{\mathcal{C}\mathrm{on}_c}\,\varphi^\sim=(\underset{i\to j}{\mathcal{C}\mathrm{on}_c}\,\varphi)^\sim$, where $\underset{i\to j}{\mathcal{C}\mathrm{on}_c}\,\varphi$ is defined in Definition XII.15.

Proposition XVI.8. *Let* $c((\xi,s),\ (\xi',s'))\in\mathcal{F}_0(2;\Sigma)$ *be an antisymmetric function.*

(i) *Let* p *be a natural number,* $m,m'\geq 0$, $n,n'\geq 1$ *and* $f\in\check{\mathcal{F}}_m(n;\Sigma)$, $f'\in\check{\mathcal{F}}_{m'}(n',\Sigma)$. *If* $(n,n')\neq(1,1)$ *then*

$$|\underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\,\mathrm{Ant}_{\mathrm{ext}}(f\otimes f')|\tilde{}_{p,\Sigma}$$

$$\leq 9|c|_{1,\Sigma}\max_{\substack{p_1+p_2=p \\ m\leq p_1<m+n \\ m'\leq p_2<m'+n'}}\min\{|f|\tilde{}_{p_1+1,\Sigma}|f'|\tilde{}_{p_2,\Sigma},|f|\tilde{}_{p_1,\Sigma}|f'|\tilde{}_{p_2+1,\Sigma}\}$$

and if $(n,n')=(1,1)$ *then*

$$|\underset{1\to n+1}{\mathcal{C}\mathrm{on}_c}\,\mathrm{Ant}_{\mathrm{ext}}(f\otimes f')|\tilde{}_{m+m'-1,\Sigma}\leq 12|c|_{1,\Sigma}|f|\tilde{}_{m,\Sigma}|f'|\tilde{}_{m',\Sigma}\,.$$

(ii) *Assume that there is a function $C(k)$ that is supported in the jth neighborhood, such that $c((\cdot,s),(\cdot,s'))$ is the Fourier transform of $\chi_s(k)C(k)\chi_{s'}(k)$ in the sense of Definition* IX.3 *and that $|C(k)| \le \frac{\varepsilon}{|\imath k_0 - e(\mathbf{k})|}$ for some $\varepsilon \ge 0$.*

Let $f \in \check{\mathcal{F}}_m(n;\Sigma)$, $n' \le n$ and set, when $n' < n$, as in Definition III.5,

$$f'(\check\eta_1,\ldots,\check\eta_m;(\xi_{n'+1},s_{n'+1}),\ldots,(\xi_n,s_n))$$
$$= \sum_{\substack{s_i\in\Sigma\\ i=1,\ldots,n'}} \iint d\xi_1\cdots d\xi_{n'} f(\check\eta_1,\ldots,\check\eta_m;$$
$$(\xi_1,s_1),\ldots,(\xi_{n'},s_{n'}),\ldots,(\xi_n,s_n))\psi(\xi_1,s_1)\cdots\psi(\xi_{n'},s_{n'})d\mu_{C_\Sigma}(\psi)$$

where

$$C_\Sigma(\psi(\xi,s),\psi(\xi',s')) = \sum_{\substack{t\cap s\ne\emptyset\\ t'\cap s'\ne\emptyset}} c((\xi,t),(\xi',t'))\,.$$

For $n' = n$, set

$$f'(\check\eta_1,\ldots,\check\eta_m)(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_m)$$
$$= \sum_{\substack{s_i\in\Sigma\\ i=1,\ldots,n}} \iint d\xi_1\cdots d\xi_n f(\check\eta_1,\ldots,\check\eta_m;(\xi_1,s_1),\ldots,(\xi_n,s_n))$$
$$\cdot\,\psi(\xi_1,s_1)\cdots\psi(\xi_n,s_n)d\mu_{C_\Sigma}(\psi)\,.$$

Then, for all natural numbers p,

$$|f'|\tilde{}_{p,\Sigma} \le \left(\varepsilon \mathrm{B}_3\frac{\mathfrak{l}}{M^j}\right)^{n'/2}\begin{cases} |f|\tilde{}_{p,\Sigma} & \text{if } n\ne n'\\ |f|\tilde{}_{p+1,\Sigma} & \text{if } n=n'\end{cases}$$

with a constant B_3 *that is independent of j and Σ.*

(iii) *Let $D(k)$, $D'(k)$ be functions obeying $|D(k)|$, $|D'(k)| \le \frac{2}{|\imath k_0 - e(\mathbf{k})|}$ and let $d((\cdot,s),(\cdot,s'))$ resp. $d'((\cdot,s),(\cdot,s'))$ be the Fourier transform of $\chi_s(k)D(k)$ $\chi_{s'}(k)$ resp. $\chi_s(k)D'(k)\chi_{s'}(k)$ in the sense of Definition* IX.3.

Let $1 \le i_1,i_2,i_3 \le n$, $1 \le i_1',i_2',i_3' \le n'$ with $i_1\ne i_2\ne i_3\ne i_1$, $i_1'\ne i_2'\ne i_3'\ne i_1'$, and let $p \ge 1$. Then there is a constant B_4 *that is independent of j and Σ such that for all $f \in \check{\mathcal{F}}_m(n;\Sigma)$, $f' \in \check{\mathcal{F}}_{m'}(n',\Sigma)$*

$$\Big|\underset{i_1\to n+i_1'}{\mathcal{C}\mathrm{on}_c}\ \underset{i_2\to n+i_2'}{\mathcal{C}\mathrm{on}_d}\ \underset{i_3\to n+i_3'}{\mathcal{C}\mathrm{on}_{d'}}(f\otimes f')\Big|\tilde{}_{p,\Sigma}$$
$$\le \left(\mathrm{B}_4\frac{\mathfrak{l}}{M^j}\right)^2 |c|_{1,\Sigma} \max_{\substack{p_1+p_2=p\\ m\le p_1<m+n\\ m'\le p_2<m'+n'}} \min\{|f|\tilde{}_{p_1+3,\Sigma}|f'|\tilde{}_{p_2,\Sigma},|f|\tilde{}_{p_1,\Sigma}|f'|\tilde{}_{p_2+3,\Sigma}\}$$

if $(n, n') \neq (3,3)$ *and*

$$
\Big| \operatorname*{\mathcal{C}on_c}_{i_1 \to n+i_1'} \operatorname*{\mathcal{C}on_d}_{i_2 \to n+i_2'} \operatorname*{\mathcal{C}on_{d'}}_{i_3 \to n+i_3'} (f \otimes f') \widetilde{|}_{m+m'-1,\Sigma}
$$

$$
\leq \left(\mathrm{B}_4 \frac{\mathfrak{l}}{M^j} \right)^2 |c|_{1,\Sigma} \min\{ |f\widetilde{|}_{m+2,\Sigma} |f'\widetilde{|}_{m',\Sigma}, |f\widetilde{|}_{m,\Sigma} |f'\widetilde{|}_{m'+2,\Sigma} \}
$$

if $(n, n') = (3,3)$.

Proof. The proofs of part (i) and part (ii), except for the case $n = n'$, is similar to that of parts (i) and (ii) of Proposition XII.16. The proof of part (iii) is similar to that of Proposition XII.18. So we only give the proof of part (ii) for the case $n = n'$.

By translation invariance,

$$
f'(\check{\eta}_1, \dots, \check{\eta}_m) = \sum_{\substack{s_i \in \Sigma \\ i=1,\dots,n}} \iint d\xi_1 \cdots d\xi_n \delta(x_{n,0}) \delta(\mathbf{x}_n) f(\check{\eta}_1, \dots, \check{\eta}_m;
$$

$$
(\xi_1, s_1), \dots, (\xi_n, s_n)) \psi(\xi_1, s_1) \cdots \psi(\xi_n, s_n) d\mu_{C_\Sigma}(\psi)
$$

where $\xi_n = (x_{n,0}, \mathbf{x}_n, \sigma_n, a_n)$. By Proposition IV.3(ii), with

$$
G = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \tilde{\chi}_s(k)^2 |C(k)| \leq \varepsilon \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{\tilde{\chi}_s(k)^2}{|\imath k_0 - e(\mathbf{k})|} \leq \mathrm{const}\, \varepsilon \frac{\mathfrak{l}}{M^j}
$$

we have, for any dd-operator D,

$$
|\mathrm{D}f'(\check{\eta}_1, \dots, \check{\eta}_m)| \leq 4G^{n/2} \sum_{\substack{s_i \in \Sigma \\ i=1,\dots,n}} |||\mathrm{D}f(\check{\eta}_1, \dots, \check{\eta}_m; (\cdot, s_1), \dots, (\cdot, s_n))|||_{1,\infty} .
$$

Hence

$$
|f\widetilde{|}_{m-1,\Sigma} \leq 4G^{n/2} |f\widetilde{|}_{m,\Sigma} . \qquad \square
$$

In Theorem XV.3, ladders played a special role. Due to the "external improvement" of Lemma XII, we needed to consider only ladders all of whose "ends" correspond to ψ fields and are integrated out at a later scale. This is not the case when we use the norms developed in this chapter. We consider ladders some of whose "ends" correspond to ψ fields and have sectorized position space variables $(\xi, s) \in \mathcal{B} \times \Sigma$, and some of whose ends correspond to ϕ fields and have momentum space variables $\check{\eta} \in \check{\mathcal{B}}$. To do this, we extend the definitions and estimates of ladders from Sec. XIV.

Definition XVI.9. (i) Let C be a propagator over $\mathcal{B}$. We define its extension $\tilde{C}$ over the disjoint union $\check{\mathcal{B}} \cup \mathcal{B}$ by

$$
\tilde{C}(x, y) = \begin{cases} C(x, y) & \text{if } x, y \in \mathcal{B} \\ 0 & \text{if } x \in \check{\mathcal{B}} \text{ or } y \in \check{\mathcal{B}} \end{cases} .
$$

(ii) Let C, D be propagators over $\mathcal{B}$ and R a rung over $\check{\mathcal{B}} \cup \mathcal{B}$. We set

$$L_\ell(R; C, D) = L_\ell(R; \tilde{C}, \tilde{D}) .$$

(iii) Let P be a bubble propagator over $\mathcal{B}$, r a rung over $\mathfrak{X}_\Sigma = \check{\mathcal{B}} \cup (\mathcal{B} \times \Sigma)$. We set

$$(r \bullet P)(y_1, y_2; x_3, x_4)$$
$$= \sum_{s_1', s_2' \in \Sigma} \int_{\mathcal{B} \times \mathcal{B}} dx_1' dx_2' r(y_1, y_2, (x_1', s_1'), (x_2', s_2')) P(x_1', x_2'; x_3, x_4) .$$

$(r \bullet P)$ is a function on $\mathfrak{X}_\Sigma^2 \times \mathcal{B}^2$. For a general function F on $\mathfrak{X}_\Sigma^2 \times \mathcal{B}^2$, define the rung $(F \bullet r)$ over $\mathfrak{X}_\Sigma$ by

$$(F \bullet r)(y_1, y_2, y_3, y_4)$$
$$= \sum_{s_1', s_2' \in \Sigma} \int_{\mathcal{B} \times \mathcal{B}} dx_1' dx_2' F(y_1, y_2; x_1', x_2') r((x_1', s_1'), (x_2', s_2'), y_3, y_4)$$

if at least one of the arguments $y_1, \ldots, y_4$ lies in $\mathcal{B} \times \Sigma \subset \mathfrak{X}_\Sigma$, and for $\check{\eta}_1, \check{\eta}_2, \check{\eta}_3, \check{\eta}_4 \in \check{\mathcal{B}} \subset \mathfrak{X}_\Sigma$

$$(F \bullet r)(\check{\eta}_1, \check{\eta}_2, \check{\eta}_3, \check{\eta}_4)(2\pi)^{d+1} \delta(\check{\eta}_1 + \check{\eta}_2 + \check{\eta}_3 + \check{\eta}_4)$$
$$= \sum_{s_1', s_2' \in \Sigma} \int_{\mathcal{B} \times \mathcal{B}} dx_1' dx_2' F(\check{\eta}_1, \check{\eta}_2; x_1', x_2') r((x_1', s_1'), (x_2', s_2'), \check{\eta}_3, \check{\eta}_4) .$$

(iv) Let $\ell \geq 1$, $r_1, \ldots, r_{\ell+1}$ rungs over $\mathfrak{X}_\Sigma$ and $P_1, \ldots, P_\ell$ bubble propagators over $\mathcal{B}$. The ladder with rungs $r_1, \ldots, r_{\ell+1}$ and bubble propagators $P_1, \ldots, P_\ell$ is defined to be

$$r_1 \bullet P_1 \bullet r_2 \bullet P_2 \bullet \cdots \bullet r_\ell \bullet P_\ell \bullet r_{\ell+1} .$$

If r is a rung over $\mathfrak{X}_\Sigma$ and A, B are propagators over $\mathcal{B}$, we define $L_\ell(r; A, B)$ as the ladder with $\ell + 1$ rungs r and ℓ bubble propagators $\mathcal{C}(A, B)$.

