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# SINGLE SCALE ANALYSIS OF MANY FERMION SYSTEMS PART 1: INSULATORS

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We construct, using fermionic functional integrals, thermodynamic Green's functions for a weakly coupled fermion gas whose Fermi energy lies in a gap. Estimates on the Green's functions are obtained that are characteristic of the size of the gap. This prepares the way for the analysis of single scale renormalization group maps for a system of fermions at temperature zero without a gap.

*Keywords*: Fermi liquid; renormalization; fermionic functional integral; insulator.

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## I. Introduction to Part 1

Consider a gas of fermions with prescribed, strictly positive, density, together with a crystal lattice of magnetic ions in $d$ space dimensions. The fermions interact with each other through a two-body potential. The lattice provides periodic scalar and vector background potentials. As well, the ions can oscillate, generating phonons and then the fermions interact with the phonons.

To start, turn off the fermion–fermion and fermion–phonon interactions. Then we have a gas of independent fermions, each with Hamiltonian

$$H_0 = \frac{1}{2m}(i\boldsymbol{\nabla} + \mathbf{A}(\mathbf{x}))^2 + \mathcal{U}(\mathbf{x})\,.$$

Assume that the vector and scalar potentials $\mathbf{A}$, $\mathcal{U}$ are periodic with respect to some lattice $\Gamma$ in $\mathbb{R}^d$. Note that it is the magnetic potential, and not just the magnetic field, that is assumed to be periodic. This forces the magnetic field to have mean zero. Here, bold face characters are $d$-component vectors. Because the Hamiltonian commutes with lattice translations it is possible to simultaneously diagonalize the Hamiltonian and the generators of lattice translations. Call the eigenvalues and eigenvectors $\varepsilon_\nu(\mathbf{k})$ and $\phi_{\nu,\mathbf{k}}(\mathbf{x})$ respectively. They obey

$$\begin{aligned} H_0\phi_{\nu,\mathbf{k}}(\mathbf{x}) &= \varepsilon_\nu(\mathbf{k})\phi_{\nu,\mathbf{k}}(\mathbf{x}) \\ \phi_{\nu,\mathbf{k}}(\mathbf{x}+\boldsymbol{\gamma}) &= e^{i\langle\mathbf{k},\boldsymbol{\gamma}\rangle}\phi_{\nu,\mathbf{k}}(\mathbf{x}) \qquad \forall\,\boldsymbol{\gamma}\in\Gamma\,. \end{aligned} \tag{I.1}$$

The crystal momentum $\mathbf{k}$ runs over $\mathbb{R}^2/\Gamma^\#$ where

$$\Gamma^\# = \{\mathbf{b}\in\mathbb{R}^2 | \langle\mathbf{b},\boldsymbol{\gamma}\rangle \in 2\pi\mathbb{Z} \text{ for all } \boldsymbol{\gamma}\in\Gamma\}$$

is the dual lattice to $\Gamma$. The band index $\nu\in\mathbb{N}$ just labels the eigenvalues for boundary condition $\mathbf{k}$ in increasing order. When $\mathbf{A}=\mathcal{U}=0$, $\varepsilon_\nu(\mathbf{k}) = \frac{1}{2m}(\mathbf{k}-\mathbf{b}_{\nu,\mathbf{k}})^2$ for some $\mathbf{b}_{\nu,\mathbf{k}}\in\Gamma^\#$.

In the grand canonical ensemble, the Hamiltonian $H$ is replaced by $H-\mu N$ where $N$ is the number operator and the chemical potential $\mu$ is used to control the density of the gas. At very low temperature, which is the physically interesting domain, only those pairs $\nu,\mathbf{k}$ for which $\varepsilon_\nu(\mathbf{k})\approx\mu$ are important. To keep things as simple as possible, we assume that $\varepsilon_\nu(\mathbf{k})\approx\mu$ only for one value $\nu_0$ of $\nu$ and we fix an ultraviolet cutoff so that we consider only those crystal momenta in a region B for which $|\varepsilon_{\nu_0}(\mathbf{k})-\mu|$ is smaller than some fixed small constant. We denote $E(\mathbf{k}) = \varepsilon_{\nu_0}(\mathbf{k})-\mu$.

When the fermion–fermion and fermion–phonon interactions are turned on, the models at temperature zero are characterized by the Euclidean Green's functions, formally defined by

$$\begin{aligned} &G_{2n}(p_1,\dots,q_n)(2\pi)^{d+1}\delta(\Sigma p_i - \Sigma q_i) \\ &\quad = \left\langle \prod_{i=1}^n \psi_{p_i}\bar\psi_{q_i}\right\rangle = \frac{\int(\prod_{i=1}^n \psi_{p_i}\bar\psi_{q_i})e^{\mathcal{A}(\psi,\bar\psi)}\prod_{k,\sigma}d\psi_{k,\sigma}d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)}\prod_{k,\sigma}d\psi_{k,\sigma}d\bar\psi_{k,\sigma}}\,. \end{aligned} \tag{I.2}$$

The action

$$\mathcal{A}(\psi,\bar{\psi}) = -\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\ (ik_0 - E(\mathbf{k}))\bar{\psi}_k\psi_k + \mathcal{V}(\psi,\bar{\psi})\,. \tag{I.3}$$

The interaction $\mathcal{V}$ will be specified shortly. We prefer to split $\mathcal{A} = \mathcal{Q} + \mathcal{V}$ where $\mathcal{Q} = -\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\ (ik_0 - E(\mathbf{k}))\bar{\psi}_k\psi_k$ and write

$$\begin{aligned}\langle f(\psi,\bar{\psi})\rangle &= \frac{\int f(\psi,\bar{\psi})e^{\mathcal{A}(\psi,\bar{\psi})}\prod_{k,\sigma} d\psi_{k,\sigma}d\bar{\psi}_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar{\psi})}\prod_{k,\sigma} d\psi_{k,\sigma}d\bar{\psi}_{k,\sigma}}\\ &= \frac{\int f(\psi,\bar{\psi})e^{\mathcal{V}(\psi,\bar{\psi})}d\mu_C(\psi,\bar{\psi})}{\int e^{\mathcal{V}(\psi,\bar{\psi})}d\mu_C(\psi,\bar{\psi})}\end{aligned}$$

where $d\mu_C$ is the Grassmann Gaussian "measure" with covariance

$$C(k) = \frac{1}{ik_0 - E(\mathbf{k})}\,.$$

We now take some time to explain (I.2). The fermion fields are vectors

$$\psi_k = \begin{bmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{bmatrix} \qquad \bar{\psi}_k = [\bar{\psi}_{k,\uparrow}\quad \bar{\psi}_{k,\downarrow}]$$

whose components $\psi_{k,\sigma},\bar{\psi}_{k,\sigma},\ k = (k_0,\mathbf{k}) \in \mathbb{R}\times \mathrm{B}, \sigma \in \{\uparrow,\downarrow\}$, are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other.

$$\overset{(-)}{\psi}_{k,\sigma}\overset{(-)}{\psi}_{p,\tau} = -\overset{(-)}{\psi}_{p,\tau}\overset{(-)}{\psi}_{k,\sigma}\ .$$

We have deliberately chosen $\bar{\psi}$ to be a row vector and $\psi$ to be a column vector so that

$$\bar{\psi}_k\psi_p = \bar{\psi}_{k,\uparrow}\psi_{p,\uparrow} + \bar{\psi}_{k,\downarrow}\psi_{p,\downarrow} \qquad \psi_k\bar{\psi}_p = \begin{bmatrix}\psi_{k,\uparrow}\bar{\psi}_{p,\uparrow} & \psi_{k,\uparrow}\bar{\psi}_{p,\downarrow}\\ \psi_{k,\downarrow}\bar{\psi}_{p,\uparrow} & \psi_{k,\downarrow}\bar{\psi}_{p,\downarrow}\end{bmatrix}.$$

In the argument $k = (k_0,\mathbf{k})$, the last $d$ components $\mathbf{k}$ are to be thought of as a crystal momentum and the first component $k_0$ as the dual variable to a temperature or imaginary time. Hence the $\sqrt{-1}$ in $ik_0 - E(\mathbf{k})$. Our ultraviolet cutoff restricts $\mathbf{k}$ to B. In the full model, $\mathbf{k}$ is replaced by $(\nu,\mathbf{k})$ with $\nu$ summed over $\mathbb{N}$ and $\mathbf{k}$ integrated over $\mathbb{R}^d/\Gamma^\#$. On the other hand, the ultraviolet cutoff does not restrict $k_0$ at all. It still runs over $\mathbb{R}$. So we could equally well express the model in terms of a Hamiltonian acting on a Fock space. We find the functional integral notation more efficient, so we use it. The relationship between the position space field $\psi_\sigma(x_0,\mathbf{x})$, with $(x_0,\mathbf{x})$ running over (imaginary) time $\times$ space, and the momentum space field $\psi_{k,\sigma}$ is really given, in our single band approximation, by

$$\psi_\sigma(x_0,\mathbf{x}) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}}e^{-ik_0x_0}\overline{\phi_{\nu_0,\mathbf{k}}(\mathbf{x})}\psi_{k,\sigma}\,.$$

We find it convenient to use a conventional Fourier transform, so we work in a "pseudo" space–time and instead define

$$\psi_\sigma(x_0, \mathbf{x}) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k,x\rangle_-} \psi_{k,\sigma}$$

$$\bar\psi_\sigma(x_0, \mathbf{x}) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{-\imath\langle k,x\rangle_-} \bar\psi_{k,\sigma}$$

where $\langle k, x\rangle_- = -k_0x_0 + \mathbf{k}\cdot\mathbf{x}$ for $k = (k_0, \mathbf{k}) \in \mathbb{R}\times\mathbb{R}^d$. Under this convention, the covariance in position space is

$$\begin{aligned} C(x, x') &= \int \psi(x)\bar\psi(x')d\mu_C(\psi, \bar\psi) \\ &= \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k,x-x'\rangle_-} C(k)\,. \end{aligned}$$

Under suitable conditions on $\phi_{\nu_0,\mathbf{k}}(\mathbf{x})$, it is easy to go from the pseudo space–time $\psi(x)$ to the real one.