Remark XVI.10. In the situation of Definition XVI.9(ii), let R' be the restriction of R to $\mathcal{B}^4$, $R_{\rm left}$ the restriction of R to $(\check{\mathcal{B}} \cup \mathcal{B})^2 \times \mathcal{B}^2$ and $R_{\rm right}$ the restriction of R to $\mathcal{B}^2 \times (\check{\mathcal{B}} \cup \mathcal{B})^2$. Then

$$L_\ell(R; C, D) = R_{\rm left} \circ \mathcal{C}(C, D) \circ R' \circ \cdots \circ R' \circ \mathcal{C}(C, D) \circ R_{\rm right} .$$

Similarly, in the situation of Definition XVI.9(iv), let r' be the restriction of r to $(\mathcal{B} \times \Sigma)^4$, $r_{\rm left}$ the restriction of r to $\mathfrak{X}_\Sigma^2 \times (\mathcal{B} \times \Sigma)^2$ and $r_{\rm right}$ the restriction of r to $(\mathcal{B} \times \Sigma)^2 \times \mathfrak{X}_\Sigma^2$. Then

$$L_\ell(r; C, D) = r_{\rm left} \bullet \mathcal{C}(C, D) \bullet r' \bullet \cdots \bullet r' \bullet \mathcal{C}(C, D) \bullet r_{\rm right} .$$

In analogy to Lemma XIV.4 we have

Lemma XVI.11. *Let c and d be propagators over $\mathcal{B} \times \Sigma$ and r a rung over $\mathfrak{X}_\Sigma$. Define the propagators C and D over $\mathcal{B}$ by*

$$C(x_1, x_2) = \sum_{t_1, t_2 \in \Sigma} c((x_1, t_1), (x_2, t_2)) \qquad D(x_1, x_2) = \sum_{t_1, t_2 \in \Sigma} d((x_1, t_1), (x_2, t_2))$$

and new propagators $\tilde{c}$ and $\tilde{d}$ over $\mathcal{B} \times \Sigma$ by

$$\tilde{c}((x_1, s_1), (x_2, s_2)) = C(x_1, x_2) \qquad \tilde{d}((x_1, s_1), (x_2, s_2)) = D(x_1, x_2)\,.$$

Then, for all $\ell \geq 1$

$$L_\ell(r; C, D) = L_\ell(r; \tilde{c}, \tilde{d})\,.$$

In analogy to Lemma XIV.5, we have

Lemma XVI.12. *Let $f \in \check{\mathcal{F}}_{4;\Sigma}$. Let $C(k)$ and $D(k)$ be functions on $\mathbb{R} \times \mathbb{R}^2$, that are supported in the jth neighborhood, and $C(\xi, \xi')$, $D(\xi, \xi')$ their Fourier transforms as in Definition* IX.3. *Furthermore, let $c((\cdot, s), (\cdot, s'))$ and $d((\cdot, s), (\cdot, s'))$ be the Fourier transform of $\chi_s(k) C(k) \chi_{s'}(k)$ and $\chi_s(k) D(k) \chi_{s'}(k)$. Define propagators over $\mathcal{B} \times \Sigma$ by*

$$c_\Sigma((\xi, s), (\xi', s')) = \sum_{\substack{t \cap s \neq \emptyset \\ t' \cap s' \neq \emptyset}} c((\xi, t), (\xi', t'))$$

$$d_\Sigma((\xi, s), (\xi', s')) = \sum_{\substack{t \cap s \neq \emptyset \\ t' \cap s' \neq \emptyset}} d((\xi, t), (\xi', t'))\,.$$

Then

$$L_\ell(f; C, D) = L_\ell(f; c_\Sigma, d_\Sigma)$$

for all $\ell \geq 1$. Here, the ladder on the right-hand side is defined as in Definition XVI.9(ii), *but with $\mathcal{B}$ replaced by $\mathcal{B} \times \Sigma$, and uses the $\circ$ product of Definition* XIV.1(iv), *while the ladder on the left-hand side is as in Definition* XVI.9(iv) *and uses the $\bullet$ product.*

Also observe that, by Remark XVI.10, for $f \in \check{\mathcal{F}}_{4;\Sigma}$

$$L_\ell(f; C, D) = \sum_{i_1, \ldots, i_4 \in \{0,1\}} f|_{(i_1, i_2, 1, 1)} \bullet \mathcal{C}(C, D) \bullet f|_{(1,1,1,1)} \bullet \cdots \bullet \mathcal{C}(C, D) \bullet f|_{(1,1,i_3,i_4)}\,. \qquad \text{(XVI.1)}$$

XVII. The Renormalization Group Map and Norms in Momentum Space

This section provides the analog of Sec. XV for the $|\ \check{|}$-norms. Again, let $j \geq 2$ and let Σ be a sectorization of scale j and length $\frac{1}{M^{j-3/2}} \leq \mathfrak{l} \leq \frac{1}{M^{(j-1)/2}}$. Fix a system $\boldsymbol{\rho} = (\rho_{m;n})$ of positive real numbers such that

$$\begin{aligned} \rho_{m;n} &\leq \rho_{m;n'} \qquad \text{if } n \leq n' \\ \rho_{m+m';n+n'-2} &\leq \rho_{m;n}\rho_{m';n'} \\ \rho_{m+1;n-1} &\leq \rho_{m;n} \,. \end{aligned} \tag{XVII.1}$$

Definition XVII.1. (i) For a function $f \in \check{\mathcal{F}}_{n;\Sigma}$ and a natural number p we set

$$|f\check{|}_{p,\Sigma,\boldsymbol{\rho}} = \rho_{n;0}|g\check{|}_{p,\Sigma} + \sum_{\substack{\boldsymbol{\imath}\in|\{0,1\}^n \\ m(\boldsymbol{\imath})<n}} \rho_{m(\boldsymbol{\imath});n-m(\boldsymbol{\imath})}|\mathrm{Ord}(f|_{\boldsymbol{\imath}})\check{|}_{p,\Sigma}$$

where g is the function on $\check{\mathcal{B}}_n$ such that

$$f|_{(0,\ldots,0)}(\check{\eta}_1,\ldots,\check{\eta}_n) = (2\pi)^{d+1}\delta(\check{\eta}_1+\cdots+\check{\eta}_n)g(\check{\eta}_1,\ldots,\check{\eta}_n)\,.$$

(ii) For $f \in \check{\mathcal{F}}_m(n;\Sigma)$ set

$$|f\check{|}_{\Sigma} = \rho_{m;n}\begin{cases} |f\check{|}_{1,\Sigma} + |f\check{|}_{2,\Sigma} + \frac{1}{\mathfrak{l}}|f\check{|}_{3,\Sigma} + \frac{1}{\mathfrak{l}}||f\check{|}_{4,\Sigma} + \frac{1}{\mathfrak{l}^2}|f\check{|}_{5,\Sigma} + \frac{1}{\mathfrak{l}^2}||f\check{|}_{6,\Sigma} & \text{if } m \neq 0 \\ |f\check{|}_{1,\Sigma} + \frac{1}{\mathfrak{l}}|f\check{|}_{3,\Sigma} + \frac{1}{\mathfrak{l}^2}|f\check{|}_{5,\Sigma} & \text{if } m = 0 \end{cases}$$

and for $f \in \check{\mathcal{F}}_{n;\Sigma}$ set

$$|f\check{|}_{\Sigma} = |g\check{|}_{\Sigma} + \sum_{\substack{\boldsymbol{\imath}\in\{0,1\}^n \\ m(\boldsymbol{\imath})<n}} |\mathrm{Ord}(f|_{\boldsymbol{\imath}})\check{|}_{\Sigma}$$

where, as in part (i), g is the function on $\check{\mathcal{B}}_n$ such that

$$f|_{(0,\ldots,0)}(\check{\eta}_1,\ldots,\check{\eta}_n) = (2\pi)^{d+1}\delta(\check{\eta}_1+\cdots+\check{\eta}_n)g(\check{\eta}_1,\ldots,\check{\eta}_n)\,.$$

(iii) With the notation introduced in Definitions XVI.1 and XVI.7(iii), every translation invariant sectorized Grassmann function w can be uniquely written in the form

$$w(\phi,\psi) = \sum_n \int_{\mathfrak{X}_\Sigma^n} dx_1\cdots dx_n f_n(x_1,\ldots,x_n)\Psi(x_1)\cdots\Psi(x_n)$$

where $f_n \in \check{\mathcal{F}}_{n;\Sigma}$ is an antisymmetric function. Set, in analogy with Definition XV.1(iii), for $\alpha > 0$ and $X \in \mathfrak{N}_{d+1}$,

$$N_j^{\sim}(w;\alpha;X,\Sigma,\boldsymbol{\rho}) = \frac{M^{2j}}{\mathfrak{l}}\mathfrak{c}_j(X)\sum_{n\geq 0}\alpha^n\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2}|f_n\check{|}_{\Sigma}$$

where $\mathrm{B} = 4\max\{8\mathrm{B}_1, \mathrm{B}_2, 32\mathrm{B}_3, 4\mathrm{B}_4\}$ with B_1, B_2 being the constants of Propositions XII.16 and XII.18 and B_3, B_4 being the constants of Proposition XVI.8.

Remark XVII.2. A sectorized Grassmann function w can also be uniquely written in the form

$$w(\phi,\psi) = \sum_{m,n} \int d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n w_{m,n}(\eta_1, \ldots, \eta_m(\xi_1, s_1), \ldots, (\xi_n, s_n))$$
$$\cdot \phi(\eta_1) \cdots \phi(\eta_m) \psi((\xi_1, s_1)) \cdots \psi((\xi_n, s_n))$$

with $w_{m,n}$ antisymmetric separately in the η variables and in the ξ variables. Then,

$$\begin{aligned} N_j^{\sim}(w;\alpha;X,\Sigma,\boldsymbol{\rho}) &= \frac{M^{2j}}{\mathfrak{l}} \mathfrak{e}_j(X) \sum_{m,n\geq 0} \alpha^{m+n} \left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2} |w^{\sim}_{m,n}|^{\sim}_{\Sigma} \\ &= \frac{M^{2j}}{\mathfrak{l}} \mathfrak{e}_j(X) \sum_{n\geq 0} \alpha^{n} \left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2} \\ &\quad \cdot \rho_{0;n} \left[|w^{\sim}_{0,n}|^{\sim}_{1,\Sigma} + \frac{1}{\mathfrak{l}} |w^{\sim}_{0,n}|^{\sim}_{3,\Sigma} + \frac{1}{\mathfrak{l}^2} |w^{\sim}_{0,n}|^{\sim}_{5,\Sigma} \right] \\ &\quad + \frac{M^{2j}}{\mathfrak{l}} \mathfrak{e}_j(X) \sum_{\substack{m,n\geq 0 \\ m\neq 0}} \alpha^{m+n} \left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2} \\ &\quad \cdot \rho_{m;n} \left[\sum_{p=1}^{6} \frac{1}{\mathfrak{l}^{[(p-1)/2]}} |w^{\sim}_{m,n}|^{\sim}_{p,\Sigma} \right]. \end{aligned}$$

Here, $w^{\sim}_{m,n}$ is the partial Fourier transform of $w_{m,n}$ of Definition IX.1(ii) and $[(p-1)/2]$ is the integer part of $\frac{p-1}{2}$. In particular,

$$N_j^{\sim}(w(\phi,0);\alpha;X,\Sigma,\boldsymbol{\rho}) = \mathfrak{e}_j(X)[\alpha^2 \rho_{2;0} \mathrm{B} M^j |w^{\sim}_{2,0}|^{\sim}_{1,\Sigma} + \alpha^4 \rho_{4;0} \mathrm{B}^2 |w^{\sim}_{4,0}|^{\sim}_{3,\Sigma}]$$

and

$$N_j^{\sim}(w(0,\psi);\alpha;X,\Sigma,\boldsymbol{\rho}) = N_j(w(0,\psi);\alpha;X,\Sigma,\boldsymbol{\rho}).$$

Theorem XVII.3. *Let $c_B > 0$. There are constants* const, const_0, α_0, γ_0 *and* τ_0 *that are independent of j, Σ, $\boldsymbol{\rho}$ such that for all $\alpha \geq \alpha_0$ and $\gamma \leq \gamma_0$ the following holds:*

Let $u((\xi,s),(\xi',s')), v((\xi,s),(\xi',s')) \in \mathcal{F}_0(2;\Sigma)$ be antisymmetric, spin independent, particle number conserving functions. Set

$$C(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)}, \quad D(k) = \frac{\nu^{(\geq j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}(k)}$$

and let $C(\xi,\xi')$, $D(\xi,\xi')$ *be the Fourier transforms of* $C(k)$, $D(k)$ *as in Definition* IX.3. *Let* $B(k)$ *be a function on* $\mathbb{R}\times\mathbb{R}^2$ *and set*

$$(\hat{B}\phi)(\xi) = \int d\xi' \hat{B}(\xi,\xi')\phi(\xi')$$

where $\hat{B}$ *was defined in Definition* IX.4. *Furthermore, let* $\mathcal{W}(\phi,\psi)$ *be an even Grassmann function and set*[e]

$$:\mathcal{W}'(\phi,\psi):_{\psi,D} = \Omega_C(:\mathcal{W}(\phi,\psi):_{\psi,C+D})(\phi,\psi+\hat{B}\phi)\,.$$

Assume that the following estimates are fulfilled:

- $\rho_{m+1;n-1} \le \gamma\rho_{m;n}$ *for all* $m\ge 0$ *and* $n\ge 1$.
- $|\check{u}(k)|, |\check{v}(k)| \le \frac{1}{2}|\imath k_0 - e(k)|$.
- $|u|_{1,\Sigma} \le \mu(\Lambda+X)\mathfrak{e}_j(X)$ *with* $X\in\mathfrak{N}_{d+1}$, $\mu,\Lambda>0$ *such that* $(1+\mu)(\Lambda+X_{\mathbf{0}}) \le \frac{\tau_0}{M^j}$.
- $\|B(k)\|^{\tilde{}} \le c_B\mathfrak{e}_j(X)$.
- $\mathcal{W}$ *has a sectorized representative*

$$w(\phi,\psi) = \sum_n \int_{\mathfrak{X}_\Sigma^n} dx_1\cdots dx_n f_n(x_1,\ldots,x_n)\Psi(x_1)\cdots\Psi(x_n)$$

with antisymmetric functions $f_n\in\check{\mathcal{F}}_{n;\Sigma}$ *such that* $f_2=0$ *and*

$$N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho}) \le \mathrm{const}_0\,\alpha + \sum_{\delta\ne 0}\infty t^\delta\,.$$

Then $\mathcal{W}'$ *has a sectorized representative* w' *such that*

$$N_j^{\sim}(w'-w;\alpha;X,\Sigma,\boldsymbol{\rho}) \le \mathrm{const}\left(\frac{1}{\alpha}+\gamma\right)\frac{N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}{1-\frac{\mathrm{const}}{\alpha}N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}\,.$$

Furthermore, if one writes $w'(\phi,\psi) = \sum_n\int_{\mathfrak{X}_\Sigma^n} dx_1\cdots dx_n f_n'(x_1,\ldots,x_n)\Psi(x_1)\cdots\Psi(x_n)$, *with antisymmetric functions* $f_n'\in\check{\mathcal{F}}_{n;\Sigma}$, *then*

$$|f_2'|^{\tilde{}}_{1,\Sigma,\boldsymbol{\rho}} \le \frac{\mathrm{const}}{\alpha^8}\frac{\mathfrak{l}}{M^j}\frac{N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})^2}{1-\frac{\mathrm{const}}{\alpha}N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}\,.$$

If one writes $w'(\phi,\psi) = w''(\phi,\psi+\hat{B}\phi)$ *where* $(\hat{B}\phi)(\xi,s) = \int d\xi'\hat{B}(\xi,\xi')\phi(\xi')$, *with abuse of notation, and expands*

$$w''(\phi,\psi) = \sum_n\int_{\mathfrak{X}_\Sigma^n} dx_1\cdots dx_n f_n''(x_1,\ldots,x_n)\Psi(x_1)\cdots\Psi(x_n)$$

[e]The definition of $\mathcal{W}'$ as an analytic function, rather than merely a formal Taylor series was explained in Remark XV.11.

with antisymmetric functions $f''_n \in \check{\mathcal{F}}_{n;\Sigma}$*, then*

$$\left| f''_4 - f_4 - \frac{1}{4}\sum_{\ell=1}^{\infty}(-1)^{\ell}(12)^{\ell+1}\,\mathrm{Ant}\,L_\ell(f_4;C,D)\right|^{\sim}_{3,\Sigma,\boldsymbol{\rho}}$$

$$\leq \frac{\mathrm{const}}{\alpha^{10}}\,\mathfrak{l}\,\frac{N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})^2}{1-\frac{\mathrm{const}}{\alpha}N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}\,.$$

Here $L_\ell(f_4;C,D)$ *is a ladder in the sense of Definition* XVI.9(iv).