For a simple spin independent two-body fermion–fermion interaction, with no phonon interaction,

$$\mathcal{V} = -\frac{1}{2} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \int dtd\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})\bar\psi_\sigma(t,\mathbf{x})\psi_\sigma(t,\mathbf{x})\bar\psi_\tau(t,\mathbf{y})\psi_\tau(t,\mathbf{y})\,.$$

The general form of the interaction is

$$\mathcal{V}(\psi, \bar\psi) = \int_{(\mathbb{R}\times\mathbb{R}^2\times\{\uparrow,\downarrow\})^4} V_0(x_1, x_2, x_3, x_4)\bar\psi(x_1)\psi(x_2)\bar\psi(x_3)\psi(x_4)dx_1dx_2dx_3dx_4$$

where, for $x = (x_0, \mathbf{x}, \sigma)$, we write $\psi(x) = \psi_\sigma(x_0, \mathbf{x})$ and $\bar\psi(x) = \bar\psi_\sigma(x_0, \mathbf{x})$. The translation invariant function $V(x_1, x_2, x_3, x_4)$ can implement both the fermion–fermion and fermion–phonon interactions.

This series of four papers provides part of the construction of an interacting Fermi liquid at temperature zero[a] in $d = 2$ space dimensions.[b] Before we give the description of the content of these four papers, we outline the main results of the full construction. For the detailed hypotheses and results, see [5]. The main assumptions concerning the interaction are contained in

**Hypothesis I.1.** The interaction is weak and short range. That is, $V_0$ is sufficiently near the origin in $\mathfrak{V}$, which is a Banach space of fairly short range, spin independent, translation invariant functions $V_0(x_1, x_2, x_3, x_4)$. See [5, Theorem I.4] for $\mathfrak{V}$'s precise norm.

For some results, we also assume that $V_0$ is "$k_0$-reversal real"

$$V_0(Rx_1, Rx_2, Rx_3, Rx_4) = \overline{V_0(x_1, x_2, x_3, x_4)} \tag{I.4}$$

[a]For results at strictly positive temperature see [1–3].
[b]For $d = 1$, the corresponding system is a Luttinger liquid. See [4] and the references therein.

where $R(x_0, \mathbf{x}, \sigma) = (x_0, -\mathbf{x}, \sigma)$ and "bar/unbar exchange invariant"

$$V_0(-x_2, -x_1, -x_4, -x_3) = V_0(x_1, x_2, x_3, x_4) \tag{I.5}$$

where $-(x_0, \mathbf{x}, \sigma) = (-x_0, -\mathbf{x}, \sigma)$. If $V_0$ corresponds to a two-body interaction $v(\mathbf{x}_1 - \mathbf{x}_3)$ with a real-valued Fourier transform, then $V_0$ obeys (I.4) and (I.5).

We prove that perturbation expansions for various objects converge. These objects depend on both $E(\mathbf{k})$ and $V_0$ and are **not** smooth in $V_0$ when $E(\mathbf{k})$ is held fixed. However, we can recover smoothness in $V_0$ by a change of variables. To do so, we split $E(\mathbf{k}) = e(\mathbf{k}) - \delta e(V_0, \mathbf{k})$ into two parts and choose $\delta e(V_0, \mathbf{k})$ to satisfy an implicit renormalization condition. This is called renormalization of the dispersion relation. Define the proper self energy $\Sigma(p)$ for the action $\mathcal{A}$ by the equation

$$(ip_0 - e(\mathbf{p}) - \Sigma(p))^{-1}\ (2\pi)^{d+1}\delta(p-q) = \frac{\int \psi_p \bar\psi_q\ e^{\mathcal{A}(\psi,\bar\psi)}\ \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)}\ \prod d\psi_{k,\sigma} d\bar\psi_{k,\sigma}}\,.$$

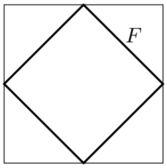
The counterterm $\delta e(V_0, \mathbf{k})$ is chosen so that $\Sigma(0, \mathbf{p})$ vanishes on the Fermi surface $F = \{\mathbf{p} | e(\mathbf{p}) = 0\}$. We take $e(\mathbf{k})$ and $V_0$, rather than the more natural, $E(\mathbf{k})$ and $V_0$ as input data. The counterterm $\delta e$ will be an output of our main theorem. It will lie in a suitable Banach space $\mathcal{E}$. While the problem of inverting the map $e \mapsto E = e - \delta e$ is reasonably well understood on a perturbative level [6], our estimates are not yet good enough to do so nonperturbatively. Our main hypotheses are imposed on $e(\mathbf{k})$.

**Hypothesis I.2.** The dispersion relation $e(\mathbf{k})$ is a real-valued, sufficiently smooth, function. We further assume that

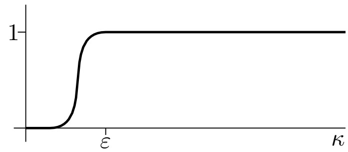
(a) the Fermi curve $F = \{\mathbf{k} \in \mathbb{R}^2 | e(\mathbf{k}) = 0\}$ is a simple closed, connected, convex curve with nowhere vanishing curvature.
(b) $\nabla e(\mathbf{k})$ does not vanish on $F$.
(c) For each $\mathbf{q} \in \mathbb{R}^2$, $F$ and $-F + \mathbf{q}$ have low degree of tangency. ($F$ is "strongly asymmetric".) Here $-F + q = \{-\mathbf{k} + \mathbf{q} | \mathbf{k} \in F\}$.

Again, for the details, see [5, Hypothesis I.12].

It is the strong asymmetry condition, Hypothesis I.2(c), that makes this class of models somewhat unusual and permits the system to remain a Fermi liquid when the interaction is turned on. If $\mathbf{A} = 0$ then, taking the complex conjugate of (I.1), we see that $\varepsilon_\nu(-\mathbf{k}) = \varepsilon_\nu(\mathbf{k})$ so that Hypothesis I.2(c) is violated for $\mathbf{q} = 0$. Hence the presence of a nonzero vector potential $\mathbf{A}$ is essential. We shall say more about the role of strong asymmetry later. For now, we just mention one model that violates these hypotheses, not only for technical reasons but because it exhibits different physics. It is the Hubbard model at half filling, whose Fermi curve is sketched below. This Fermi curve is not smooth, violating Hypothesis I.2(b), has zero curvature almost everywhere, violating Hypothesis I.2(a), and is invariant under $\mathbf{k} \to -\mathbf{k}$ so that $F = -F$, violating Hypothesis I.2(c) with $\mathbf{q} = 0$.




To give a rigorous definition of (I.2) one must introduce cutoffs and then take the limit in which the cutoffs are removed. To impose an infrared cutoff in the spatial directions one could put the system in a finite periodic box $\mathbb{R}^2/L\Gamma$. To impose an ultraviolet cutoff in the spatial directions one may put the system on a lattice. By also imposing infrared and ultraviolet cutoffs in the temporal direction, we could arrange to start from a finite dimensional Grassmann algebra. We choose not to do so. We prove that formal renormalized perturbation expansions converge. The coefficients in those expansions are well-defined even without a finite volume cutoff. So we choose to start with $x$ running over all $\mathbb{R}^3$. We impose a (permanent) ultraviolet cutoff through a smooth, compactly supported function $U(\mathbf{k})$. This keeps $\mathbf{k}$ permanently bounded. We impose a (temporary) infrared cutoff through a function $\nu_\varepsilon(k_0^2 + e(\mathbf{k})^2)$ where $\nu_\varepsilon(\kappa)$ looks like




When $\varepsilon > 0$ and $\nu_\varepsilon(k_0^2 + e(\mathbf{k})^2) > 0$, $|ik_0 - e(\mathbf{k})|$ is at least of order $\varepsilon$. The coefficients of the perturbation expansion (either renormalized or not) of the cutoff Euclidean Green's functions

$$G_{2n;\varepsilon}(x_1, \sigma_1, \ldots, y_n, \tau_n) = \left\langle \prod_{i=1}^{n} \psi_{\sigma_i}(x_i)\bar{\psi}_{\tau_i}(y_i) \right\rangle_\varepsilon$$

where

$$\langle f \rangle_\varepsilon = \frac{\int f(\psi, \bar{\psi}) e^{\mathcal{V}(\psi,\bar{\psi})} d\mu_{C_\varepsilon}(\psi, \bar{\psi})}{\int e^{\mathcal{V}(\psi,\bar{\psi})} d\mu_{C_\varepsilon}(\psi, \bar{\psi})} \quad \text{with} \quad C_\varepsilon(k; \delta e) = \frac{U(\mathbf{k})\nu_\varepsilon(k_0^2 + e(\mathbf{k})^2)}{ik_0 - e(\mathbf{k}) + \delta e(\mathbf{k})}$$

are well-defined. Our main result is

**Theorem [5, Theorem I.4].** *Assume that* $d = 2$ *and that* $e(\mathbf{k})$ *fulfils Hypothesis* I.2. *There is*

- *a nontrivial open ball* $\mathcal{B} \subset \mathfrak{V}$, *centered on the origin, and*
- *an analytic*[c] *function* $V \in \mathcal{B} \mapsto \delta e(V) \in \mathcal{E}$, *that vanishes for* $V = 0$,

[c]For an elementary discussion of analytic maps between Banach spaces see, for example, [7, Appendix A].

*such that*:

- *for any* $\varepsilon > 0$ *and* $n \in \mathbb{N}$, *the formal Taylor series for the Green's functions* $G_{2n;\varepsilon}$ *converges to an analytic function on* $\mathcal{B}$,
- *as* $\varepsilon \to 0$, $G_{2n;\varepsilon}$ *converges uniformly, in* $x_1, \ldots, y_n$ *and* $V \in \mathcal{B}$, *to a translation invariant, spin independent, particle number conserving function* $G_{2n}$ *that is analytic in* $V$.

*If, in addition,* $V$ *is* $k_0$*-reversal real, as in* (I.4), *then* $\delta e(\mathbf{k}; V)$ *is real for all* $\mathbf{k}$.

**Theorem [5, Theorem I.5].** *Under the hypotheses of* [5, *Theorem* I.4] *and the assumption that* $V \in \mathcal{B}$ *obeys the symmetries* (I.4) *and* (I.5), *the Fourier transform*

$$\begin{aligned}\check{G}_2(k_0, \mathbf{k}) &= \int dx_0 d^2\mathbf{x}\ e^{\imath\langle k,x\rangle_-} G_2((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)) \\ &= \int dx_0 d^2\mathbf{x}\ e^{\imath\langle k,x\rangle_-} G_2((0,0,\downarrow),(x_0,\mathbf{x},\downarrow)) \\ &= \frac{1}{ik_0 - e(\mathbf{k}) - \Sigma(k)} \quad \textit{when } U(\mathbf{k}) = 1\end{aligned}$$

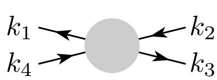
*of the two-point function exists and is continuous, except on the Fermi curve* (*precisely, except when* $k_0 = 0$ *and* $e(\mathbf{k}) = 0$). *The momentum distribution function*

$$n(\mathbf{k}) = \lim_{\tau \to 0+} \int \frac{dk_0}{2\pi}\ e^{\imath k_0 \tau}\ \check{G}_2(k_0, \mathbf{k})$$

*is continuous except on the Fermi curve* $F$. *If* $\bar{\mathbf{k}} \in F$, *then* $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k})$ *and* $\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k})$ *exist and obey*

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})<0}} n(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}} \\ e(\mathbf{k})>0}} n(\mathbf{k}) = 1 + O(V) > \frac{1}{2}\,.$$

**Theorem [5, Theorem I.7].** *Let*

$\check{G}_4(k_1, k_2, k_3, k_4) =$ 


(*spin dropped from notation*) *be the Fourier transform of the four-point function and*

$$\check{G}_4^A(k_1, k_2, k_3, k_4) = \check{G}_4(k_1, k_2, k_3, k_4) \prod_{\ell=1}^{4} \frac{1}{\check{G}_2(k_\ell)}$$

*its amputation by the physical propagator. Under the hypotheses of* [5, *Theorem* I.5], $\check{G}_4^A$ *has a decomposition*

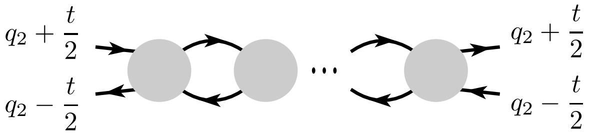
$$\begin{aligned}\check{G}_4^A(k_1, k_2, k_3, k_4) = N(k_1, k_2, k_3, k_4) &+ \frac{1}{2} L\left(\frac{k_1 + k_2}{2}, \frac{k_3 + k_4}{2}, k_2 - k_1\right) \\ &- \frac{1}{2} L\left(\frac{k_3 + k_2}{2}, \frac{k_1 + k_4}{2}, k_2 - k_3\right)\end{aligned}$$

*with*

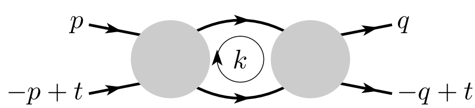
- ○ $N$ *continuous*
- ○ $L(q_1, q_2, t)$ *continuous except at* $t = 0$
- ○ $\lim_{t_0 \to 0} L(q_1, q_2, t)$ *continuous*
- ○ $\lim_{\mathbf{t} \to 0} L(q_1, q_2, t)$ *continuous.*

Think of $L$ as a particle–hole ladder

$L(q_1, q_2, t) =$




We now discuss further the role of the geometric conditions of Hypothesis I.2 in blocking the Cooper channel. When you turn on the interaction $V$, the system itself effectively replaces $V$ by more complicated "effective interaction". The (dominant) contribution




to the strength of the effective interaction between two particles of total momentum $t = p_1 + p_2 = q_1 + q_2$ is

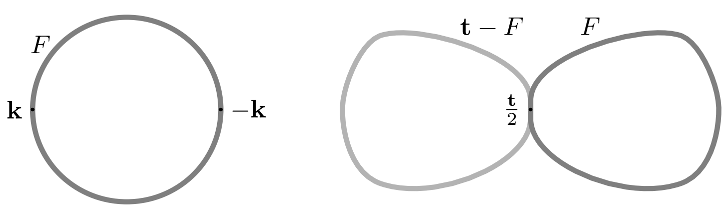
$$\int dk \frac{\text{stuff}}{[ik_0 - e(\mathbf{k})][i(-k_0 + t_0) - e(-\mathbf{k} + \mathbf{t})]} \,.$$

Note that

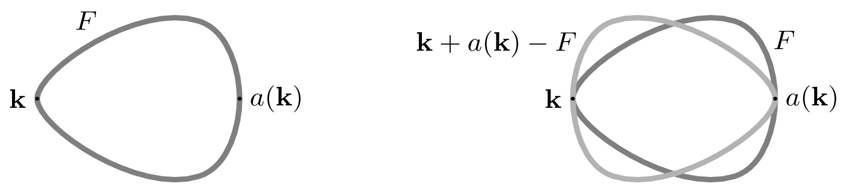
$$[ik_0 - e(\mathbf{k})] = 0 \Longleftrightarrow k_0 = 0\,, \quad e(\mathbf{k}) = 0 \quad \Longleftrightarrow \quad k_0 = 0, \quad \mathbf{k} \in F$$

$$[i(-k_0 + t_0) - e(-\mathbf{k} + \mathbf{t})] = 0 \Longleftrightarrow k_0 = t_0\,, \quad e(-\mathbf{k} + \mathbf{t}) = 0 \quad \Longleftrightarrow \quad k_0 = t_0, \quad \mathbf{k} \in \mathbf{t} - F\,.$$

We can transform $\frac{1}{ik_0 - e(\mathbf{k})}$ locally to $\frac{1}{ik_0 - k_1}$ by a simple change of variables. Thus $\frac{1}{ik_0 - e(\mathbf{k})}$ is locally integrable, but is not locally $L^2$. So the strength of the effective interaction diverges when the total momentum $t$ obeys $t_0 = 0$ and $F = \mathbf{t} - F$, because then the singular locus of $\frac{1}{ik_0 - e(\mathbf{k})}$ coincides with the singular locus of $\frac{1}{i(-k_0 + t_0) - e(-\mathbf{k} + \mathbf{t})}$. This always happens when $F = -F$ (for example, when $F$ is a circle) and $t = 0$. Similarly the strength of the effective interaction diverges when $F$ has a flat piece and $\mathbf{t}/2$ lies in that flat piece, as in the figure on the right below. On the other hand, when $F$ is strongly asymmetric, $F$ and $\mathbf{t} - F$ always



intersect only at isolated points. A "worst" case is illustrated below. There the antipode, $a(\mathbf{k})$, of $\mathbf{k} \in F$, is the unique point of $F$, different from $\mathbf{k}$, such that the tangents to $F$ at $\mathbf{k}$ and $a(\mathbf{k})$ are parallel.




For strongly asymmetric Fermi curves, $\frac{1}{[ik_0 - e(\mathbf{k})][i(-k_0+t_0) - e(-\mathbf{k}+\mathbf{t})]}$ remains locally integrable in $k$ for each fixed $\mathbf{t}$ and strength of the effective interaction remains bounded.

The Green's functions $G_{2n}$ are constructed using a multiscale analysis and renormalization. The multiscale analysis is introduced by choosing a parameter $M > 1$ and decomposing momentum space into a family of shells, with the $j$th shell consisting of those momenta $k$ obeying $|ik_0 - e(\mathbf{k})| \approx \frac{1}{M^j}$. Correspondingly, we write the covariance as a telescoping series $C(k) = \sum_{j=0}^{\infty} C^{(j)}(k)$ where, for $j \geq 1$,

$$C^{(j)} = C_{M^{-j}} - C_{M^{-j+1}}$$

is the "covariance at scale $j$". By construction $C^{(j)}(k)$ vanishes unless $\sqrt{k_0^2 + e(\mathbf{k})^2}$ is of order $M^{-j}$, and $\|C^{(j)}(k)\|_{L^\infty} \approx M^j$.