The proof of Theorem XVII.3 is similar that of Theorems XV.3 and X.12. Recall that w and w' are elements of the Grassmann algebra over the vector space, $\tilde{V}$, generated by $\check\phi(\check\eta)$, $\check\eta \in \check{\mathcal{B}}$, $\psi(\xi,s)$, $(\xi,s)\in(\mathcal{B}\times\Sigma)$. Let $c((\cdot,s),(\cdot,s'))$ and $d((\cdot,s),(\cdot,s'))$ be the Fourier transform of $\chi_s(k)C(k)\chi_{s'}(k)$ and $\chi_s(k)D(k)\chi_{s'}(k)$ in the sense of Definition IX.3. Then c and d define covariances on $\tilde{V}$ by

$$\tilde{C}_\Sigma(\check\phi(\check\eta),\check\phi(\check\eta'))=0\,,\qquad \tilde{D}_\Sigma(\check\phi(\check\eta),\check\phi(\check\eta'))=0$$

$$\tilde{C}_\Sigma(\check\phi(\check\eta),\psi((\xi,s)))=0\,,\qquad \tilde{D}_\Sigma(\check\phi(\check\eta),\psi((\xi,s)))=0$$

and

$$\tilde{C}_\Sigma(\psi(\xi,s),\psi(\xi',s'))=c_\Sigma((\xi,s),(\xi',s'))=\sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}}c((\xi,t),(\xi',t'))$$

$$\tilde{D}_\Sigma(\psi(\xi,s),\psi(\xi',s'))=d_\Sigma((\xi,s),(\xi',s'))=\sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}}d((\xi,t),(\xi',t'))\,.$$

The restriction of $\tilde{C}_\Sigma$ resp. $\tilde{D}_\Sigma$ to the vector space, V_Σ, generated by $\psi(\xi,s)$, $(\xi,s)\in(\mathcal{B}\times\Sigma)$, coincides with the C_Σ resp. D_Σ of Proposition XII, while the subspace V_{ext}, generated by $\check\phi(\check\eta)$, $\check\eta\in\check{\mathcal{B}}$, is isotropic and perpendicular to V_Σ with respect to both $\tilde{C}_\Sigma$ and $\tilde{D}_\Sigma$.

For $f\in\check{\mathcal{F}}_m(n;\Sigma)$ set

$$|f|^{\sim}_{\mathrm{impr},\Sigma}=\rho_{m;n}\begin{cases}|f|^{\sim}_{1,\Sigma}+|f|^{\sim}_{2,\Sigma}+\frac{1}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma}+\frac{1}{\mathfrak{l}}|f|^{\sim}_{4,\Sigma} & \text{if } m\neq 0\\ |f|^{\sim}_{1,\Sigma}+\frac{1}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma} & \text{if } m=0\end{cases}\qquad\text{(XVII.2)}$$

and for $f\in\check{\mathcal{F}}_{n;\Sigma}$ set

$$|f|^{\sim}_{\mathrm{impr},\Sigma}=|g|^{\sim}_{\mathrm{impr},\Sigma}+\sum_{\substack{\boldsymbol{\imath}\in\{0,1\}^n\\ m(\boldsymbol{\imath})<n}}|\mathrm{Ord}(f|_{\boldsymbol{\imath}})|^{\sim}_{\mathrm{impr},\Sigma}$$

where g is the function on $\check{\mathcal{B}}_n$ such that

$$f|_{(0,\ldots,0)}(\check\eta_1,\ldots,\check\eta_n)=(2\pi)^{d+1}\delta(\check\eta_1+\cdots+\check\eta_n)g(\check\eta_1,\ldots,\check\eta_n)\,.$$

The seminorms $|\cdot|^{\sim}_{\rm impr,\Sigma}$ (and $|\cdot|'_{\rm impr}$, $N^{\sim}_{\rm impr}(\cdot;\alpha)$, to be introduced shortly) are used only locally, between this point and the end of the proof of Theorem XVII.3.

Lemma XVII.4. *Under the hypotheses of Theorem* XVII.3, *there exists a constant* const_1 *that is independent of* j *and* Σ *such that the covariances* $(\tilde C_\Sigma, \tilde D_\Sigma)$ *have improved integration constants*

$$\mathfrak{c} = \mathrm{const}_1\, M^j \mathfrak{e}_j(X)\,, \quad \frac{1}{2}\mathrm{b} = \sqrt{\frac{\mathrm{B}\,\mathfrak{l}}{4M^j}}\,, \quad J = \mathfrak{l}$$

for the families $|\cdot|^{\sim}_{\Sigma}$ *and* $|\cdot|^{\sim}_{\rm impr,\Sigma}$ *of seminorms* (*in the sense of* [3, *Definition* VI.1]).

Proof. By Proposition XVI.8 and (XVII.1), the covariances $(\tilde C_\Sigma, \tilde D_\Sigma)$ have integration constants

$$\mathfrak{c}' = 12|c|_{1,\Sigma} \qquad \mathrm{b}' = \sqrt{\max\{8\mathrm{B}_3, \mathrm{B}_4\}\frac{\mathfrak{l}}{M^j}}$$

for the configuration $|\cdot|^{\sim}_{1,\Sigma,\boldsymbol\rho}, |\cdot|^{\sim}_{2,\Sigma,\boldsymbol\rho}, \ldots, |\cdot|^{\sim}_{6,\Sigma,\boldsymbol\rho}$ of seminorms, in the sense of [3, Definition VI.11]. Hence, by [3, Lemma VI.12], $(\tilde C_\Sigma, \tilde D_\Sigma)$ have improved integration constants $\mathfrak{c}'$, b' and $J = \mathfrak{l}$ for the families

$$|f|' = |f|^{\sim}_{1,\Sigma,\boldsymbol\rho} + |f|^{\sim}_{2,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}}|f|^{\sim}_{4,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}^2}|f|^{\sim}_{5,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}^2}|f|^{\sim}_{6,\Sigma,\boldsymbol\rho}$$

$$|f|'_{\rm impr} = |f|^{\sim}_{1,\Sigma,\boldsymbol\rho} + |f|^{\sim}_{2,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}}|f|^{\sim}_{3,\Sigma,\boldsymbol\rho} + \frac{1}{\mathfrak{l}}|f|^{\sim}_{4,\Sigma,\boldsymbol\rho}\,.$$

When $f \in \check{\mathcal{F}}_0(n;\Sigma)$, $|f|^{\sim}_{p+1,\Sigma,\boldsymbol\rho} \le |f|^{\sim}_{p,\Sigma,\boldsymbol\rho}$ for all odd p so that

$$|f|^{\sim}_{\Sigma} \le |f|' \le |2|f|^{\sim}_{\Sigma} \qquad |f|^{\sim}_{\rm impr,\Sigma} \le |f|'_{\rm impr} \le |2|f|^{\sim}_{\rm impr,\Sigma}\,.$$

Hence $(\tilde C_\Sigma, \tilde D_\Sigma)$ have improved integration constants $4\mathfrak{c}'$, $2\mathrm{b}'$ and $J = \mathfrak{l}$ for the families $|\cdot|^{\sim}_{\Sigma}$ and $|\cdot|^{\sim}_{\rm impr,\Sigma}$ of seminorms. As in Lemma XV.5, $\mathfrak{c}' \le \mathrm{const}\, M^j \mathfrak{e}_j(X)$ and the lemma follows. □

Lemma XVII.5. *Let* $c_B > 0$. *Then there are constants* const *and* γ_0, *independent of* M, j, Σ, $\boldsymbol\rho$ *such that the following holds for all* $\gamma \le \gamma_0$ *and all* $X, X_B \in \mathfrak{N}_{d+1}$. *Let* $g(\phi,\psi)$ *be a sectorized Grassmann function and set*

$$g'(\phi,\psi) = g(\phi, \psi + \hat B\phi)\,.$$

Assume that $\|B(k)\|^{\sim} \le c_B \mathfrak{e}_j(X)$ *and* $\|B(k)\|^{\sim} \le c_B X_B \mathfrak{e}_j(X)$. *If* $\rho_{m+1;n-1} \le \gamma\rho_{m;n}$ *for all* $m \ge 0$ *and* $n \ge 1$, *then*

$$N_j^{\sim}(g'-g;\alpha;X,\Sigma,\boldsymbol\rho) \le \mathit{const}\,\gamma X_B N_j^{\sim}(g;2\alpha;X,\Sigma,\boldsymbol\rho)\,.$$

Let $G_{m,n}$, *resp.* $G'_{m,n}$, *be the kernel of the part of* g, *resp.* g', *that is of degree* m *in* ϕ *and degree* n *in* ψ. *Then, for* $p \in \{1,3\}$,

$$\sum_{\substack{m,n\\ m+n=p+1}} |G'^{\sim}_{m,n} - G^{\sim}_{m,n}|_{p,\Sigma,\boldsymbol\rho} \le \mathit{const}\,\gamma X_B \mathfrak{e}_j(X) \sum_{\substack{m,n\\ m+n=p+1}} |G^{\sim}_{m,n}|_{p,\Sigma,\boldsymbol\rho}\,.$$

Proof. Let $\varphi \in \check{\mathcal{F}}_m(n;\Sigma)$, $1 \le i \le n$ and set, for $\check\eta_{m+1} = (k_{m+1}, \sigma_{m+1}, a_{m+1})$,

$$\begin{aligned}
&\varphi'(\check\eta_1, \ldots, \check\eta_{m+1}; (\xi_1, s_1), \ldots, (\xi_{n-1}, s_{n-1})) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} \int d\zeta B(k_{m+1}) E_+(\check\eta_{m+1}, \zeta) \varphi(\check\eta_1, \ldots, \check\eta_m; \\
&\qquad (\xi_1, s_1), \ldots, (\xi_{i-1}, s_{i-1}), (\zeta, s), (\xi_i, s_i), \ldots, (\xi_{n-1}, s_{n-1}))
\end{aligned}$$

if $n \ge 2$, and

$$\begin{aligned}
&\varphi'(\check\eta_1, \ldots, \check\eta_{m+1})(2\pi)^{d+1}\delta(k_1 + \cdots + k_{m+1}) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} \int d\zeta B(k_{m+1}) E_+(\check\eta_{m+1}, \zeta) \varphi(\check\eta_1, \ldots, \check\eta_m; (\zeta, s)) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} B(k_{m+1}) \varphi(\check\eta_1, \ldots, \check\eta_m; (0, \sigma_{m+1}, a_{m+1}, s)) \\
&\qquad \cdot (2\pi)^{d+1}\delta(k_1 + \cdots + k_{m+1})
\end{aligned}$$

if $n = 1$. For any fixed $\check\eta_1, \ldots, \check\eta_{m+1}$

$$\begin{aligned}
&|||\varphi'(\check\eta_1, \ldots, \check\eta_{m+1}; (\xi_1, s_1), \ldots, (\xi_{n-1}, s_{n-1}))|||_{1,\infty} \\
&\quad \le 2 \sup_{k,s} |B(k)|\, |||\varphi(\check\eta_1, \ldots, \check\eta_m; (\xi_1, s_1), \ldots, (\zeta, s), \ldots, (\xi_{n-1}, s_{n-1}))|||_{1,\infty}
\end{aligned}$$

when $n \ge 2$, since $|E_+(\check\eta_{m+1}, \zeta)| \le 1$ and the requirement that k_{m+1} be in the sector s restricts the choice of s to at most two different sectors. For $n = 1$,

$$|\varphi'(\check\eta_1, \ldots, \check\eta_{m+1})| \le 2 \sup_{k,s} |B(k)|\, |\varphi(\check\eta_1, \ldots, \check\eta_m; (0, \sigma_{m+1}, a_{m+1}, s))|\,.$$

Since $\mathrm{D}^\delta_{m+1} E_+(\check\eta_{m+1}, \zeta) = \zeta^\delta E_+(\check\eta_{m+1}, \zeta)$, Leibniz and [6, Corollary A.5(ii)] implies that, for both $X' = X_B$ and $X' = 1$,

$$\mathfrak{e}_j(X)|\varphi'|^{\sim}_{p,\Sigma} \le \mathit{const}\, \mathfrak{e}_j(X)\|B(k)\|^{\sim} |\varphi|^{\sim}_{p,\Sigma} \le \mathit{const}\, c_B X' \mathfrak{e}_j(X)|\varphi|^{\sim}_{p,\Sigma} \tag{XVII.3}$$

so that $\mathfrak{e}_j(X)|\varphi'|^{\sim}_{p,\Sigma,\boldsymbol\rho} \le \mathit{const}\, c_B \gamma X' \mathfrak{e}_j(X)|\varphi|^{\sim}_{p,\Sigma,\boldsymbol\rho}$ and

$$\mathfrak{e}_j(X)|\varphi'|^{\sim}_{\Sigma} \le \mathit{const}\, c_B \gamma X' \mathfrak{e}_j(X)|\varphi|^{\sim}_{\Sigma}\,. \tag{XVII.4}$$

Write $g(\phi, \psi) = \sum_{m,n} g_{m,n}(\phi, \psi)$, with $g_{m,n}$ of degree m in ϕ and degree n in ψ, and

$$g(\phi, \psi + \zeta) = \sum_{m,n} g_{m,n}(\phi, \psi + \zeta) = \sum_{m,n} \sum_{\ell=0}^{n} g_{m,n-\ell,\ell}(\phi, \psi, \zeta)$$

with $g_{m,n-\ell,\ell}$ of degrees m in ϕ, $n - \ell$ in ψ and ℓ in ζ. Let $G_{m,n}$ and $G_{m,n-\ell,\ell}$ be the kernels of $g_{m,n}(\phi, \psi)$ and $g_{m,n-\ell,\ell}(\phi, \psi, \hat B\phi)$ respectively. By the binomial theorem and repeated application of (XVII.4), $\ell - 1$ times with $X' = 1$ and once with $X' = X_B$,

$$\mathfrak{e}_j(X)|G^{\sim}_{m,n-\ell,\ell}\widetilde{\rceil}_{\Sigma} \le \left({\scriptstyle const}\, c_B\gamma\right)^{\ell} \binom{n}{\ell} X_B \mathfrak{e}_j(X)|G^{\sim}_{m,n}\widetilde{\rceil}_{\Sigma}$$

if $\ell \ge 1$. Then,

$$g'(\phi,\psi) - g(\phi,\psi) = g(\phi,\psi+\hat{B}\phi) - g(\phi,\psi) = \sum_{m,n\ge 0}\sum_{\ell=1}^{n} g_{m,n-\ell,\ell}(\phi,\psi,\hat{B}\phi)$$

and

$$\begin{aligned}
&N^{\sim}_j(g'-g;\alpha;X,\Sigma,\boldsymbol{\rho})\\
&\quad\le \frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)\sum_{m,n\ge 0}\sum_{\ell=1}^{n}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n-\ell,\ell}\widetilde{\rceil}_{\Sigma}\\
&\quad\le \frac{M^{2j}}{\mathfrak{l}}X_B\mathfrak{e}_j(X)\sum_{m,n\ge 0}\sum_{\ell=1}^{n}\binom{n}{\ell}\left({\scriptstyle const}\, c_B\gamma\right)^{\ell}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\widetilde{\rceil}_{\Sigma}\\
&\quad= \frac{M^{2j}}{\mathfrak{l}}X_B\mathfrak{e}_j(X)\sum_{m,n\ge 0}\left[(1+{\scriptstyle const}\, c_B\gamma)^n - 1\right]\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\widetilde{\rceil}_{\Sigma}\,.
\end{aligned}$$

If ${\scriptstyle const}\, c_B\gamma \le \frac{1}{3}$

$$\begin{aligned}
(1+{\scriptstyle const}\, c_B\gamma)^n - 1 &\le {\scriptstyle const}\, c_B\gamma n(1+{\scriptstyle const}\, c_B\gamma)^{n-1}\\
&\le {\scriptstyle const}\, c_B\gamma\left(\frac{3}{2}\right)^n(1+{\scriptstyle const}\, c_B\gamma)^{n-1}\\
&\le {\scriptstyle const}\, c_B\gamma 2^n\\
&\le {\scriptstyle const}\,\gamma 2^n
\end{aligned}$$

and

$$N^{\sim}_j(g'-g;\alpha;X,\Sigma,\boldsymbol{\rho}) \le {\scriptstyle const}\,\gamma X_B N^{\sim}_j(g;2\alpha;X,\Sigma,\boldsymbol{\rho})\,.$$

The proof of the second claim is similar but uses

$$|G^{\sim}_{m,n-\ell,\ell}\widetilde{\rceil}_{p,\Sigma,\boldsymbol{\rho}} \le \left({\scriptstyle const}\, c_B\gamma\right)^{\ell}\binom{n}{\ell} X_B\mathfrak{e}_j(X)|G^{\sim}_{m,n}\widetilde{\rceil}_{p,\Sigma,\boldsymbol{\rho}}$$

and $\left({\scriptstyle const}\, c_B\gamma\right)^{\ell}\binom{n}{\ell} \le {\scriptstyle const}\,\gamma$ for $\ell \ge 1$, $n \le 4$. □

Proof of Theorem XVII.3. For a sectorized Grassmann function $v = \sum_n v_n$ with $v_n \in \bigwedge^n \tilde{V}$ let

$$N^\sim(v;\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_n \alpha^n \mathrm{b}^n |v_n|^\sim_\Sigma$$

$$N^\sim_{\mathrm{impr}}(v;\alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_n \alpha^n \mathrm{b}^n |v_n|^\sim_{\mathrm{impr},\Sigma}$$

be the quantities introduced in [1, Definition II.23] and just after [3, Lemma VI.2]. Then

$$N^\sim(v;\alpha) = \frac{\mathrm{const}_1}{\mathrm{B}} N^\sim_j(v;\alpha;X,\Sigma,\boldsymbol{\rho})$$

where const_1 is the constant of Lemma XVII.4.

If $:w'':_{\psi,\tilde{D}_\Sigma} = \Omega_{\tilde{C}_\Sigma}(:w:_{\psi,\tilde{C}_\Sigma+\tilde{D}_\Sigma})$, then, by Proposition XII, parts (ii) and (iii), and [1, Proposition A.2(ii)]

$$w' = w''(\phi, \psi + \hat{B}\phi)$$

is a sectorized representative for $\mathcal{W}'$. We apply [3, Theorem VI.6] to get estimates on w''. Choosing $\mathrm{const}_0 = \frac{\mathrm{B}}{8\,\mathrm{const}_1}$, the hypotheses of this theorem are fulfilled by Lemma XVII.4. Consequently,

$$N^\sim(w''-w;\alpha) \le \frac{1}{2\alpha^2}\frac{N^\sim(w;32\alpha)^2}{1-\frac{1}{\alpha^2}N^\sim(w;32\alpha)} \tag{XVII.5}$$

$$\alpha^2\mathfrak{c}|f_2''|^\sim_{\mathrm{impr},\Sigma} \le \frac{2^{10}\mathfrak{l}}{\alpha^6}\frac{N^\sim(w;64\alpha)^2}{1-\frac{8}{\alpha}N^\sim(w;64\alpha)} \tag{XVII.6}$$

$$\alpha^4\mathrm{b}^2\mathfrak{c}\left| f_4'' - f_4 - \frac{1}{4}\sum_{\ell\ge 1}(-1)^\ell (12)^{\ell+1}\mathrm{Ant}\, L_\ell(f_4;c_\Sigma,d_\Sigma)\right|^\sim_{\mathrm{impr},\Sigma}$$
$$\le \frac{2^{10}\mathfrak{l}}{\alpha^6}\frac{N^\sim(w;64\alpha)^2}{1-\frac{8}{\alpha}N^\sim(w;64\alpha)}\,. \tag{XVII.7}$$

In (XVII.7), we used the description of ladders in terms of kernels given in [3, Proposition C.4].

By Lemma XVII.5, with $X_B = 1$,

$$\begin{aligned}
N^\sim(w'-w;\alpha) &= N^\sim(w''(\phi,\psi+\hat{B}\phi) - w(\phi,\psi);\alpha)\\
&\le N^\sim(w''(\phi,\psi+\hat{B}\phi) - w''(\phi,\psi);\alpha) + N^\sim(w''(\phi,\psi)-w(\phi,\psi);\alpha)\\
&\le \mathrm{const}\,\gamma N^\sim(w'';2\alpha) + N^\sim(w''-w;\alpha)\\
&\le \mathrm{const}\,\gamma N^\sim(w;2\alpha) + (1+\mathrm{const}\,\gamma)N^\sim(w''-w;2\alpha)\\
&\le \mathrm{const}\,\gamma N^\sim(w;2\alpha) + (1+\mathrm{const}\,\gamma)\frac{1}{8\alpha^2}\frac{N^\sim(w;64\alpha)^2}{1-\frac{1}{\alpha^2}N^\sim(w;64\alpha)}\\
&\le \mathrm{const}\left(\frac{1}{\alpha}+\gamma\right)\frac{N^\sim_j(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}{1-\frac{\mathrm{const}}{\alpha}N^\sim_j(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}\,.
\end{aligned}$$

By (XVII.6),

$$M^j\alpha^2\mathfrak{e}_j(X)|f_2''|^{\sim}_{1,\Sigma,\boldsymbol{\rho}} \le M^j\alpha^2\mathfrak{e}_j(X)|f_2''|^{\sim}_{\mathrm{impr},\Sigma} \le \mathrm{const}\,\frac{\mathfrak{l}}{\alpha^6}\frac{N^{\sim}(w;64\alpha)^2}{1-\frac{8}{\alpha}N^{\sim}(w;64\alpha)}\,.$$

Applying Lemma XVII.5 to the part of w'' that is homogeneous of degree two in ψ and ϕ combined yields

$$|f_2'|^{\sim}_{1,\Sigma,\boldsymbol{\rho}} \le 4\mathfrak{e}_j(X)|f_2''|^{\sim}_{1,\Sigma,\boldsymbol{\rho}}$$

and hence

$$|f_2'|^{\sim}_{1,\Sigma,\boldsymbol{\rho}} \le \frac{\mathrm{const}}{\alpha^8}\frac{\mathfrak{l}}{M^j}\frac{N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})^2}{1-\frac{\mathrm{const}}{\alpha}N_j^{\sim}(w;64\alpha;X,\Sigma,\boldsymbol{\rho})}\,.$$

By Lemma XVI.12 and (XVII.7)

$$\left|f_4''-f_4-\frac{1}{4}\sum_{\ell=1}^{\infty}(-1)^{\ell}(12)^{\ell+1}\mathrm{Ant}\,L_\ell(f_4;C,D)\right|^{\sim}_{3,\Sigma,\boldsymbol{\rho}}$$

$$\le \frac{\mathrm{const}}{\alpha^{10}}\mathfrak{l}\frac{N_j^{\sim}(w;64\alpha)^2}{1-\frac{\mathrm{const}}{\alpha}}N_j^{\sim}(w;64\alpha)\,. \qquad \square$$

Theorem XVII.6. *Let* $c_B>0$. *There are constants* const, const_0, α_0, γ_0 *and* τ_0 *that are independent of* j, Σ, $\boldsymbol{\rho}$ *such that for all* $\alpha\ge\alpha_0$, $\varepsilon>0$ *and* $\gamma\le\gamma_0$ *the following holds*:

Let, for κ *in a neighborhood of zero,* $u_\kappa, v_\kappa \in \mathcal{F}_0(2;\Sigma)$ *be antisymmetric, spin independent, particle number conserving functions. Set*

$$C_\kappa(k)=\frac{\nu^{(j)}(k)}{\imath k_0-e(\mathbf{k})-\check{u}_\kappa(k)}\,,\quad D_\kappa(k)=\frac{\nu^{(\ge j+1)}(k)}{\imath k_0-e(\mathbf{k})-\check{v}_\kappa(k)}$$

and let $C_\kappa(\xi,\xi')$, $D_\kappa(\xi,\xi')$ *be the Fourier transforms of* $C_\kappa(k)$, $D_\kappa(k)$. *Let* $B_\kappa(k)$ *be a function on* $\mathbb{R}\times\mathbb{R}^2$ *and set*

$$(\hat{B}_\kappa\phi)(\xi)=\int d\xi'\,\hat{B}_\kappa(\xi,\xi')\phi(\xi')\,.$$

Furthermore, let, for κ *in a neighborhood of zero,* $\mathcal{W}_\kappa(\phi,\psi)$ *be an even Grassmann function and set*

$$:\mathcal{W}_\kappa'(\phi,\psi):_{\psi,D_\kappa}=\Omega_{C_\kappa}(:\mathcal{W}_\kappa(\phi,\psi):_{\psi,C_\kappa+D_\kappa})(\phi,\psi+\hat{B}_\kappa\phi)\,.$$

Assume that the following estimates are fulfilled:

- $\rho_{m+1;n-1}\le\gamma\rho_{m;n}$ *for all* $m\ge0$ *and* $n\ge1$.
- $|\check{u}_0(k)|,|\check{v}_0(k)|\le\frac{1}{2}|\imath k_0-e(k)|$ *and* $|\frac{d}{d\kappa}\check{v}_\kappa(k)|_{\kappa=0}|\le\varepsilon|\imath k_0-e(k)|$.
- $|u_0|_{1,\Sigma}\le\mu(\Lambda+X)\mathfrak{e}_j(X)$ *and* $|\frac{d}{d\kappa}u_\kappa|_{\kappa=0}|_{1,\Sigma}\le\mathfrak{e}_j(X)Y$ *with* $X,Y\in\mathfrak{N}_{d+1}$, $\mu,\Lambda>0$ *such that* $(1+\mu)(\Lambda+X_{\mathbf{0}})\le\frac{\tau_0}{M^j}$.
- $\|B_0(k)\|^{\sim}\le c_B\mathfrak{e}_j(X)$ *and* $\|\frac{d}{d\kappa}B_\kappa(k)\|^{\sim}\le c_B\mathfrak{e}_j(X)Z$ *with* $Z\in\mathfrak{N}_{d+1}$.

- *$\mathcal{W}_\kappa$ has a sectorized representative w_κ with*

$$\mathfrak{n} \equiv N_j^{\sim}(w_0; 64\alpha; X, \Sigma, \boldsymbol{\rho}) \le \mathrm{const}_0\, \alpha + \sum_{\delta \ne 0} \infty t^\delta\,.$$

Then $\mathcal{W}'_\kappa$ has a sectorized representative w'_κ such that

$$N_j^{\sim}\left(\frac{d}{d\kappa}[w'_\kappa - w_\kappa]_{\kappa=0}; \alpha; X, \Sigma, \boldsymbol{\rho}\right)$$

$$\le \mathrm{const}\left\{\gamma + \frac{1}{\alpha^2}\frac{\mathfrak{n}}{1 - \frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\right\} N_j^{\sim}\left(\frac{d}{d\kappa}w_\kappa\bigg|_{\kappa=0}; 16\alpha; X, \Sigma, \boldsymbol{\rho}\right)$$

$$+\,\mathrm{const}\,\frac{\mathfrak{n}}{1 - \frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\left\{\frac{1}{\alpha^2}M^j Y \mathfrak{n} + \frac{\varepsilon}{\alpha^2}\mathfrak{n} + \gamma Z\right\}.$$

Lemma XVII.7. *Under the hypotheses of Theorem* XVII.6, *there exists a constant* const_2 *that is independent of j and Σ such that $\tilde{C}_{0,\Sigma}$ has contraction bound $\mathfrak{c}$, $\tilde{C}_{0,\Sigma}$ and $D_{0,\Sigma}$ have integral bound $\frac{1}{2}\mathrm{b}$ and*

$$\frac{d}{d\kappa}\tilde{C}_{\kappa,\Sigma}\bigg|_{\kappa=0} \quad \textit{has contraction bound } \mathfrak{c}' = \mathrm{const}_2\, M^{2j}\mathfrak{e}_j(X)Y$$

$$\frac{d}{d\kappa}\tilde{D}_{\kappa,\Sigma}\bigg|_{\kappa=0} \quad \textit{has integral bound } \frac{1}{2}\mathrm{b}' = \sqrt{\varepsilon}\mathrm{b}$$

for the family $|\cdot|^{\sim}_{\Sigma}$ of symmetric seminorms.

Proof. The contraction and integral bounds on $\tilde{C}_{0,\Sigma}$ and $\tilde{D}_{0,\Sigma}$ were proven in Lemma XVII.4. Clearly, the function

$$\frac{d}{d\kappa}D_\kappa(k) = \frac{d}{d\kappa}\frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}_\kappa(k)} = \frac{\nu^{(\ge j+1)}(k)}{[\imath k_0 - e(\mathbf{k}) - \check{v}_\kappa(k)]^2}\frac{d}{d\kappa}\check{v}_\kappa(k)$$

is supported on the jth neighborhood and obeys $|\frac{d}{d\kappa}D_\kappa(k)|_{\kappa=0}| \le \frac{4\varepsilon}{|\imath k_0 - e(\mathbf{k})|}$. By Proposition XVI.8(ii) and the first property of (XVII.1), $2\sqrt{4\mathrm{B}_3\varepsilon\frac{\mathfrak{l}}{M^j}} \le \sqrt{\varepsilon}\mathrm{b}$ is an integral bound for $\frac{d}{d\kappa}\tilde{D}_{\kappa,\Sigma}|_{\kappa=0}$.