We consider, for each $j$, the cutoff amputated Euclidean Green's functions $G^{\mathrm{amp}}_{2n;M^{-j}}$. They are related to the previously defined Green's functions by

$$G_{2n;\varepsilon}(x_1, y_1, \ldots, x_n, y_n)$$
$$= \int \prod_{i=1}^{n} dx'_i dy'_i \left( \prod_{i=1}^{n} C_\varepsilon(x_i, x'_i) C_\varepsilon(y'_i, y_i) \right) G^{\mathrm{amp}}_{2n;\varepsilon}(x'_1, y'_1, \ldots, x'_n, y'_n)$$

for $n \geq 2$, and

$$G_{2;\varepsilon}(x, y) - C_\varepsilon(x, y) = \int dx' dy' C_\varepsilon(x, x') C_\varepsilon(y', y) G^{\mathrm{amp}}_{2;\varepsilon}(x', y')$$

where $\varepsilon = M^{-j}$. The amputated Green's functions are the coefficients in the expansion of

$$\mathcal{G}_{\mathrm{amp},j}(\phi, \bar{\phi}) = \log \frac{1}{Z_j} \int e^{\mathcal{V}(\phi+\psi, \bar{\phi}+\bar{\psi})} d\mu_{C_{M^{-j}}}(\psi, \bar{\psi}),$$

$$Z_j = \int e^{\mathcal{V}(\psi, \bar{\psi})} d\mu_{C_{M^{-j}}}(\psi, \bar{\psi})$$

in powers of $\phi$. That is,

$$\mathcal{G}_{\mathrm{amp,j}}(\phi,\bar\phi) = \sum_{n=1}^{\infty}\frac{1}{(n!)^2}\int\prod_{i=1}^{n}dx_i dy_i G^{\mathrm{amp}}_{2n,M^{-j}}(x_1,y_1,\ldots,x_n,y_n)\prod_{i=1}^{n}\bar\phi(x_i)\phi(y_i)\,.$$

The generating functionals $\mathcal{G}_{\mathrm{amp},j}$ are controlled using the renormalization group map

$$\Omega_S(\mathcal{W})(\phi,\bar\phi,\psi,\bar\psi) = \log\frac{1}{Z}\int e^{\mathcal{W}(\phi,\bar\phi,\psi+\zeta,\bar\psi+\bar\zeta)}d\mu_S(\zeta,\bar\zeta)$$

which is defined for any covariance $S$. Here, $\mathcal{W}$ is a Grassmann function and the partition function is $Z=\int e^{\mathcal{W}(0,0,\zeta,\bar\zeta)}d\mu_S(\zeta,\bar\zeta)$. $\Omega_S$ maps Grassmann functions in the variables $\phi,\bar\phi,\psi,\bar\psi$ to Grassmann functions in the same variables. Clearly

$$\mathcal{G}_{\mathrm{amp},j}(\phi,\bar\phi) = \Omega_{C_{M^{-j}}}(\mathcal{V})(0,0,\phi,\bar\phi) \tag{I.6}$$

where we view $\mathcal{V}(\psi,\bar\psi)$ as a function of the four variables $\phi,\bar\phi,\psi,\bar\psi$ that happens to be independent of $\phi,\bar\phi$.

The renormalization group map is discussed in a general setting in great detail in [8, 9]. It obeys the semigroup property

$$\Omega_{S_1+S_2} = \Omega_{S_1}\circ\Omega_{S_2}\,.$$

Therefore

$$\mathcal{G}_{\mathrm{amp},j}(\phi,\bar\phi) = \Omega_{C^{(j)}}(\mathcal{G}_{\mathrm{amp},j-1}(\psi,\bar\psi))(0,0,\phi,\bar\phi) \tag{I.7}$$

where we again view $\mathcal{G}_{\mathrm{amp},j-1}(\psi,\bar\psi)$ as a function of $\phi,\bar\phi,\psi,\bar\psi$ that happens to be independent of $\phi,\bar\phi$. The limiting Green's functions are controlled by tracking the renormalization group flow (I.7).

One of the main inputs from this series of four papers to the proof of the theorems stated above is a detailed analysis, with bounds, of the map $\Omega_{C^{(j)}}$. This is the content of the third paper in this series. In this first paper of the series, we apply the general results of [8] to simple many fermion systems. We introduce concrete norms that fulfill the conditions of [8, Sec. II.4] and develop contraction and integral bounds for them. Then, we apply [8, Theorem II.28] and (I.6) to models for which the dispersion relation is both infrared and ultraviolet finite (insulators). For these models, no scale decomposition is necessary.

In the second paper of this series, we introduce scales and apply the results of Part 1 to integrate out the first few scales. It turns out that for higher scales the norms introduced in Parts 1 and 2 are inadequate and, in particular, power count poorly. Using sectors (see [5, Sec. II, Subsec. 8]), we introduce finer norms that, in dimension two,[d] power count appropriately. For these sectorized norms, passing from one scale to the next is not completely trivial. This question is dealt with in

[d]This is the only part of the construction that is restricted to $d=2$. We believe that the difficulties preventing the extension to $d=3$ are technical rather than physical. Indeed, there has already been some progress in this direction [10, 11].

Part 4. Cumulative notation tables are provided at the end of each paper of this series.

## II. Norms

Let $A$ be the Grassmann algebra freely generated by the fields $\phi(y), \bar{\phi}(y)$ with $y \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$. The generating functional for the connected Greens functions is a Grassmann Gaussian integral in the Grassmann algebra with coefficients in $A$ that is generated by the fields $\psi(x), \bar{\psi}(x)$ with $x \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$. We want to apply the results of [8] to this situation.

To simplify notation we define, for

$$\xi = (x_0, \mathbf{x}, \sigma, a) = (x, a) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\},$$

the internal fields

$$\psi(\xi) = \begin{cases} \psi(x) & \text{if } a = 0 \\ \bar{\psi}(x) & \text{if } a = 1 \end{cases}.$$

Similarly, we define for an external variable $\eta = (y_0, \mathbf{y}, \tau, b) = (y, b) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$, the source fields

$$\phi(\eta) = \begin{cases} \phi(y) & \text{if } b = 0 \\ \bar{\phi}(y) & \text{if } b = 1 \end{cases}.$$

$\mathcal{B} = \mathbb{R} \times \mathbb{R}^2 \times \{\uparrow, \downarrow\} \times \{0, 1\}$ is called the "base space" parameterizing the fields. The Grassmann algebra $A$ is the direct sum of the vector spaces $A_m$ generated by the products $\phi(\eta_1) \cdots \phi(\eta_m)$. Let $V$ be the vector space generated by $\psi(\xi)$, $\xi \in \mathcal{B}$. An antisymmetric function $C(\xi, \xi')$ on $\mathcal{B} \times \mathcal{B}$ defines a covariance on $V$ by $C(\psi(\xi), \psi(\xi')) = C(\xi, \xi')$. The Grassmann Gaussian integral with this covariance, $\int \cdot \, d\mu_C(\psi)$, is a linear functional on the Grassmann algebra $\bigwedge_A V$ with values in $A$.

We shall define norms on $\bigwedge_A V$ by specifying norms on the spaces of functions on $\mathcal{B}^m \times \mathcal{B}^n$, $m, n \geq 0$. The rudiments of such norms and simple examples are discussed in this section. In the next section we recall the results of [8] in the current concrete situation.

The norms we construct are $(d+1)$-dimensional seminorms in the sense of [8, Definition II.15]. They measure the spatial decay of the functions, i.e. derivatives of their Fourier transforms.

**Definition II.1 (Multi-indices).** (i) A multi-index is an element $\delta = (\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{N}_0 \times \mathbb{N}_0^d$. The length of a multi-index $\delta = (\delta_0, \delta_1, \ldots, \delta_d)$ is $|\delta| = \delta_0 + \delta_1 + \cdots + \delta_d$ and its factorial is $\delta! = \delta_0! \delta_1! \cdots \delta_d!$. For two multi-indices $\delta, \delta'$ we say that $\delta \leq \delta'$ if $\delta_i \leq \delta_i'$ for $i = 0, 1, \ldots, d$. The spatial part of the multi-index $\delta = (\delta_0, \delta_1, \ldots, \delta_d)$ is $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d) \in \mathbb{N}_0^d$. It has length $|\boldsymbol{\delta}| = \delta_1 + \cdots + \delta_d$.

(ii) Let $\delta, \delta^{(1)}, \ldots, \delta^{(r)}$ be multi-indices such that $\delta = \delta^{(1)} + \cdots + \delta^{(r)}$. Then by definition

$$\binom{\delta}{\delta^{(1)}, \ldots, \delta^{(r)}} = \frac{\delta!}{\delta^{(1)}! \cdots \delta^{(r)}!}.$$

(iii) For a multi-index $\delta$ and $x = (x_0, \mathbf{x}, \sigma),\ x' = (x'_0, \mathbf{x}', \sigma') \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$ set

$$(x - x')^\delta = (x_0 - x'_0)^{\delta_0} (\mathbf{x}_1 - \mathbf{x}'_1)^{\delta_1} \cdots (\mathbf{x}_d - \mathbf{x}'_d)^{\delta_d} .$$

If $\xi = (x, a),\ \xi' = (x', a') \in \mathcal{B}$ we define $(\xi - \xi')^\delta = (x - x')^\delta$.