Set $\mathfrak{c}'' = 12|\frac{d}{d\kappa}c_\kappa|_{\kappa=0}|_{1,\Sigma}$. By Proposition XVI.8(i) (see also [3, Lemma VI.15]) and the second property of $\boldsymbol{\rho}$ in (XVII.1), $(\frac{d}{d\kappa}c_\kappa|_{\kappa=0})_\Sigma$ has contraction bound $\mathfrak{c}''$. We showed in Lemma XV.8 that

$$\mathfrak{c}'' \le \mathrm{const}\, M^{2j}\mathfrak{e}_j(X)Y\,. \qquad \square$$

Lemma XVII.8. *Let $g(\phi, \psi)$ be a sectorized Grassmann function and set*

$$g'_\kappa(\phi, \psi) = g(\phi, \psi + \hat{B}_\kappa\phi)\,.$$

Under the hypotheses of Theorem XVII.6,

$$N_j^{\sim}\left(\frac{d}{d\kappa}g'_\kappa\bigg|_{\kappa=0}; \alpha; X, \Sigma, \boldsymbol{\rho}\right) \le \mathrm{const}\,\gamma Z N_j^{\sim}(g; 2\alpha; X, \Sigma, \boldsymbol{\rho})\,.$$

Proof. Define, as in Lemma XVII.5, for $\check\eta_{m+1} = (k_{m+1}, \sigma_{m+1}, a_{m+1})$,

$$\begin{aligned}
&\varphi'_\kappa(\check\eta_1, \dots, \check\eta_{m+1}; (\xi_1, s_1), \dots, (\xi_{n-1}, s_{n-1})) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} \int d\zeta B_\kappa(k_{m+1}) E_+(\check\eta_{m+1}, \zeta) \varphi(\check\eta_1, \dots, \check\eta_m\,; \\
&\qquad (\xi_1, s_1), \dots, (\xi_{i-1}, s_{i-1}), (\zeta, s), (\xi_i, s_i), \dots, (\xi_{n-1}, s_{n-1}))
\end{aligned}$$

if $n \geq 2$, and

$$\begin{aligned}
&\varphi'_\kappa(\check\eta_1, \dots, \check\eta_{m+1})(2\pi)^{d+1}\delta(k_1 + \cdots + k_{m+1}) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} \int d\zeta B_\kappa(k_{m+1}) E_+(\check\eta_{m+1}, \zeta) \varphi(\check\eta_1, \dots, \check\eta_m; (\zeta, s)) \\
&\quad = \mathrm{Ant}_{\mathrm{ext}} \sum_{s\in\Sigma} B_\kappa(k_{m+1}) \varphi(\check\eta_1, \dots, \check\eta_m; (0, \sigma_{m+1}, a_{m+1}, s)) \\
&\qquad \cdot (2\pi)^{d+1}\delta(k_1 + \cdots + k_{m+1})
\end{aligned}$$

if $n = 1$. By (XVII.4), with $X' = X_B = 1$,

$$\mathfrak{e}_j(X) |\varphi'_0|^{\sim}_{\Sigma} \leq \mathrm{const}\,\gamma \mathfrak{e}_j(X) |\varphi|^{\sim}_{\Sigma} \tag{XVII.8}$$

and by the same derivation as led to (XVII.4), but with $X' = X_B = Z$,

$$\mathfrak{e}_j(X) \left| \frac{d}{d\kappa} \varphi'_\kappa \Big|_{\kappa=0} \right|^{\sim}_{\Sigma} \leq \mathrm{const}\,\gamma \mathfrak{e}_j(X) Z |\varphi|^{\sim}_{\Sigma}\,. \tag{XVII.9}$$

As in Lemma XVII.5, write $g(\phi, \psi) = \sum_{m,n} g_{m,n}(\phi, \psi)$, with $g_{m,n}$ of degree m in ϕ and degree n in ψ, and

$$g(\phi, \psi + \zeta) = \sum_{m,n} g_{m,n}(\phi, \psi + \zeta) = \sum_{m,n} \sum_{\ell=0}^{n} g_{m,n-\ell,\ell}(\phi, \psi, \zeta)$$

with $g_{m,n-\ell,\ell}$ of degrees m in ϕ, $n - \ell$ in ψ and ℓ in ζ. Let $G_{m,n}$ and $G_{\kappa;m,n-\ell,\ell}$ be the kernels of $g_{m,n}(\phi, \psi)$ and $g_{m,n-\ell,\ell}(\phi, \psi, \hat B_\kappa \phi)$ respectively. By the binomial theorem, Leibniz, one application of (XVII.9) and $\ell - 1$ applications of (XVII.8),

$$\mathfrak{e}_j(X) \left| \frac{d}{d\kappa} G^{\sim}_{\kappa;m,n-\ell,\ell} \Big|_{\kappa=0} \right|^{\sim}_{\Sigma} \leq (\mathrm{const}\,\gamma)^\ell \ell \binom{n}{\ell} \mathfrak{e}_j(X) Z |G^{\sim}_{m,n}|^{\sim}_{\Sigma}\,.$$

Since $G^{\sim}_{\kappa;m,n,0}$ is independent of κ,

$$\begin{aligned}
&N^{\sim}_j \left(\frac{d}{d\kappa} g'_\kappa \Big|_{\kappa=0}; \alpha; X, \Sigma, \boldsymbol{\rho} \right) \\
&\quad \leq \frac{M^{2j}}{\mathfrak{l}} \mathfrak{e}_j(X) \sum_{m,n\geq 0} \sum_{\ell=1}^{n} \alpha^{m+n} \left(\frac{\mathfrak{l}\mathrm{B}}{M^j} \right)^{(m+n)/2} \left| \frac{d}{d\kappa} G^{\sim}_{\kappa;m,n-\ell,\ell} \Big|_{\kappa=0} \right|^{\sim}_{\Sigma}
\end{aligned}$$

$$\le \frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)Z\sum_{m,n\ge 0}\sum_{\ell=1}^{n}\ell\binom{n}{\ell}(\mathrm{const}\,\gamma)^{\ell}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\tilde{|}_{\Sigma}$$

$$= \frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)Z\sum_{m,n\ge 0}\sum_{\ell=1}^{n}n\binom{n-1}{\ell-1}(\mathrm{const}\,\gamma)^{\ell}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\tilde{|}_{\Sigma}$$

$$= \frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)Z\sum_{m,n\ge 0}\mathrm{const}\,\gamma n(1+\mathrm{const}\,\gamma)^{n-1}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\tilde{|}_{\Sigma}$$

$$\le \frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)Z\sum_{m,n\ge 0}\mathrm{const}\,\gamma 2^{n}\alpha^{m+n}\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{(m+n)/2}|G^{\sim}_{m,n}\tilde{|}_{\Sigma}$$

$$\le \mathrm{const}\,\gamma Z N_j^{\sim}(g;2\alpha;X,\Sigma,\boldsymbol{\rho})\,. \qquad \square$$

Proof of Theorem XVII.6. As in the proof of Theorem XVII.3, let, for a sectorized Grassmann function $v=\sum_n v_n$ with $v_n\in\bigwedge^n\tilde{V}$,

$$N^{\sim}(v;\alpha)=\frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_n\alpha^n\mathrm{b}^n|v_n\tilde{|}_{\Sigma}=\frac{\mathrm{const}_1}{\mathrm{B}}N_j^{\sim}(v;\alpha;X,\Sigma,\boldsymbol{\rho})$$

and

$$:w''_\kappa:_{\psi,\tilde{D}_{\kappa,\Sigma}}=\Omega_{\tilde{C}_{\kappa,\Sigma}}(:w_\kappa:_{\psi,\tilde{C}_{\kappa,\Sigma}+\tilde{D}_{\kappa,\Sigma}})\,.$$

By Proposition XII, parts (ii) and (iii), and [1, Proposition A.2(ii)],

$$w'_\kappa=w''_\kappa(\phi,\psi+\hat{B}_\kappa\phi)$$

is a sectorized representative for $\mathcal{W}'_\kappa$. By the chain rule and the triangle inequality

$$\begin{aligned}N^{\sim}\left(\frac{d}{d\kappa}[w'_\kappa-w_\kappa]_{\kappa=0};\alpha\right)&\le N^{\sim}\left(\frac{d}{d\kappa}w''_0(\phi,\psi+\hat{B}_\kappa\phi)\Big|_{\kappa=0};\alpha\right)\\&\quad+N^{\sim}\left(\frac{d}{d\kappa}[w''_\kappa(\phi,\psi+\hat{B}_0\phi)-w''_\kappa(\phi,\psi)]_{\kappa=0};\alpha\right)\\&\quad+N^{\sim}\left(\frac{d}{d\kappa}[w''_\kappa(\phi,\psi)-w_\kappa(\phi,\psi)]_{\kappa=0};\alpha\right).\end{aligned}\qquad\text{(XVII.10)}$$

By Lemma XVII.8,

$$N^{\sim}\left(\frac{d}{d\kappa}w''_0(\phi,\psi+\hat{B}_\kappa\phi)\Big|_{\kappa=0};\alpha\right)\le\mathrm{const}\,\gamma ZN_j^{\sim}(w''_0;2\alpha;X,\Sigma,\boldsymbol{\rho})\,.$$

By (XVII.5),

$$\begin{aligned}N_j^{\sim}(w''_0;2\alpha;X,\Sigma,\boldsymbol{\rho})&\le N_j^{\sim}(w_0;2\alpha;X,\Sigma,\boldsymbol{\rho})+N_j^{\sim}(w''_0-w_0;2\alpha;X,\Sigma,\boldsymbol{\rho})\\&\le N_j^{\sim}(w_0;2\alpha;X,\Sigma,\boldsymbol{\rho})+\frac{\mathrm{B}}{\mathrm{const}_1}\frac{1}{8\alpha^2}\frac{N^{\sim}(w_0;64\alpha)^2}{1-\frac{1}{4\alpha^2}N^{\sim}(w_0;64\alpha)}\end{aligned}$$

$$\le N_j^{\sim}(w_0; 2\alpha; X, \Sigma, \boldsymbol{\rho}) + \frac{\text{const}}{\alpha^2} \frac{N_j^{\sim}(w_0; 64\alpha; X, \Sigma, \boldsymbol{\rho})^2}{1 - \frac{\text{const}}{\alpha^2} N_j^{\sim}(w_0; 64\alpha; X, \Sigma, \boldsymbol{\rho})}$$

$$\le \text{const}\, \frac{N_j^{\sim}(w_0; 64\alpha; X, \Sigma, \boldsymbol{\rho})}{1 - \frac{\text{const}}{\alpha^2} N_j^{\sim}(w_0; 64\alpha; X, \Sigma, \boldsymbol{\rho})}$$

so that

$$N_j^{\sim}\left(\frac{d}{d\kappa} w_0''(\phi, \psi + \hat{B}_\kappa \phi)\Big|_{\kappa=0}; \alpha; X, \Sigma, \boldsymbol{\rho} \right) \le \text{const}\, \gamma \frac{\mathfrak{n}}{1 - \frac{\text{const}}{\alpha^2}\mathfrak{n}} Z\,. \qquad \text{(XVII.11)}$$

By Lemma XVII.5, with $g = \frac{d}{d\kappa} w_\kappa''|_{\kappa=0}$, $B = B_0$ and $X_B = 1$,

$$N_j^{\sim}\left(\frac{d}{d\kappa}[w_\kappa''(\phi, \psi + \hat{B}_0\phi) - w_\kappa''(\phi, \psi)]_{\kappa=0}; \alpha; X, \Sigma, \boldsymbol{\rho} \right)$$

$$\le \text{const}\, \gamma N_j^{\sim}\left(\frac{d}{d\kappa} w_\kappa''\Big|_{\kappa=0}; 2\alpha; X, \Sigma, \boldsymbol{\rho} \right)$$

$$\le \text{const}\, \gamma N_j^{\sim}\left(\frac{d}{d\kappa} w_\kappa\Big|_{\kappa=0}; 2\alpha; X, \Sigma, \boldsymbol{\rho} \right)$$

$$+ \text{const}\, \gamma N_j^{\sim}\left(\frac{d}{d\kappa}[w_\kappa'' - w_\kappa]_{\kappa=0}; 2\alpha; X, \Sigma, \boldsymbol{\rho} \right)\,. \qquad \text{(XVII.12)}$$

By [1, Theorem IV.4], with $\mu = M^j$ (and assuming that we have chosen $\text{const}_1 \ge 1$),

$$N^{\sim}\left(\frac{d}{d\kappa}[w_\kappa'' - w_\kappa]_{\kappa=0}; \alpha \right)$$

$$\le \frac{1}{2\alpha^2} \frac{N^{\sim}(w_0; 32\alpha)}{1 - \frac{1}{\alpha^2} N^{\sim}(w_0; 32\alpha)} N^{\sim}\left(\frac{d}{d\kappa} w_\kappa\Big|_{\kappa=0}; 8\alpha \right)$$

$$+ \frac{1}{2\alpha^2} \frac{N^{\sim}(w_0; 32\alpha)^2}{1 - \frac{1}{\alpha^2} N^{\sim}(w_0; 32\alpha)} \left\{ \frac{1}{4M^j}\, \text{const}_2\, M^{2j} \mathfrak{e}_j(X) Y + 4\varepsilon \right\}$$

$$\le \frac{\text{const}}{\alpha^2} \frac{\mathfrak{n}}{1 - \frac{\text{const}}{\alpha^2}\mathfrak{n}} \left\{ N^{\sim}\left(\frac{d}{d\kappa} w_\kappa\Big|_{\kappa=0}; 8\alpha \right) + M^j Y \mathfrak{n} + 4\varepsilon\mathfrak{n} \right\} \qquad \text{(XVII.13)}$$

since $\mathfrak{e}_j(X) N^{\sim}(w_0; 32\alpha) \le \text{const}\, N^{\sim}(w_0; 32\alpha)$. Also

$$N_j^{\sim}\left(\frac{d}{d\kappa}[w_\kappa'' - w_\kappa]_{\kappa=0}; 2\alpha; X, \Sigma, \boldsymbol{\rho} \right)$$

$$\le \frac{\text{const}}{\alpha^2} \frac{\mathfrak{n}}{1 - \frac{\text{const}}{\alpha^2}\mathfrak{n}} \left\{ N^{\sim}\left(\frac{d}{d\kappa} w_\kappa\Big|_{\kappa=0}; 16\alpha \right) + M^j Y \mathfrak{n} + 4\varepsilon\mathfrak{n} \right\}.$$

Substituting (XVII.11)–(XVII.13) into (XVII.10),

$$N^{\sim}\left(\frac{d}{d\kappa}[w'_\kappa - w_\kappa]_{\kappa=0};\alpha\right)$$

$$\le \mathrm{const}\,\gamma\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}Z + \mathrm{const}\,\gamma N^{\sim}\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};2\alpha\right)$$

$$+(1+\gamma)\frac{\mathrm{const}}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\left\{N^{\sim}\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha\right)+M^jY\mathfrak{n}+4\varepsilon\mathfrak{n}\right\}$$

$$\le \mathrm{const}\left\{\gamma+\frac{1}{\alpha^2}\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\right\}N_j^{\sim}\left(\frac{d}{d\kappa}w_\kappa\Big|_{\kappa=0};16\alpha;X,\Sigma,\boldsymbol{\rho}\right)$$

$$+\mathrm{const}\,\frac{\mathfrak{n}}{1-\frac{\mathrm{const}}{\alpha^2}\mathfrak{n}}\left\{\frac{1}{\alpha^2}M^jY\mathfrak{n}+\frac{\varepsilon}{\alpha^2}\mathfrak{n}+\gamma Z\right\}. \qquad \square$$

Remark XVII.9. In Theorem XVII.3, the sectorized representative w' of $\mathcal{W}'$ may be obtained from the sectorized representative w of $\mathcal{W}$ by

$$:w':_{\psi,\tilde{D}_\Sigma} = \Omega_{\tilde{C}_\Sigma}(:w:_{\psi,\tilde{C}_\Sigma} + \tilde{D}_\Sigma)(\phi,\psi+\hat{B}\phi)\,.$$

The obvious analog of this statement applies to Theorem XVII.6.

Appendices

D. *Naive ladder estimates*

Let $j \ge 2$ and let Σ be a sectorization of scale j and length $\frac{1}{M^{j-3/2}} \le \mathfrak{l} \le \frac{1}{M^{(j-1)/2}}$. To systematically treat ladders, we introduce an auxiliary channel norm, similar to the $|\cdot|_{2,\Sigma}^{\sim}$ norm, but with only the leftmost momenta held fixed.

Definition D.1. (i) Let $0 \le r \le 2$ and $f \in \check{\mathcal{F}}_r(4-r,\Sigma)$. We set

$$|f|_{ch,\Sigma}^{\sim} = \sup_{\substack{\check\eta_1,\ldots,\check\eta_r\in\check{\mathcal{B}}\\ s_1,\ldots,s_{2-r}\in\Sigma}} \sum_{s_{3-r},s_{4-r}\in\Sigma}\ \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2}\frac{1}{\delta!}\max_{\substack{\mathrm{D}\ \text{dd-operator}\\ \text{with }\delta(\mathrm{D})=\delta}}$$

$$\cdot\, |||\mathrm{D}f(\check\eta_1,\ldots,\check\eta_r;(\xi_1,s_1),\ldots,(\xi_{4-r},s_{4-r}))|||_{1,\infty}t^\delta\,.$$

The norm $|||\cdot|||_{1,\infty}$ of Example II.6 refers to the variables $\xi_1,\ldots,\xi_{4-r}$. If $r=0$, we also write $|f|_{ch,\Sigma}$ instead of $|f|_{ch,\Sigma}^{\sim}$.

(ii) If $f \in \check{\mathcal{F}}_{4,\Sigma}$, we set

$$|f|_{ch,\Sigma}^{\sim} = \sum_{i_1,i_2\in\{0,1\}} |\mathrm{Ord}\,f|_{(i_1,i_2,1,1)}|_{ch,\Sigma}^{\sim}\,.$$

Lemma D.2. *There is a constant* const, *independent of j and M such that the following hold. Let $0 \le r \le 2$ and $f \in \check{\mathcal{F}}_r(4-r,\Sigma)$.*

(i)

$$|f\tilde{|}_{ch,\Sigma} \le |f\tilde{|}_{1,\Sigma} \qquad \textit{if } \ r \le 1$$

$$|f\tilde{|}_{ch,\Sigma} \le \frac{\mathit{const}}{\mathfrak{l}} |f\tilde{|}_{3,\Sigma} \,.$$

(ii)

$$|f\tilde{|}_{4,\Sigma} \le |f\tilde{|}_{3,\Sigma}$$

$$|f\tilde{|}_{3,\Sigma} \le \mathit{const}\, |f\tilde{|}_{4,\Sigma} \,.$$

(iii) *If* $r = 1$ *or if* $r = 0$ *and* f *is antisymmetric, then*

$$|f\tilde{|}_{1,\Sigma} \le \frac{\mathit{const}}{\mathfrak{l}} |f\tilde{|}_{ch,\Sigma} \,.$$

Proof. Set

$$F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D}\ \text{dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \cdot |||\mathrm{D} f(\check\eta_1, \ldots, \check\eta_r; (\xi_1, s_1), \ldots, (\xi_{4-r}, s_{4-r}))|||_{1,\infty} t^\delta \,.$$

(i) Then

$$\begin{aligned} |f\tilde{|}_{ch,\Sigma} &= \sup_{\substack{\check\eta_1,\ldots,\check\eta_r \in \check{\mathcal{B}} \\ s_1,\ldots,s_{2-r} \in \Sigma}} \sum_{s_{3-r}, s_{4-r} \in \Sigma} F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) \\ &\le \sup_{\substack{1 \le i_1 < \cdots < i_{1-r} \le 4-r \\ s_{i_1},\ldots,s_{i_{1-r}} \in \Sigma \\ \check\eta_1,\ldots,\check\eta_r \in \check{\mathcal{B}}}} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \ne i_1,\ldots,i_{1-r}}} F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) = |f\tilde{|}_{1,\Sigma} \end{aligned}$$

if $r \le 1$ and, since Σ contains at most $\frac{\mathit{const}}{\mathfrak{l}}$ elements,

$$\begin{aligned} |f\tilde{|}_{ch,\Sigma} &= \sup_{\substack{\check\eta_1,\ldots,\check\eta_r \in \check{\mathcal{B}} \\ s_1,\ldots,s_{2-r} \in \Sigma}} \sum_{s_{3-r}, s_{4-r} \in \Sigma} F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) \\ &\le \frac{\mathit{const}}{\mathfrak{l}} \sup_{\substack{\check\eta_1,\ldots,\check\eta_r \in \check{\mathcal{B}} \\ s_1,\ldots,s_{3-r} \in \Sigma}} \sum_{s_{4-r} \in \Sigma} F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) \\ &\le \frac{\mathit{const}}{\mathfrak{l}} \sup_{\substack{1 \le i_1 < \cdots < i_{3-r} \le 4-r \\ s_{i_1},\ldots,s_{i_{3-r}} \in \Sigma \\ \check\eta_1,\ldots,\check\eta_r \in \check{\mathcal{B}}}} \sum_{\substack{s_i \in \Sigma \text{ for} \\ i \ne i_1,\ldots,i_{3-r}}} F(\check\eta_1, \ldots, \check\eta_r; s_1, \ldots, s_{4-r}) \\ &= \frac{\mathit{const}}{\mathfrak{l}} |f\tilde{|}_{3,\Sigma} \,. \end{aligned}$$

(ii)

$$
\begin{aligned}
|f\widetilde{|}_{4,\Sigma} &= \sup_{\substack{s_1,\dots,s_{4-r}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) \\
&\le \sup_{\substack{1\le i_1<\cdots<i_{3-r}\le 4-r \\ s_{i_1},\dots,s_{i_{3-r}}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} \sum_{\substack{s_i\in\Sigma \text{ for} \\ i\ne i_1,\dots,i_{3-r}}} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) = |f\widetilde{|}_{3,\Sigma} \\
&\le \mathit{const} \sup_{\substack{s_1,\dots,s_{4-r}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) = \mathit{const}\,|f\widetilde{|}_{4,\Sigma}
\end{aligned}
$$

since, by conservation of momentum, for any fixed $s_{i_1},\dots,s_{i_{3-r}}$ and $\check\eta_1,\dots,\check\eta_r$, there are at most const choices of s_i, $i\ne i_1,\dots,i_{3-r}$ for which $F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r})$ does not vanish.

(iii) If $r=1$ or if $r=0$ and f is antisymmetric, then

$$
\begin{aligned}
|f\widetilde{|}_{1,\Sigma} &= \sup_{\substack{1\le i_1<\cdots<i_{1-r}\le 4-r \\ s_{i_1},\dots,s_{i_{1-r}}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} \sum_{\substack{s_i\in\Sigma \text{ for} \\ i\ne i_1,\dots,i_{1-r}}} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) \\
&= \sup_{\substack{s_1,\dots,s_{1-r}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} \sum_{s_{2-r},\dots,s_{4-r}\in\Sigma} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) \\
&\le \frac{\mathit{const}}{\mathfrak{l}} \sup_{\substack{s_1,\dots,s_{2-r}\in\Sigma \\ \check\eta_1,\dots,\check\eta_r\in\check{\mathcal{B}}}} \sum_{s_{3-r},s_{4-r}\in\Sigma} F(\check\eta_1,\dots,\check\eta_r;s_1,\dots,s_{4-r}) \\
&= \frac{\mathit{const}}{\mathfrak{l}} |f\widetilde{|}_{ch,\Sigma}\,. \qquad \square
\end{aligned}
$$

Corollary D.3. *There is a constant* const, *independent of* j *and* M *such that for all* $f\in\check{\mathcal{F}}_{4;\Sigma}$

$$
|f\widetilde{|}_{ch,\Sigma} \le \frac{\mathit{const}}{\mathfrak{l}} |f\widetilde{|}_{3,\Sigma}\,.
$$

Lemma D.4. *Let* $f_1,f_2\in\check{\mathcal{F}}_{4;\Sigma}$ *and* $c,d\in\mathcal{F}_0(2;\Sigma)$. *Define propagators*

$$
c_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\ne\emptyset \\ t'\cap s'\ne\emptyset}} c((\xi,t),(\xi',t'))
$$

$$
d_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\ne\emptyset \\ t'\cap s'\ne\emptyset}} d((\xi,t),(\xi',t'))
$$

over $\mathcal{B} \times \Sigma$. *Then*

$$|f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|^{\sim}_{1,\Sigma} \le \mathit{const}\, |||d|||_\infty |c|_{1,\Sigma} |f_1|^{\sim}_{1,\Sigma} |f_2|^{\sim}_{1,\Sigma}$$

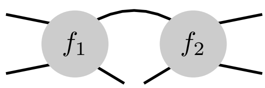
$$|f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|^{\sim}_{ch,\Sigma} \le \mathit{const}\, |||d|||_\infty |c|_{1,\Sigma} |f_1|^{\sim}_{ch,\Sigma} |f_2|^{\sim}_{ch,\Sigma}$$

$$|f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|^{\sim}_{3,\Sigma} \le \mathit{const}\, |||d|||_\infty |c|_{1,\Sigma} |f_1|^{\sim}_{ch,\Sigma} |f_2|^{\sim}_{3,\Sigma}$$

where $|||d|||_\infty = \max_{s,s' \in \Sigma} \sup_{\xi,\xi'} |d((\xi,s),(\xi',s'))|$.

Proof. Set

$$\begin{aligned} f(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s')) &= \sum_{s_1',s_3' \in \Sigma} \int d\zeta_1 d\zeta_3 f_1(\cdot,\cdot,(\zeta_3,s_3'),(\xi,s)) \\ &\quad \cdot c_\Sigma((\zeta_3,s_3'),(\zeta_1,s_1')) f_2((\zeta_1,s_1'),(\xi',s'),\cdot,\cdot) \\ &= \sum_{\substack{s_3',s_3'' \in \Sigma \\ s_1',s_1'' \in \Sigma}} \int d\zeta_1 d\zeta_3 f_1(\cdot,\cdot,(\zeta_3,s_3'),(\xi,s)) \\ &\quad \cdot c((\zeta_3,s_3''),(\zeta_1,s_1'')) f_2((\zeta_1,s_1'),(\xi',s'),\cdot,\cdot)\,. \end{aligned}$$



By iterated application of Lemma XVI.6,

$$|f|^{\sim}_{1,\Sigma} \le \mathit{const}\, ||c|_{1,\Sigma} |f_1|^{\sim}_{1,\Sigma} |f_2|^{\sim}_{1,\Sigma}\,.$$

Since

$$\begin{aligned} f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2 &= \sum_{s,s' \in \Sigma} \int d\xi d\xi' f(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s')) d_\Sigma((\xi,s),(\xi',s')) \\ &= \sum_{\substack{s,s',t,t' \in \Sigma \\ \tilde{s} \cap \tilde{t} \neq \emptyset,\, \tilde{s}' \cap \tilde{t}' \neq \emptyset}} \int d\xi d\xi' f(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s')) d((\xi,t),(\xi',t')) \end{aligned}$$

we have

$$|f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|^{\sim}_{1,\Sigma} \le \mathit{const}\, |||d|||_\infty |f|^{\sim}_{1,\Sigma}\,.$$

This proves the first inequality of the lemma. To prove the third inequality, set

$$g(\cdot,\cdot,\cdot,\cdot;(\xi,s),\xi') = \sum_{\tilde{s}' \cap \tilde{s} \neq \emptyset} f(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s'))\,.$$

By conservation of momentum

$$f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2 = \sum_{\substack{s,t,t' \in \Sigma \\ \tilde{s}\cap\tilde{t}\neq\emptyset, \tilde{t}\cap\tilde{t}'\neq\emptyset}} \int d\xi d\xi' g(\cdot,\cdot,\cdot,\cdot;(\xi,s),\xi') d((\xi,t),(\xi',t')) \,. \quad \text{(D.1)}$$

Fix any $\boldsymbol{\imath} = (i_1,\dots,i_4) \in \{0,1\}^4$ and let

$$f_{\boldsymbol{\imath}}(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s')) = \sum_{\substack{s_3',s_3''\in\Sigma \\ s_1',s_1''\in\Sigma}} \int d\zeta_1 d\zeta_3 f_1|_{(i_1,i_2,1,1)}(\cdot,\cdot,(\zeta_3,s_3'),(\xi,s))$$
$$\cdot\, c((\zeta_3,s_3''),(\zeta_1,s_1''))f_2|_{(1,1,i_3,i_4)}((\zeta_1,s_1'),(\xi',s'),\cdot,\cdot)$$

$$g_{\boldsymbol{\imath}}(\cdot,\cdot,\cdot,\cdot;(\xi,s),\xi') = \sum_{\tilde{s}'\cap\tilde{s}\neq\emptyset} f_{\boldsymbol{\imath}}(\cdot,\cdot,\cdot,\cdot;(\xi,s),(\xi',s'))\,.$$

By (D.1)

$$f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}}$$
$$= \sum_{\substack{s,t,t' \in \Sigma \\ \tilde{s}\cap\tilde{t}\neq\emptyset, \tilde{t}\cap\tilde{t}'\neq\emptyset}} \int d\xi d\xi' g_{\boldsymbol{\imath}}(\cdot,\cdot,\cdot,\cdot;(\xi,s),\xi') d((\xi,t),(\xi',t')) \,. \quad \text{(D.2)}$$