(iv) For a function $f(\xi_1, \ldots, \xi_n)$ on $\mathcal{B}^n$, a multi-index $\delta$, and $1 \le i, j \le n$; $i \ne j$ set

$$\mathcal{D}^\delta_{i,j} f(\xi_1, \ldots, \xi_n) = (\xi_i - \xi_j)^\delta f(\xi_1, \ldots, \xi_n) .$$

**Lemma II.2 (Leibniz's rule).** *Let $f(\xi_1, \ldots, \xi_n)$ be a function on $\mathcal{B}^n$ and $f'(\xi_1, \ldots, \xi_m)$ a function on $\mathcal{B}^m$. Set*

$$g(\xi_1, \ldots, \xi_{n+m-2}) = \int_{\mathcal{B}} d\eta f(\xi_1, \ldots, \xi_{n-1}, \eta) f'(\eta, \xi_n, \ldots, \xi_{n+m-2}) .$$

*Let $\delta$ be a multi-index and $1 \le i \le n-1,\ n \le j \le n+m-2$. Then*

$$\mathcal{D}^\delta_{i,j} g(\xi_1, \ldots, \xi_{n+m-2}) = \sum_{\delta' \le \delta} \binom{\delta}{\delta', \delta - \delta'} \int_{\mathcal{B}} d\eta \times \mathcal{D}^{\delta'}_{i,n} f(\xi_1, \ldots, \xi_{n-1}, \eta) \mathcal{D}^{\delta - \delta'}_{1, j-n+2} f'(\eta, \xi_n, \ldots, \xi_{n+m-2}) .$$

**Proof.** For each $\eta \in \mathcal{B}$

$$(\xi_i - \xi_j)^\delta = ((\xi_i - \eta) + (\eta - \xi_j))^\delta = \sum_{\delta' \le \delta} \binom{\delta}{\delta', \delta - \delta'} (\xi_i - \eta)^{\delta'} (\eta - \xi_j)^{\delta - \delta'} .$$

□

**Definition II.3 (Decay operators).** Let $n$ be a positive integer. A decay operator $\mathcal{D}$ on the set of functions on $\mathcal{B}^n$ is an operator of the form

$$\mathcal{D} = \mathcal{D}^{\delta^{(1)}}_{u_1, v_1} \cdots \mathcal{D}^{\delta^{(k)}}_{u_k, v_k}$$

with multi-indices $\delta^{(1)}, \ldots, \delta^{(k)}$ and $1 \le u_j,\ v_j \le n,\ u_j \ne v_j$. The indices $u_j, v_j$ are called variable indices. The total order of derivatives in $\mathcal{D}$ is

$$\delta(\mathcal{D}) = \delta^{(1)} + \cdots + \delta^{(k)} .$$

In a similar way, we define the action of a decay operator on the set of functions on $(\mathbb{R} \times \mathbb{R}^d)^n$ or on $(\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\})^n$.

**Definition II.4.** (i) On $\mathbb{R}_+ \cup \{\infty\} = \{x \in \mathbb{R} | x \ge 0\} \cup \{+\infty\}$, addition and the total ordering $\le$ are defined in the standard way. With the convention that $0 \cdot \infty = \infty$, multiplication is also defined in the standard way.

(ii) Let $d \ge -1$. For $d \ge 0$, the $(d+1)$-dimensional norm domain $\mathfrak{N}_{d+1}$ is the set of all formal power series

$$X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta\, t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$$

in the variables $t_0, t_1, \ldots, t_d$ with coefficients $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$. To shorten notation, we set $t^\delta = t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d}$. Addition and partial ordering on $\mathfrak{N}_{d+1}$ are defined componentwise. Multiplication is defined by

$$(X \cdot X')_\delta = \sum_{\beta+\gamma=\delta} X_\beta X'_\gamma \,.$$

The max and min of two elements of $\mathfrak{N}_{d+1}$ are again defined componentwise.

The zero-dimensional norm domain $\mathfrak{N}_0$ is defined to be $\mathbb{R}_+ \cup \{\infty\}$. We also identify $\mathbb{R}_+ \cup \{\infty\}$ with the set of all $X \in \mathfrak{N}_{d+1}$ with $X_\delta = 0$ for all $\delta \neq \mathbf{0} = (0, \ldots, 0)$.

If $a > 0$, $X_{\mathbf{0}} \neq \infty$ and $a - X_{\mathbf{0}} > 0$ then $(a - X)^{-1}$ is defined as

$$(a - X)^{-1} = \frac{1}{a - X_{\mathbf{0}}} \sum_{n=0}^{\infty} \left( \frac{X - X_{\mathbf{0}}}{a - X_{\mathbf{0}}} \right)^n .$$

For an element $X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta \, t^\delta$ of $\mathfrak{N}_{d+1}$ and $0 \leq j \leq d$ the formal derivative $\frac{\partial}{\partial t_j} X$ is defined as

$$\frac{\partial}{\partial t_j} X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} (\delta_j + 1) X_{\delta + \epsilon_j} t^\delta$$

where $\epsilon_j$ is the $j$th unit vector.

**Definition II.5.** Let $E$ be a complex vector space. A $(d+1)$-dimensional seminorm on $E$ is a map $\| \cdot \| : E \to \mathfrak{N}_{d+1}$ such that

$$\|e_1 + e_2\| \leq \|e_1\| + \|e_2\| \,, \quad \|\lambda e\| = |\lambda| \, \|e\|$$

for all $e, e_1, e_2 \in E$ and $\lambda \in \mathbb{C}$.

**Example II.6.** For a function $f$ on $\mathcal{B}^m \times \mathcal{B}^n$ we define the (scalar valued) $L_1$–$L_\infty$-norm as

$$|||f|||_{1,\infty} = \begin{cases} \displaystyle\max_{1 \leq j_0 \leq n} \sup_{\xi_{j_0} \in \mathcal{B}} \int \prod_{\substack{j=1,\ldots,n \\ j \neq j_0}} d\xi_j \, |f(\xi_1, \ldots, \xi_n)| & \text{if } m = 0 \\ \displaystyle\sup_{\eta_1, \ldots, \eta_m \in \mathcal{B}} \int \prod_{j=1,\ldots,n} d\xi_j \, |f(\eta_1, \ldots, \eta_m;\ \xi_1, \ldots, \xi_n)| & \text{if } m \neq 0 \end{cases}$$

and the $(d+1)$-dimensional $L_1$–$L_\infty$ seminorm

$$\|f\|_{1,\infty} = \begin{cases} \displaystyle\sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} \frac{1}{\delta!} \left( \max_{\substack{\mathcal{D} \text{ decay operator} \\ \text{with } \delta(\mathcal{D}) = \delta}} |||\mathcal{D} f|||_{1,\infty} \right) t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d} & \text{if } m = 0 \\ |||f|||_{1,\infty} & \text{if } m \neq 0 \end{cases} .$$

Here $|||f|||_{1,\infty}$ stands for the formal power series with constant coefficient $|||f|||_{1,\infty}$ and all other coefficients zero.

**Lemma II.7.** *Let $f$ be a function on $\mathcal{B}^m \times \mathcal{B}^n$ and $f'$ a function on $\mathcal{B}^{m'} \times \mathcal{B}^{n'}$. Let $1 \le \mu \le n,\ 1 \le \nu \le n'$. Define the function $g$ on $\mathcal{B}^{m+m'} \times \mathcal{B}^{n+n'-2}$ by*

$$g(\eta_1, \dots, \eta_{m+m'};\ \xi_1, \dots, \xi_{\mu-1}, \xi_{\mu+1}, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \dots, \xi_{n+n'})$$
$$= \int_{\mathcal{B}} d\zeta f(\eta_1, \dots, \eta_m;\ \xi_1, \dots, \xi_{\mu-1}, \zeta, \xi_{\mu+1}, \dots, \xi_n)$$
$$\times f'(\eta_{m+1}, \dots, \eta_{m+m'};\ \xi_{n+1}, \dots, \xi_{n+\nu-1}, \zeta, \xi_{n+\nu+1}, \dots, \xi_{n+n'})\,.$$

*If $m = 0$ or $m' = 0$*

$$|||g|||_{1,\infty} \le |||f|||_{1,\infty}\ |||f'|||_{1,\infty}$$
$$\|g\|_{1,\infty} \le \|f\|_{1,\infty}\ \|f'\|_{1,\infty}\,.$$

**Proof.** We first consider the norm $|||\cdot|||_{1,\infty}$. In the case $m \ne 0, m' = 0$, for all $\eta_1, \dots, \eta_m \in \mathcal{B}$

$$\Bigg|\int \prod_{\substack{j=1\\ j\ne\mu,n+\nu}}^{n+n'} d\xi_j\ g(\eta_1, \dots, \eta_m;\ \xi_1, \dots, \xi_{\mu-1}, \xi_{\mu+1}, \dots, \xi_n\,,$$
$$\times\ \xi_{n+1}, \dots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \dots, \xi_{n+n'})\Bigg|$$
$$\le \left|\int d\xi_1 \cdots d\xi_n\ f(\eta_1, \dots, \eta_m;\ \xi_1, \dots, \xi_n)\right|$$
$$\times \sup_{\zeta\in\mathcal{B}} \left|\int \prod_{\substack{j=1\\ j\ne\nu}}^{n'} d\xi'_j\ f'(\xi'_1, \dots, \xi'_{\nu-1}, \zeta, \xi'_{\nu+1}, \dots, \xi'_{n'})\right|$$
$$\le |||f|||_{1,\infty}\ |||f'|||_{1,\infty}\,.$$

The case $m = 0,\ m' \ne 0$ is similar. In the case $m = m' = 0$, first fix $j_0 \in \{1, \dots, n\} \setminus \{\mu\}$, and fix $\xi_{j_0} \in \mathcal{B}$. As in the case $m \ne 0, m' = 0$ one shows that

$$\left|\int \prod_{\substack{j=1\\ j\ne j_0,\mu,n+\nu}}^{n+n'} d\xi_j\ g(\xi_1, \dots, \xi_{\mu-1}, \xi_{\mu+1}, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{n+\nu-1}, \xi_{n+\nu+1}, \dots, \xi_{n+n'})\right|$$
$$\le \left|\int \prod_{\substack{j=1\\ j\ne j_0}}^{n} d\xi_j\ f(\xi_1, \dots, \xi_n)\right| \sup_{\zeta\in\mathcal{B}} \left|\int \prod_{\substack{j=1\\ j\ne\nu}}^{n'} d\xi'_j\ f'(\xi'_1, \dots, \xi'_{\nu-1}, \zeta, \xi'_{\nu+1}, \dots, \xi'_{n'})\right|$$
$$\le |||f|||_{1,\infty}\ |||f'|||_{1,\infty}\,.$$

If one fixes one of the variables $\xi_{j_0}$ with $j_0 \in \{n+1, \ldots, n+n'\} \setminus \{n+\nu\}$, the argument is similar.