For each $\nu = 1,\dots,4$, fix $\check{\eta}_\nu \in \check{\mathcal{B}}$ when $i_\nu = 0$ and $s_\nu \in \Sigma$ when $i_\nu = 1$. Let

$$z_\nu = \begin{cases} \check{\eta}_\nu & \text{if } i_\nu = 0 \\ (\xi_\nu, s_\nu) & \text{if } i_\nu = 1 \end{cases}$$

and

$$G_{\boldsymbol{\imath}} = \sum_{s\in\Sigma} \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\text{D dd-operator} \\ \text{with } \delta(\text{D})=\delta}} |||\text{D} g_{\boldsymbol{\imath}}(z_1,\dots,z_4;(\xi,s),\xi')|||_{1,\infty} t^\delta \,.$$

By iterated application of Leibniz's rule and Lemma D.2(ii),

$$\begin{aligned} G_{\boldsymbol{\imath}} &\le \mathit{const}\, |c|_{1,\Sigma} |f_1|_{(i_1,i_2,1,1)}\tilde{|}_{ch,\Sigma} |f_2|_{(1,1,i_3,i_4)}\tilde{|}_{4,\Sigma} \\ &\le \mathit{const}\, |c|_{1,\Sigma} |f_1|_{(i_1,i_2,1,1)}\tilde{|}_{ch,\Sigma} |f_2|_{(1,1,i_3,i_4)}\tilde{|}_{3,\Sigma} \\ &\le \mathit{const}\, |c|_{1,\Sigma} |f_1\tilde{|}_{ch,\Sigma} |f_2\tilde{|}_{3,\Sigma} \,. \end{aligned}$$

Furthermore, by (D.2),

$$\sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\text{D dd-operator} \\ \text{with } \delta(\text{D})=\delta}} |||\text{D} f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}}(z_1,\dots,z_4)|||_{1,\infty} t^\delta \le 9 |||d|||_\infty G_{\boldsymbol{\imath}}\,.$$

This, together with Lemma D.2(ii), shows that

$$\begin{aligned} |f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}}\tilde{|}_{3,\Sigma} &\le \mathit{const}\, |f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}}\tilde{|}_{4,\Sigma} \\ &\le \mathit{const}\, |||d|||_\infty |c|_{1,\Sigma} |f_1\tilde{|}_{ch,\Sigma} |f_2\tilde{|}_{3,\Sigma} \,. \end{aligned}$$

The proof of the second inequality is similar to that of the third. Choose $\boldsymbol{\imath} = (i_1, i_2, 1, 1)$ with $i_1, i_2 \in \{0, 1\}$. For each $\nu = 1, 2$, fix $\check\eta_\nu \in \check{\mathcal{B}}$ when $i_\nu = 0$ and $s_\nu \in \Sigma$ when $i_\nu = 1$. Let

$$z_\nu = \begin{cases} \check\eta_\nu & \text{if } i_\nu = 0 \\ (\xi_\nu, s_\nu) & \text{if } i_\nu = 1 \end{cases}$$

and

$$G'_{\boldsymbol{\imath}} = \sum_{s_3, s_4, s \in \Sigma} \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D} \text{ dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \cdot |||\mathrm{D} g_{\boldsymbol{\imath}}(z_1, z_2, (\xi_3, s_3), (\xi_4, s_4); (\xi, s), \xi')|||_{1,\infty} t^\delta \, .$$

By iterated application of Leibniz's rule,

$$\begin{aligned} G'_{\boldsymbol{\imath}} &\le \mathit{const}\, |c|_{1,\Sigma} |f_1|_{(i_1,i_2,1,1)} \tilde{\lceil}_{ch,\Sigma} |f_2|_{(1,1,1,1)} \tilde{\lceil}_{ch,\Sigma} \\ &\le \mathit{const}\, |c|_{1,\Sigma} |f_1\tilde{\lceil}_{ch,\Sigma} |f_2\tilde{\lceil}_{ch,\Sigma} \, . \end{aligned}$$

Furthermore, by (D.2),

$$\begin{aligned} &\sum_{s_3, s_4 \in \Sigma} \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^2} \frac{1}{\delta!} \max_{\substack{\mathrm{D} \text{ dd-operator} \\ \text{with } \delta(\mathrm{D}) = \delta}} \\ &\qquad \cdot |||\mathrm{D} f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}}(z_1, z_2, (\xi_3, s_3), (\xi_4, s_4))|||_{1,\infty} t^\delta \\ &\le 9 |||d|||_\infty G'_{\boldsymbol{\imath}} \, . \end{aligned}$$

This shows that

$$|f_1 \circ (c_\Sigma \otimes d_\Sigma) \circ f_2|_{\boldsymbol{\imath}} \tilde{\lceil}_{ch,\Sigma} \le \mathit{const}\, |||d|||_\infty |c|_{1,\Sigma} |f_1\tilde{\lceil}_{ch,\Sigma} |f_2\tilde{\lceil}_{ch,\Sigma} \, . \qquad \square$$

Lemma D.5. *Let $f_1, f_2 \in \check{\mathcal{F}}_{4;\Sigma}$ be momentum conserving functions. Also let $u, v, u', v' \in \mathcal{F}_0(2; \Sigma)$ be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms obey $|\check u(k)|, |\check v(k)|, |\check u'(k)|, |\check v'(k)| \le \frac{1}{2}|\imath k_0 - e(k)|$. Let $X \in \mathfrak{N}_3$ and $0 < \varepsilon \le 1$ such that*

$$|u|_{1,\Sigma}, |u'|_{1,\Sigma} \le \frac{1}{2M^j} X \qquad |u - u'|_{1,\Sigma} \le \frac{\varepsilon}{M^j} X$$

and

$$|\check u(k) - \check u'(k)| \, , \quad |\check v(k) - \check v'(k)| \le \varepsilon |\imath k_0 - e(k)| \, .$$

Set

$$C(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u(k)} \, , \qquad D(k) = \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check v(k)}$$

$$C'(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check u'(k)} \, , \qquad D'(k) = \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check v'(k)}$$

and let $C(\xi,\xi')$, $D(\xi,\xi')$, $C'(\xi,\xi')$, $D'(\xi,\xi')$ *be their Fourier transforms as in Definition* IX.3. *Assume that* $X_{\mathbf{0}} \le \min\{\tau_1,\tau_2\}$, *where* τ_1 *and* τ_2 *are the constants of Proposition* XIII.5 *and Lemma* XIII.6, *respectively.*

(i)

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2\tilde{|}_{1,\Sigma} \le \text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1\tilde{|}_{1,\Sigma}|f_2\tilde{|}_{1,\Sigma}$$

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2\tilde{|}_{ch,\Sigma} \le \text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1\tilde{|}_{ch,\Sigma}|f_2|_{ch,\Sigma}$$

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2\tilde{|}_{3,\Sigma} \le \text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1\tilde{|}_{ch,\Sigma}|f_2\tilde{|}_{3,\Sigma}\,.$$

(ii)

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2 - f_1 \bullet \mathcal{C}(C',D') \bullet f_2\tilde{|}_{1,\Sigma} \le \text{const}\,\varepsilon\mathfrak{l}\frac{\mathfrak{c}_j(1+X)}{1-X}|f_1\tilde{|}_{1,\Sigma}|f_2\tilde{|}_{1,\Sigma}$$

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2 - f_1 \bullet \mathcal{C}(C',D') \bullet f_2\tilde{|}_{ch,\Sigma} \le \text{const}\,\varepsilon\mathfrak{l}\frac{\mathfrak{c}_j(1+X)}{1-X}|f_1\tilde{|}_{ch,\Sigma}|f_2|_{ch,\Sigma}$$

$$|f_1 \bullet \mathcal{C}(C,D) \bullet f_2 - f_1 \bullet \mathcal{C}(C',D') \bullet f_2\tilde{|}_{3,\Sigma} \le \text{const}\,\varepsilon\mathfrak{l}\frac{\mathfrak{c}_j(1+X)}{1-X}|f_1\tilde{|}_{ch,\Sigma}|f_2\tilde{|}_{3,\Sigma}\,.$$

Proof. Let $c((\cdot,s),(\cdot,s'))$, $d((\cdot,s),(\cdot,s'))$, $c'((\cdot,s),(\cdot,s'))$, $d'((\cdot,s),(\cdot,s'))$ be the Fourier transforms of $\chi_s(k)\,C(k)\,\chi_{s'}(k)$, $\chi_s(k)\,D(k)\,\chi_{s'}(k)$, $\chi_s(k)\,C'(k)\,\chi_{s'}(k)$, $\chi_s(k)\,D'(k)\,\chi_{s'}(k)$ in the sense of Definition IX.3. By Proposition XIII.5(ii) and Lemma XIII.6(i)

$$\begin{aligned}|c|_{1,\Sigma} &\le \text{const}\,\frac{M^j\mathfrak{c}_j}{1-X}\\ |c-c'|_{1,\Sigma} &\le \text{const}\,\varepsilon\frac{M^j\mathfrak{c}_jX}{1-X}\,.\end{aligned} \tag{D.3}$$

For all $s,s'\in\Sigma$, the L^1-norm of $\chi_s(k)D(k)\chi_{s'}(k)$ is bounded by const $\frac{\mathfrak{l}}{M^{2j}}M^j =$ const $\frac{\mathfrak{l}}{M^j}$. The same holds for $\chi_s(k)D'(k)\chi_{s'}(k)$. Also, the L^1-norm of

$$\chi_s(k)D(k)\chi_{s'}(k) - \chi_s(k)D'(k)\chi_{s'}(k) = \chi_s(k)(v(k)-v'(k))D(k)D'(k)\chi_{s'}(k)$$

is bounded by $\varepsilon\,\text{const}\,\frac{\mathfrak{l}}{M^j}$. The same bounds apply when D is replaced by C. Consequently

$$\begin{aligned}&|\!|\!|d|\!|\!|_\infty \le \text{const}\,\frac{\mathfrak{l}}{M^j}\,, \quad |\!|\!|d'|\!|\!|_\infty \le \text{const}\,\frac{\mathfrak{l}}{M^j}\,, \quad |\!|\!|d-d'|\!|\!|_\infty \le \varepsilon\,\text{const}\,\frac{\mathfrak{l}}{M^j}\,,\\ &|\!|\!|c|\!|\!|_\infty \le \text{const}\,\frac{\mathfrak{l}}{M^j}\,, \quad |\!|\!|c'|\!|\!|_\infty \le \text{const}\,\frac{\mathfrak{l}}{M^j}\,, \quad |\!|\!|c-c'|\!|\!|_\infty \le \varepsilon\,\text{const}\,\frac{\mathfrak{l}}{M^j}\,.\end{aligned} \tag{D.4}$$

Also recall from Lemma XVI.12 that if

$$c_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} c((\xi,t),(\xi',t'))$$

$$c'_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} c'((\xi,t),(\xi',t'))$$

$$d_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} d((\xi,t),(\xi',t'))$$

$$d'_\Sigma((\xi,s),(\xi',s')) = \sum_{\substack{t\cap s\neq\emptyset\\ t'\cap s'\neq\emptyset}} d'((\xi,t),(\xi',t'))$$

then

$$f_1 \bullet (C\otimes D)\bullet f_2 = f_1\circ(c_\Sigma\otimes d_\Sigma)\circ f_2\,,\quad f_1\bullet(C'\otimes D')\bullet f_2 = f_1\circ(c'_\Sigma\otimes d'_\Sigma)\circ f_2\,.$$

To prove the first inequality in part (i), observe that by Lemma D.4, (D.3) and (D.4)

$$|f_1\circ(c_\Sigma\otimes d_\Sigma)\circ f_2|^{\sim}_{1,\Sigma} \le \mathrm{const}\,\frac{M^j\mathfrak{c}_j}{1-X}\frac{\mathfrak{l}}{M^j}|f_1|^{\sim}_{1,\Sigma}|f_2|^{\sim}_{1,\Sigma} = \mathrm{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1|^{\sim}_{1,\Sigma}|f_2|^{\sim}_{1,\Sigma}\,.$$

Similarly

$$|f_1\circ(d_\Sigma\otimes c_\Sigma)\circ f_2|^{\sim}_{1,\Sigma} \le \mathrm{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1|^{\sim}_{1,\Sigma}|f_2|^{\sim}_{1,\Sigma}$$

$$|f_1\circ(c_\Sigma\otimes c_\Sigma)\circ f_2|^{\sim}_{1,\Sigma} \le \mathrm{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}|f_1|^{\sim}_{1,\Sigma}|f_2|^{\sim}_{1,\Sigma}\,.$$

By Definition XIV.1(iii), $\mathcal{C}(C,D) = C\otimes D + D\otimes C + C\otimes C$. Therefore, the first inequality of part (i) follows. The proof of the other inequalities in part (i) is similar.

To prove the first inequality of part (ii), it suffices by Definition XIV.1(iii) to bound each of the quantities

$$|f_1\bullet(C\otimes D)\bullet f_2 - f_1\bullet(C'\otimes D')\bullet f_2|^{\sim}_{1,\Sigma}$$

$$|f_1\bullet(D\otimes C)\bullet f_2 - f_1\bullet(D'\otimes C')\bullet f_2|^{\sim}_{1,\Sigma}$$

$$|f_1\bullet(C\otimes C)\bullet f_2 - f_1\bullet(C'\otimes C')\bullet f_2|^{\sim}_{1,\Sigma}$$

by $\mathrm{const}\,\varepsilon\mathfrak{l}\frac{\mathfrak{c}_j(1+X)}{1-X}|f_1|^{\sim}_{1,\Sigma}|f_2|^{\sim}_{1,\Sigma}$. Again we only bound the first quantity; the other two are similar. As above, by Lemma D.4, (D.3) and (D.4)

$$\begin{aligned}&|f_1\bullet(C\otimes D)\bullet f_2 - f_1\bullet(C'\otimes D')\bullet f_2|^{\sim}_{1,\Sigma}\\ &\qquad\le |f_1\circ(c_\Sigma\otimes(d_\Sigma - d'_\Sigma))\circ f_2|^{\sim}_{1,\Sigma} + |f_1\circ((c_\Sigma - c'_\Sigma)\otimes d'_\Sigma)\circ f_2|^{\sim}_{1,\Sigma}\end{aligned}$$

$$\le \text{const}\left(\frac{M^j\mathfrak{c}_j}{1-X}\varepsilon\frac{\mathfrak{l}}{M^j}+\varepsilon\frac{M^j\mathfrak{c}_j X}{1-X}\frac{\mathfrak{l}}{M^j}\right)|f_1|\tilde{}_{1,\Sigma}|f_2|\tilde{}_{1,\Sigma}$$

$$\le \text{const}\,\varepsilon\mathfrak{l}\frac{\mathfrak{c}_j(1+X)}{1-X}|f_1|\tilde{}_{1,\Sigma}|f_2|\tilde{}_{1,\Sigma}\,.$$

The proof of the other inequalities in part (ii) of the lemma is similar. □

Corollary D.6. *Let $f \in \check{\mathcal{F}}_{4;\Sigma}$ and let C, D, C', D' be as in Lemma* D.5. *Then*

(i)

$$|L_\ell(f;C,D)|\tilde{}_{1,\Sigma} \le \left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^\ell |f|\tilde{}^{\,\ell+1}_{1,\Sigma}$$

$$|L_\ell(f;C,D)|\tilde{}_{ch,\Sigma} \le \left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^\ell |f|\tilde{}^{\,\ell+1}_{ch,\Sigma}\,.$$

(ii)

$$|L_\ell(f;C,D)-L_\ell(f;C',D')|\tilde{}_{1,\Sigma} \le \varepsilon(1+X)\left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^\ell |f|\tilde{}^{\,\ell+1}_{1,\Sigma}$$

$$|L_\ell(f;C,D)-L_\ell(f;C',D')|\tilde{}_{ch,\Sigma} \le \varepsilon(1+X)\left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^\ell |f|\tilde{}^{\,\ell+1}_{ch,\Sigma}\,.$$

Proof. Part (i) follows by induction on ℓ from the first two inequalities of Lemma D.5(i) using

$$L_\ell(f;C,D) = L_{\ell-1}(f;C,D)\bullet\mathcal{C}(C,D)\bullet f\,. \tag{D.5}$$

To prove part (ii), observe that

$$\begin{aligned} &L_\ell(f;C,D)-L_\ell(f;C',D')\\ &\quad= [L_{\ell-1}(f;C,D)-L_{\ell-1}(f;C',D')]\bullet\mathcal{C}(C,D)\bullet f\\ &\qquad+L_{\ell-1}(f;C',D')\bullet[\mathcal{C}(C,D)-\mathcal{C}(C',D')]\bullet f \end{aligned} \tag{D.6}$$

and again apply induction on ℓ, using part (i) and the first two inequalities of Lemma D.5(ii). □

Proposition D.7. *Let $f \in \check{\mathcal{F}}_{4;\Sigma}$. Also let $u, v, u', v' \in \mathcal{F}_0(2;\Sigma)$ be antisymmetric, spin independent, particle number conserving functions whose Fourier transforms obey $|\check{u}(k)|, |\check{v}(k)|, |\check{u}'(k)|, |\check{v}'(k)| \le \frac{1}{2}|\imath k_0 - e(k)|$. Let $X \in \mathfrak{N}_3$ and $0 < \varepsilon \le 1$ such that*

$$|u|_{1,\Sigma}, |u'|_{1,\Sigma} \le \frac{1}{2M^j}X \qquad |u-u'|_{1,\Sigma} \le \frac{\varepsilon}{M^j}X$$

and

$$|\check{u}(k) - \check{u}'(k)|\,, \quad |\check{v}(k) - \check{v}'(k)| \le \varepsilon |\imath k_0 - e(k)|\,.$$

Set

$$C(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)}\,, \qquad D(k) = \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}(k)}$$

$$C'(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}'(k)}\,, \qquad D'(k) = \frac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{v}'(k)}$$

and let $C(\xi,\xi')$, $D(\xi,\xi')$, $C'(\xi,\xi')$, $D'(\xi,\xi')$ *be their Fourier transforms as in Definition* IX.3. *Assume that* $X_{\mathbf{0}} \le \min\{\tau_1, \tau_2\}$, *where* τ_1 *and* τ_2 *are the constants of Proposition* XIII.5 *and Lemma* XIII.6, *respectively. Then for all* $\ell \ge 1$

(i)

$$|L_\ell(f;C,D)|^{\sim}_{1,\Sigma} \le \left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^{\ell} |f|^{\sim\,\ell+1}_{1,\Sigma}$$

$$|L_\ell(f;C,D)|^{\sim}_{3,\Sigma} \le \left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^{\ell} |f|^{\sim\,\ell}_{ch,\Sigma} |f|^{\sim}_{3,\Sigma}\,.$$

(ii)

$$|L_\ell(f;C,D) - L_\ell(f;C',D')|^{\sim}_{3,\Sigma} \le \varepsilon(1+X)\left(\text{const}\,\frac{\mathfrak{c}_j}{1-X}\right)^{\ell} |f|^{\sim\,\ell+1}_{3,\Sigma}\,.$$

Proof. The first inequality of part (i) was already stated in Corollary D.6(i). By (D.5), the second inequality of part (i) follows from Corollary D.6(i) and the third inequality of Lemma D.5(i).

With an argument as above, using Lemma D.5 and Corollary D.6, one deduces from (D.6) that

$$|L_\ell(f;C,D) - L_\ell(f;C',D')|^{\sim}_{3,\Sigma} \le \varepsilon(1+X)\left(\text{const}\,\frac{\mathfrak{l}\,\mathfrak{c}_j}{1-X}\right)^{\ell} |f|^{\sim\,\ell}_{ch,\Sigma} |f|^{\sim}_{3,\Sigma}$$

the claim now follows Corollary D.3. □

Remark D.8. Using Corollary D.3, one also sees that in the situation of Proposition D.7,

$$|L_\ell(f;C,D)|^{\sim}_{3,\Sigma} \le \left(\text{const}\,\frac{\mathfrak{c}_j}{1-X}\right)^{\ell} |f|^{\sim\,\ell+1}_{3,\Sigma}\,.$$

Notation

Norms

Norm	Characteristics	Reference
$\vert\vert\vert \cdot \vert\vert\vert_{1,\infty}$	no derivatives, external positions, acts on functions	Example II.6
$\\|\cdot\\|_{1,\infty}$	derivatives, external positions, acts on functions	Example II.6
$\\|\cdot\\|^{\sim}_{\infty}$	derivatives, external momenta, acts on functions	Definition IV.6
$\vert\vert\vert \cdot \vert\vert\vert_{\infty}$	no derivatives, external positions, acts on functions	Example III.4
$\\|\cdot\\|^{\sim}_{1}$	derivatives, external momenta, acts on functions	Definition IV.6
$\\|\cdot\\|^{\sim}_{\infty,B}$	derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$	Definition IV.6
$\\|\cdot\\|^{\sim}_{1,B}$	derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$	Definition IV.6
$\\|\cdot\\|$	$\rho_{m;n}\\|\cdot\\|_{1,\infty}$	Lemma V.1
$N(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha)$	$\frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n\ge 0}\alpha^n\mathrm{b}^n\\|\mathcal{W}_{m,n}\\|$	Definition III.9
		Theorem V.2
$N_0(\mathcal{W};\beta;X,\boldsymbol{\rho})$	$\mathfrak{e}_0(X)\sum_{m+n\in 2\mathbb{N}}\beta^n\rho_{m;n}\\|\mathcal{W}_{m,n}\\|_{1,\infty}$	Theorem VIII.6
$\\|\cdot\\|_{L^1}$	derivatives, acts on functions on $\mathbb{R}\times\mathbb{R}^d$	before Lemma IX.6
$\\|\cdot\\|^{\sim}$	derivatives, external momenta, acts on functions	Definition X.4
$N_0^{\sim}(\mathcal{W};\beta;X,\boldsymbol{\rho})$	$\mathfrak{e}_0(X)\sum_{m+n\in 2\mathbb{N}}\beta^{m+n}\rho_{m;n}\\|W^{\sim}_{m,n}\vert^{\sim}$	before Lemma X.11
$\vert\cdot\vert^{\sim}$	like $\rho_{m;n}\\|\cdot\\|^{\sim}$ but acts on $\tilde{V}^{\otimes n}$	Theorem X.12
$N^{\sim}(\mathcal{W};\mathfrak{c},\mathrm{b},\alpha)$	$\frac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n}\alpha^{m+n}\mathrm{b}^{m+n}\vert W^{\sim}_{m,n}\vert^{\sim}$	Theorem X.12
$\vert\cdot\vert_{p,\Sigma}$	derivatives, external positions, all but p sectors summed	Definition XII.9
$\vert\varphi\vert_{\Sigma}$	$\rho_{m;n}\begin{cases}\vert\varphi\vert_{1,\Sigma}+\frac{1}{\mathfrak{l}}\vert\varphi\vert_{3,\Sigma}+\frac{1}{\mathfrak{l}^2}\vert\varphi\vert_{5,\Sigma} & \text{if } m=0\\\\ \frac{\mathfrak{l}}{M^{2j}}\vert\varphi\vert_{1,\Sigma} & \text{if } m\ne 0\end{cases}$	Definition XV.1
$N_j(w;\alpha;X,\Sigma,\boldsymbol{\rho})$	$\frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)\sum_{m,n\ge 0}\alpha^n\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2}\vert w_{m,n}\vert_{\Sigma}$	Definition XV.1
$\vert\cdot\vert^{\sim}_{p,\Sigma}$	derivatives, external momenta, all but p sectors summed	Definition XVI.4
$\vert\cdot\vert^{\sim}_{p,\Sigma,\boldsymbol{\rho}}$	weighted variant of $\vert\cdot\vert^{\sim}_{p,\Sigma}$	Definition XVII.1(i)
$\vert f\vert^{\sim}_{\Sigma}$	$\rho_{m;n}\begin{cases}\vert f\vert^{\sim}_{1,\Sigma}+\frac{1}{\mathfrak{l}}\vert f\vert^{\sim}_{3,\Sigma}+\frac{1}{\mathfrak{l}^2}\vert f\vert^{\sim}_{5,\Sigma} & \text{if } m=0\\\\ \sum_{p=1}^{6}\frac{1}{\mathfrak{l}^{[(p-1)/2]}}\vert f\vert^{\sim}_{p,\Sigma} & \text{if } m\ne 0\end{cases}$	Definition XVII.1(ii)
$N_j^{\sim}(w;\alpha;X,\Sigma,\boldsymbol{\rho})$	$\frac{M^{2j}}{\mathfrak{l}}\mathfrak{e}_j(X)\sum_{n\ge 0}\alpha^n\left(\frac{\mathfrak{l}\mathrm{B}}{M^j}\right)^{n/2}\vert f_n\vert^{\sim}_{\Sigma}$	Definition XVII.1(iii)
$\vert\cdot\vert^{\sim}_{ch,\Sigma}$	channel variant of $\vert\cdot\vert^{\sim}_{2,\Sigma}$ for ladders	Definition D.1
$\vert\cdot\vert_{ch,\Sigma}$	channel variant of $\vert\cdot\vert_{2,\Sigma}$ for ladders	Definition D.1

Other notation

Notation	Description	Reference
$\Omega_S(\mathcal{W})(\phi,\psi)$	$\log \frac{1}{Z}\int e^{\mathcal{W}(\phi,\psi+\zeta)}d\mu_S(\zeta)$	before (I.6)
J	particle/hole swap operator	(VI.1)
$\tilde\Omega_C(\mathcal{W})(\phi,\psi)$	$\log \frac{1}{Z}\int e^{\phi J\zeta}e^{\mathcal{W}(\phi,\psi+\zeta)}d\mu_C(\zeta)$	Definition VII.1
r_0	number of k_0 derivatives tracked	Sec. VI
r	number of $\mathbf{k}$ derivatives tracked	Sec. VI
M	scale parameter, $M>1$	before Definition VIII.1
$const$	generic constant, independent of scale	
const	generic constant, independent of scale and M	
$\nu^{(j)}(k)$	jth scale function	Definition VIII.1
$\tilde\nu^{(j)}(k)$	jth extended scale function	Definition VIII.4(i)
$\nu^{(\ge j)}(k)$	$\varphi(M^{2j-1}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.1
$\tilde\nu^{(\ge j)}(k)$	$\varphi(M^{2j-2}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.4(ii)
$\bar\nu^{(\ge j)}(k)$	$\varphi(M^{2j-3}(k_0^2+e(\mathbf{k})^2))$	Definition VIII.4(iii)
$\mathfrak{l}$	length of sectors	Definition XII.1
Σ	sectorization	Definition XII.1
$S(C)$	$\sup_m \sup_{\xi_1,\dots,\xi_m\in\mathcal{B}} \left(\left\vert\int \psi(\xi_1)\cdots\psi(\xi_m)d\mu_C(\psi)\right\vert\right)^{1/m}$	Definition IV.1
B	j-independent constant	Definitions XV.1, XVII.1
$\mathfrak{c}_j$	$=\sum_{\substack{\vert\boldsymbol{\delta}\vert\le r\\ \vert\delta_0\vert\le r_0}} M^{j\vert\delta\vert}t^\delta + \sum_{\substack{\vert\boldsymbol{\delta}\vert>r\\ \text{or }\vert\delta_0\vert>r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}$	Definition XII.2
$\mathfrak{e}_j(X)$	$=\frac{\mathfrak{c}_j}{1-M^jX}$	Definition XV.1(ii)
$*$	convolution	before (XIII.6)
$\circ$	ladder convolution	Definition XIV.1(iv)
$\bullet$	ladder convolution	Definitions XIV.3, XVI.9
$\check f$	Fourier transform	Definition IX.1(i)
$\check u$	Fourier transform for sectorized u	Definition XII(iv)
$f^\sim$	partial Fourier transform	Definition IX.1(ii)
$\hat\chi$	Fourier transform	Definition IX.4
$\mathcal{B}$	$\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\times\{0,1\}$ viewed as position space	beginning of Sec. II
$\check{\mathcal{B}}$	$\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\times\{0,1\}$ viewed as momentum space	beginning of Sec. IX
$\check{\mathcal{B}}_m$	$\{(\check\eta_1,\dots,\check\eta_m)\in\check{\mathcal{B}}^m \vert \check\eta_1+\cdots+\check\eta_m=0\}$	before Definition X.1
$\mathfrak{X}_\Sigma$	$\check{\mathcal{B}}\cup(\mathcal{B}\times\Sigma)$	Definition XVI.1
$\mathcal{F}_m(n)$	functions on $\mathcal{B}^m\times\mathcal{B}^n$, antisymmetric in $\mathcal{B}^m$ arguments	Definition II.9
$\check{\mathcal{F}}_m(n)$	functions on $\check{\mathcal{B}}^m\times\mathcal{B}^n$, antisymmetric in $\check{\mathcal{B}}^m$ arguments	Definition X.8
$\mathcal{F}_m(n;\Sigma)$	functions on $\mathcal{B}^m\times(\mathcal{B}\times\Sigma)^n$, internal momenta in sectors	Definition XII(ii)
$\check{\mathcal{F}}_m(n;\Sigma)$	functions on $\check{\mathcal{B}}^m\times(\mathcal{B}\times\Sigma)^n$, internal momenta in sectors	Definition XVI.7(i)
$\check{\mathcal{F}}_{n;\Sigma}$	functions on $\mathfrak{X}_\Sigma^n$ that reorder to $\check{\mathcal{F}}_m(n-m;\Sigma)$'s	Definition XVI.7(iii)

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