We now consider the norm $\|\cdot\|_{1,\infty}$. If $m \neq 0$ or $m' \neq 0$ this follows from the first part of this lemma and

$$\|g\|_{1,\infty} = |||g|||_{1,\infty} \leq |||f|||_{1,\infty}\ |||f'|||_{1,\infty} \leq \|f\|_{1,\infty}\ \|f'\|_{1,\infty}\,.$$

So assume that $m = m' = 0$. Set

$$\|f\|_{1,\infty} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!} X_\delta t^\delta \qquad \|f'\|_{1,\infty} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!} X'_\delta t^\delta$$

$$\|g\|_{1,\infty} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!} Y_\delta t^\delta$$

with $X_\delta, X'_\delta, Y_\delta \in \mathbb{R}_+ \cup \{\infty\}$. Let $\mathcal{D}$ be a decay operator of degree $\delta$ acting on $g$. The variable indices for $g$ lie in the set $I \cup I'$, where

$$I = \{1, \ldots, \mu-1, \mu+1, \ldots, n\}$$

$$I' = \{n+1, \ldots, n+\nu-1, n+\nu+1, \ldots, n+n'\}\,.$$

We can factor the decay operator $\mathcal{D}$ in the form

$$\mathcal{D} = \pm\tilde{\mathcal{D}}\mathcal{D}_2\mathcal{D}_1$$

where all variable indices of $\mathcal{D}_1$ lie in $I$, all variable indices of $\mathcal{D}_2$ lie in $I'$, and

$$\tilde{\mathcal{D}} = \mathcal{D}^{\delta^{(1)}}_{u_1,v_1} \cdots \mathcal{D}^{\delta^{(k)}}_{u_k,v_k}$$

with $u_1, \ldots, u_k \in I,\ v_1, \ldots, v_k \in I'$. Set $h = \mathcal{D}_1 f$ and $h' = \mathcal{D}_2 f'$. By Leibniz's rule

$$\begin{aligned}
\mathcal{D}g &= \pm\tilde{\mathcal{D}} \int_{\mathcal{B}} d\zeta h(\xi_1, \ldots, \xi_{\mu-1}, \zeta, \xi_{\mu+1}, \ldots, \xi_n) \\
&\qquad \times h'(\xi_{n+1}, \ldots, \xi_{n+\nu-1}, \zeta, \xi_{n+\nu+1}, \ldots, \xi_{n+n'}) \\
&= \pm \sum_{\substack{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)} \\ \text{for } i=1,\ldots,k}} \left(\prod_{i=1}^k \binom{\delta^{(i)}}{\alpha^{(i)}, \beta^{(i)}}\right) \int d\zeta \left(\prod_{i=1}^k \mathcal{D}^{\alpha^{(i)}}_{u_i,\mu} h\right) \\
&\qquad \times (\xi_1, \ldots, \xi_{\mu-1}, \zeta, \xi_{\mu+1}, \ldots, \xi_n) \\
&\qquad \times \left(\prod_{i=1}^k \mathcal{D}^{\beta^{(i)}}_{\nu,v_i} h'\right) (\xi_{n+1}, \ldots, \xi_{n+\nu-1}, \zeta, \xi_{n+\nu+1}, \ldots, \xi_{n+n'})\,.
\end{aligned}$$

By the first part of this lemma, the $L_1$–$L_\infty$-norm of each integral on the right-hand side is bounded by

$$\left|\left|\left| \prod_{i=1}^k \mathcal{D}^{\alpha^{(i)}}_{u_i,\mu} h \right|\right|\right|_{1,\infty} \left|\left|\left| \prod_{i=1}^k \mathcal{D}^{\beta^{(i)}}_{\nu,v_i} h' \right|\right|\right|_{1,\infty}\,.$$

Therefore, setting $\tilde\delta = \delta(\tilde{\mathcal{D}}) = \delta^{(1)} + \cdots + \delta^{(k)}$,

$$\begin{aligned}
|||\mathcal{D}g|||_{1,\infty} t^\delta \le & \sum_{\substack{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)} \\ \text{for } i=1,\ldots,k}} \left(\prod_{i=1}^k \binom{\delta^{(i)}}{\alpha^{(i)},\beta^{(i)}}\right) t^{\delta(\mathcal{D}_1)} t^{\alpha^{(1)}+\cdots+\alpha^{(k)}} \\
& \times \left|\left|\left| \prod_{i=1}^k \mathcal{D}_{u_i,\mu}^{\alpha^{(i)}}(\mathcal{D}_1 f)\right|\right|\right|_{1,\infty} t^{\delta(\mathcal{D}_2)} t^{\beta^{(1)}+\cdots+\beta^{(k)}} \left|\left|\left| \prod_{i=1}^k \mathcal{D}_{\nu,v_i}^{\beta^{(i)}}(\mathcal{D}_2 f')\right|\right|\right|_{1,\infty} \\
\le & \sum_{\alpha+\beta=\tilde\delta} \sum_{\substack{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)} \\ \alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \beta^{(1)}+\cdots+\beta^{(k)}=\beta}} \left(\prod_{i=1}^k \binom{\delta^{(i)}}{\alpha^{(i)},\beta^{(i)}}\right) \\
& \times X_{\delta(\mathcal{D}_1)+\alpha} t^{\delta(\mathcal{D}_1)+\alpha}\ X'_{\delta(\mathcal{D}_2)+\beta} t^{\delta(\mathcal{D}_2)+\beta} \\
= & \sum_{\alpha+\beta=\tilde\delta} \binom{\tilde\delta}{\alpha,\ \beta} X_{\delta(\mathcal{D}_1)+\alpha} t^{\delta(\mathcal{D}_1)+\alpha}\ X'_{\delta(\mathcal{D}_2)+\beta} t^{\delta(\mathcal{D}_2)+\beta} \\
\le & \sum_{\alpha+\beta=\tilde\delta} \binom{\delta}{\delta(\mathcal{D}_1)+\alpha,\ \delta(\mathcal{D}_2)+\beta} \\
& \times X_{\delta(\mathcal{D}_1)+\alpha} t^{\delta(\mathcal{D}_1)+\alpha} X'_{\delta(\mathcal{D}_2)+\beta} t^{\delta(\mathcal{D}_2)+\beta}\,. \qquad \text{(II.1)}
\end{aligned}$$

In the equality, we used the fact that for each pair of multi-indices $\alpha, \beta$ with $\alpha+\beta = \tilde\delta$ and each $k$-tuple of multi-indices $\delta^{(i)},\ 1 \le i \le k$, with $\sum_i \delta^{(i)} = \tilde\delta$

$$\sum_{\substack{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)} \\ \alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \beta^{(1)}+\cdots+\beta^{(k)}=\beta}} \prod_{i=1}^k \binom{\delta^{(i)}}{\alpha^{(i)},\beta^{(i)}} = \binom{\tilde\delta}{\alpha,\ \beta}.$$

This standard combinatorial identity follows from

$$\begin{aligned}
\sum_{\alpha+\beta=\tilde\delta} \binom{\tilde\delta}{\alpha,\ \beta} x^\alpha y^\beta = (x+y)^{\tilde\delta} &= \prod_{i=1}^k (x+y)^{\delta^{(i)}} \\
&= \prod_{i=1}^k \left( \sum_{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)}} \binom{\delta^{(i)}}{\alpha^{(i)},\beta^{(i)}} x^{\alpha^{(i)}} y^{\beta^{(i)}} \right) \\
&= \sum_{\substack{\alpha^{(i)}+\beta^{(i)}=\delta^{(i)} \\ i=1,\ldots,k}} \left[\prod_{i=1}^k \binom{\delta^{(i)}}{\alpha^{(i)},\beta^{(i)}}\right] x^{\alpha^{(1)}+\cdots+\alpha^{(k)}} y^{\beta^{(1)}+\cdots+\beta^{(k)}}
\end{aligned}$$

by matching the coefficients of $x^\alpha y^\beta$.

It follows from (II.1) that

$$\frac{1}{\delta!}Y_\delta t^\delta \le \sum_{\alpha'+\beta'=\delta} \frac{1}{\alpha'!}X_{\alpha'}t^{\alpha'} \ \frac{1}{\beta'!}X'_{\beta'}t^{\beta'}$$

and

$$\|g\|_{1,\infty} \le \sum_\delta \sum_{\alpha'+\beta'=\delta} \frac{1}{\alpha'!}X_{\alpha'}t^{\alpha'} \qquad \frac{1}{\beta'!}X'_{\beta'}t^{\beta'} = \|f\|_{1,\infty}\ \|f'\|_{1,\infty}\,.$$

□

**Corollary II.8.** *Let $f$ be a function on $\mathcal{B}^n$, $f'$ a function on $\mathcal{B}^{n'}$ and $C_2, C_3$ functions on $\mathcal{B}^2$. Set*

$$h(\xi_4,\dots,\xi_n,\xi'_4,\dots,\xi'_{n'}) = \int d\zeta d\xi_2 d\xi'_2 d\xi_3 d\xi'_3 f(\zeta,\xi_2,\xi_3,\xi_4,\dots,\xi_n) \times C_2(\xi_2,\xi'_2)C_3(\xi_3,\xi'_3)\ f'(\zeta,\xi'_2,\xi'_3,\xi'_4,\dots,\xi'_{n'})\,.$$

*Then*

$$\|h\|_{1,\infty} \le \sup_{\xi,\xi'}|C_2(\xi,\xi')|\ \sup_{\eta,\eta'}|C_3(\eta,\eta')|\ \|f\|_{1,\infty}\ \|f'\|_{1,\infty}\,.$$

**Proof.** Set

$$g(\xi_2,\dots,\xi_n,\xi'_2,\dots,\xi'_{n'}) = \int d\zeta f(\zeta,\xi_2,\xi_3,\xi_4,\dots,\xi_n)f'(\zeta,\xi'_2,\xi'_3,\xi'_4,\dots,\xi'_{n'})\,.$$

Let $\mathcal{D}$ be a decay operator acting on $h$. Then

$$\mathcal{D}h = \int d\xi_2 d\xi'_2 d\xi_3 d\xi'_3 C_2(\xi_2,\xi'_2)C_3(\xi_3,\xi'_3)\mathcal{D}g(\xi_2,\dots,\xi_n,\xi'_2,\dots,\xi'_{n'})\,.$$

Consequently

$$|||\mathcal{D}h|||_{1,\infty} \le \sup|C_2|\ \sup|C_3|\ |||\mathcal{D}g|||_{1,\infty}$$

and therefore

$$\|h\|_{1,\infty} \le \sup|C_2|\ \sup|C_3|\ \|g\|_{1,\infty}\,.$$

The corollary now follows from Lemma II.7. □

**Definition II.9.** Let $\mathcal{F}_m(n)$ be the space of all functions $f(\eta_1,\dots,\eta_m;\ \xi_1,\dots,\xi_n)$ on $\mathcal{B}^m\times\mathcal{B}^n$ that are antisymmetric in the $\eta$ variables. If $f(\eta_1,\dots,\eta_m;\ \xi_1,\dots,\xi_n)$ is any function on $\mathcal{B}^m\times\mathcal{B}^n$, its antisymmetrization in the external variables is

$$\mathrm{Ant}_{\mathrm{ext}} f(\eta_1,\dots,\eta_m;\ \xi_1,\dots,\xi_n) = \frac{1}{m!}\sum_{\pi\in S_m}\mathrm{sgn}(\pi)f(\eta_{\pi(1)},\dots,\eta_{\pi(m)};\ \xi_1,\dots,\xi_n)\,.$$

For $m, n\ge 0$, the symmetric group $S_n$ acts on $\mathcal{F}_m(n)$ from the right by

$$f^\pi(\eta_1,\dots,\eta_m;\ \xi_1,\dots,\xi_n) = f(\eta_1,\dots,\eta_m;\ \xi_{\pi(1)},\dots,\xi_{\pi(n)}) \quad \text{for } \pi\in S_n\,.$$

**Definition II.10.** A seminorm $\|\cdot\|$ on $\mathcal{F}_m(n)$ is called symmetric, if for every $f \in \mathcal{F}_m(n)$ and $\pi \in S_n$

$$\|f^\pi\| = \|f\|$$

and $\|f\| = 0$ if $m = n = 0$.

For example, the seminorms $\|\cdot\|_{1,\infty}$ of Example II.6 are symmetric.

## III. Covariances and the Renormalization Group Map

**Definition III.1 (Contraction).** Let $C(\xi, \xi')$ be any skew symmetric function on $\mathcal{B} \times \mathcal{B}$. Let $m, n \geq 0$ and $1 \leq i < j \leq n$. For $f \in \mathcal{F}_m(n)$ the contraction $\underset{i\to j}{\mathcal{C}\text{on}_C} f \in \mathcal{F}_m(n-2)$ is defined as

$$\begin{aligned}
&\underset{i\to j}{\mathcal{C}\text{on}_C}\, f(\eta_1, \dots, \eta_m;\ \xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n) \\
&\qquad = (-1)^{j-i+1} \int d\zeta d\zeta'\ C(\zeta, \zeta') \\
&\qquad\quad \times f(\eta_1, \dots, \eta_m;\ \xi_1, \dots, \xi_{i-1}, \zeta, \xi_{i+1}, \dots, \xi_{j-1}, \zeta', \xi_{j+1}, \dots, \xi_n)\,.
\end{aligned}$$

**Definition III.2 (Contraction Bound).** Let $\|\cdot\|$ be a family of symmetric seminorms on the spaces $\mathcal{F}_m(n)$. We say that $\mathfrak{c} \in \mathfrak{N}_{d+1}$ is a contraction bound for the covariance $C$ with respect to this family of seminorms, if for all $m, n, m', n' \geq 0$ there exist $i$ and $j$ with $1 \leq i \leq n,\ 1 \leq j \leq n'$ such that

$$\left\| \underset{i\to j}{\mathcal{C}\text{on}_C} (\mathrm{Ant}_{\mathrm{ext}}(f \otimes f')) \right\| \leq \mathfrak{c} \|f\|\, \|f'\|$$

for all $f \in \mathcal{F}_m(n),\ f' \in \mathcal{F}_{m'}(n')$. Observe that $f \otimes f'$ is a function on $(\mathcal{B}^m \times \mathcal{B}^n) \times (\mathcal{B}^{m'} \times \mathcal{B}^{n'}) \cong \mathcal{B}^{m+m'} \times \mathcal{B}^{n+n'}$, so that $\mathrm{Ant}_{\mathrm{ext}}(f \otimes f') \in \mathcal{F}_{m+m'}(n+n')$.

**Remark III.3.** If $\mathfrak{c}$ is a contraction bound for the covariance $C$ with respect to a family of symmetric seminorms, then, by symmetry,

$$\left\| \underset{i\to n+j}{\mathcal{C}\text{on}_C} (\mathrm{Ant}_{\mathrm{ext}}(f \otimes f')) \right\| \leq \mathfrak{c} \|f\|\, \|f'\|$$

for **all** $1 \leq i \leq n,\ 1 \leq j \leq n'$ and all $f \in \mathcal{F}_m(n),\ f' \in \mathcal{F}_{m'}(n')$.

**Example III.4.** The $L_1$–$L_\infty$-norm introduced in Example II.6 has $\max\{\|C\|_{1,\infty}, |||C|||_\infty\}$ as a contraction bound for covariance $C$. Here, $|||C|||_\infty$ is the element of $\mathfrak{N}_{d+1}$ whose constant term is $\sup_{\xi,\xi'} |C(\xi, \xi')|$ and is the only nonzero term. This is easily proven by iterated application of Lemma II.7. See also [8, Example II.26]. A more general statement will be formulated and proven in Lemma V.1(iii).

**Definition III.5 (Integral Bound).** Let $\|\cdot\|$ be a family of symmetric seminorms on the spaces $\mathcal{F}_m(n)$. We say that $\mathrm{b} \in \mathbb{R}_+$ is an integral bound for the covariance $C$ with respect to this family of seminorms, if the following holds:

Let $m \geq 0$, $1 \leq n' \leq n$. For $f \in \mathcal{F}_m(n)$ define $f' \in \mathcal{F}_m(n-n')$ by

$$\begin{aligned} f'(\eta_1, \ldots, \eta_m;\ \xi_{n'+1}, \ldots, \xi_n) \\ &= \iint_{\mathcal{B}^{n'}} d\xi_1 \cdots d\xi_{n'}\ f(\eta_1, \ldots, \eta_m;\ \xi_1, \ldots, \xi_{n'}, \xi_{n'+1}, \ldots, \xi_n) \\ &\quad \times \psi(\xi_1) \cdots \psi(\xi_{n'}) d\mu_C(\psi)\,. \end{aligned}$$

Then

$$\|f'\| \leq (\mathrm{b}/2)^{n'}\ \|f\|\,.$$

**Remark III.6.** Suppose that there is a constant $S$ such that

$$\left| \int \psi(\xi_1) \cdots \psi(\xi_n) d\mu_C(\psi) \right| \leq S^n$$

for all $\xi_1, \ldots, \xi_n \in \mathcal{B}$. Then $2S$ is an integral bound for $C$ with respect to the $L_1$–$L_\infty$-norm introduced in Example II.6.

**Definition III.7.** (i) We define $A_m[n]$ as the subspace of the Grassmann algebra $\bigwedge_A V$ that consists of all elements of the form

$$\begin{aligned} Gr(f) = \int d\eta_1 \cdots d\eta_m d\xi_1 \cdots d\xi_n\ f(\eta_1, \ldots, \eta_m;\ \xi_1, \ldots, \xi_n) \\ \times \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \cdots \psi(\xi_n) \end{aligned}$$

with a function $f$ on $\mathcal{B}^m \times \mathcal{B}^n$.

(ii) Every element of $A_m[n]$ has a unique representation of the form $Gr(f)$ with a function $f(\eta_1, \ldots, \eta_m;\ \xi_1, \ldots, \xi_n) \in \mathcal{F}_m(n)$ that is antisymmetric in its $\xi$ variables. Therefore a seminorm $\|\cdot\|$ on $\mathcal{F}_m(n)$ defines a canonical seminorm on $A_m[n]$, which we denote by the same symbol $\|\cdot\|$.

**Remark III.8.** For $F \in A_m[n]$

$$\|F\| \leq \|f\| \quad \text{for all } f \in \mathcal{F}_m(n) \text{ with } Gr(f) = F\,.$$

**Proof.** Let $f \in \mathcal{F}_m(n)$. Then $f' = \frac{1}{n!} \sum_{\pi \in S_n} \mathrm{sgn}(\pi)\, f^\pi$ is the unique element of $\mathcal{F}_m(n)$ that is antisymmetric in its $\xi$ variables such that $Gr(f') = Gr(f)$. Therefore

$$\|Gr(f)\| = \|f'\| \leq \frac{1}{n!} \sum_{\pi \in S_n} \|f^\pi\| = \frac{1}{n!} \sum_{\pi \in S_n} \|f\| = \|f\|$$

since the seminorm is symmetric. □

**Definition III.9.** Let $\|\cdot\|$ be a family of symmetric seminorms, and let $\mathcal{W}(\phi, \psi)$ be a Grassmann function. Write

$$\mathcal{W} = \sum_{m,n \geq 0} \mathcal{W}_{m,n}$$

with $\mathcal{W}_{m,n} \in A_m[n]$. For any constants $\mathfrak{c} \in \mathfrak{N}_{d+1}$, $\mathrm{b} > 0$ and $\alpha \geq 1$ set

$$N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, \alpha) = \frac{1}{\mathrm{b}^2}\mathfrak{c} \sum_{m,n\geq 0} \alpha^n \mathrm{b}^n \|\mathcal{W}_{m,n}\|\,.$$

In practice, the quantities $\mathrm{b}, \mathfrak{c}$ will reflect the "power counting" of $\mathcal{W}$ with respect to the covariance $C$ and the number $\alpha$ is proportional to an inverse power of the largest allowed modulus of the coupling constant.

In this paper, we will derive bounds on the renormalization group map for several kinds of seminorms. The main ingredients from [8] are

**Theorem III.10.** *Let $\|\cdot\|$ be a family of symmetric seminorms and let $C$ be a covariance on $V$ with contraction bound $\mathfrak{c}$ and integral bound* $\mathrm{b}$. *Then the formal Taylor series $\Omega_C(:\mathcal{W}:)$ converges to an analytic map on $\{\mathcal{W} | \mathcal{W}$ even, $N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, 8\alpha)_{\mathbf{0}} < \frac{\alpha^2}{4}\}$. Furthermore, if $\mathcal{W}(\phi,\psi)$ is an even Grassmann function such that*

$$N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, 8\alpha)_{\mathbf{0}} < \frac{\alpha^2}{4}$$

*then*

$$N(\Omega_C(:\mathcal{W}:) - \mathcal{W};\, \mathfrak{c}, \mathrm{b}, \alpha) \leq \frac{2}{\alpha^2} \frac{N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, 8\alpha)^2}{1 - \frac{4}{\alpha^2} N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, 8\alpha)}\,.$$

*Here, $: \cdot :$ denotes Wick ordering with respect to the covariance $C$.*

In Sec. V we will use this theorem to discuss the situation of an insulator. More generally we have:

**Theorem III.11.** *Let, for $\kappa$ in a neighborhood of* 0, *$\mathcal{W}_\kappa(\phi,\psi)$ be an even Grassmann function and $C_\kappa, D_\kappa$ be antisymmetric functions on $\mathcal{B}\times\mathcal{B}$. Assume that $\alpha \geq 1$ and*

$$N(\mathcal{W}_0;\, \mathfrak{c}, \mathrm{b}, 32\alpha)_{\mathbf{0}} < \alpha^2$$

*and that*

| | |
|---|---|
| *$C_0$ has contraction bound $\mathfrak{c}$* | *$\frac{1}{2}\mathrm{b}$ is an integral bound for $C_0, D_0$* |
| *$\left.\frac{d}{d\kappa}C_\kappa\right\|_{\kappa=0}$ has contraction bound $\mathfrak{c}'$* | *$\frac{1}{2}\mathrm{b}'$ is an integral bound for $\left.\frac{d}{d\kappa}D_\kappa\right\|_{\kappa=0}$* |

*and that $\mathfrak{c} \leq \frac{1}{\mu}\mathfrak{c}^2$. Set*

$$:\tilde{\mathcal{W}}_\kappa(\phi,\psi):_{\psi,D_\kappa} = \Omega_{C_\kappa}(:\mathcal{W}_\kappa:_{\psi,C_\kappa+D_\kappa})\,.$$

*Then*

$$\begin{aligned} &N\left(\frac{d}{d\kappa}[\tilde{\mathcal{W}}_\kappa - \mathcal{W}_\kappa]_{\kappa=0};\, \mathfrak{c}, \mathrm{b}, \alpha\right) \\ &\quad \leq \frac{1}{2\alpha^2} \frac{N(\mathcal{W}_0;\, \mathfrak{c}, \mathrm{b}, 32\alpha)}{1 - \frac{1}{\alpha^2}N(\mathcal{W}_0;\, \mathfrak{c}, \mathrm{b}, 32\alpha)} N\left(\left.\frac{d}{d\kappa}\mathcal{W}_\kappa\right|_{\kappa=0};\, \mathfrak{c}, \mathrm{b}, 8\alpha\right) \\ &\qquad + \frac{1}{2\alpha^2} \frac{N(\mathcal{W}_0;\, \mathfrak{c}, \mathrm{b}, 32\alpha)^2}{1 - \frac{1}{\alpha^2}N(\mathcal{W}_0;\, \mathfrak{c}, \mathrm{b}, 32\alpha)} \left\{\frac{1}{4\mu}\mathfrak{c}' + \left(\frac{\mathrm{b}'}{\mathrm{b}}\right)^2\right\}. \end{aligned}$$

**Proof of Theorems III.10 and III.11.** If $f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$ is a function on $\mathcal{B}^m \times \mathcal{B}^n$ we define the corresponding element of $A_m \otimes V^{\otimes n}$ as

$$\mathrm{Tens}(f) = \int \prod_{i=1}^{m} d\eta_i \prod_{j=1}^{n} d\xi_j \; f(\eta_1, \ldots, \eta_m; \xi_1, \ldots, \xi_n)$$
$$\times \phi(\eta_1) \cdots \phi(\eta_m) \psi(\xi_1) \otimes \cdots \otimes \psi(\xi_n) \,.$$

Each element of $A_m \otimes V^{\otimes n}$ can be uniquely written in the form $\mathrm{Tens}(f)$ with a function $f \in \mathcal{F}_m(n)$. Therefore a seminorm on $\mathcal{F}_m(n)$ defines a seminorm on $A_m \otimes V^{\otimes n}$ and conversely. Under this correspondence, symmetric seminorms on $\mathcal{F}_m(n)$ in the sense of Definition II.10 correspond to symmetric seminorms on $A_m \otimes V^{\otimes n}$ in the sense of [8, Definition II.18], contraction bounds as in Definition III.2 correspond, by Remark III.3, to contraction bounds as in [8, Definition II.25(i)] and integral bounds as in Definition III.5 correspond to integral bounds as in [8, Definition II.25(ii)]. Furthermore the norms on the spaces $A_m[n]$ defined in Definition II.7(ii) agrees with those of [8, Lemma II.22]. Therefore [8, Theorem III.10] follows directly from [8, Theorem II.28] and Theorem III.11 follows from [8, Theorem IV.4]. □

## IV. Bounds for Covariances

### *Integral bounds*

**Definition IV.1.** For any covariance $C = C(\xi, \xi')$ we define

$$S(C) = \sup_{m} \sup_{\xi_1, \ldots, \xi_m \in \mathcal{B}} \left( \left| \int \psi(\xi_1) \cdots \psi(\xi_m) d\mu_C(\psi) \right| \right)^{1/m} .$$

**Remark IV.2.** (i) By Remark III.6, $2\,S(C)$ is an integral bound for $C$ with respect to the $L_1$–$L_\infty$-norms introduced in Example II.6.

(ii) For any two covariances $C, C'$

$$S(C + C') \le S(C) + S(C') \,.$$

**Proof of (ii).** For $\xi_1, \ldots, \xi_m \in \mathcal{B}$

$$\int \psi(\xi_1) \cdots \psi(\xi_m) d\mu_{(C+C')}(\psi)$$
$$= \int (\psi(\xi_1) + \psi'(\xi_1)) \cdots (\psi(\xi_m) + \psi'(\xi_m)) d\mu_C(\psi) d\mu_{C'}(\psi') \,.$$

Multiplying out one sees that

$$(\psi(\xi_1) + \psi'(\xi_1)) \cdots (\psi(\xi_m) + \psi'(\xi_m)) = \sum_{p=0}^{m} \sum_{\substack{I \subset \{1, \ldots, m\} \\ |I| = p}} \mathcal{M}(p, I)$$

with

$$\mathcal{M}(p, I) = \pm \prod_{i \in I} \psi(\xi_i) \prod_{j \notin I} \psi'(\xi_j) \,.$$

Therefore

$$\begin{aligned}\left|\int \psi(\xi_1)\cdots\psi(\xi_m)d\mu_{(C+C')}(\psi)\right| &\le \sum_{p=0}^{m}\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=p}}\left|\iint \mathcal{M}(p,I)d\mu_C(\psi)d\mu_{C'}(\psi')\right|\\ &\le \sum_{p=0}^{m}\sum_{\substack{I\subset\{1,\ldots,m\}\\|I|=p}} S(C)^p S(C')^{m-p}\\ &= (S(C)+S(C'))^m\,. \qquad \square\end{aligned}$$

In this section, we assume that there is a function $C(k)$ such that for $\xi=(x,a)=(x_0,\mathbf{x},\sigma,a),\ \xi'=(x',a')=(x_0',\mathbf{x}',\sigma',a')\in\mathcal{B}$

$$C(\xi,\xi')=\begin{cases}\delta_{\sigma,\sigma'}\displaystyle\int\frac{d^{d+1}k}{(2\pi)^{d+1}}e^{\imath\langle k,x-x'\rangle_-}C(k) & \text{if } a=0,\ a'=1\\[2ex] -\delta_{\sigma,\sigma'}\displaystyle\int\frac{d^{d+1}k}{(2\pi)^{d+1}}e^{\imath\langle k,x'-x>_-}C(k) & \text{if } a=1,\ a'=0\\[2ex] 0 & \text{if } a=a'\end{cases}\tag{IV.1}$$

(as usual, the case $x_0=x_0'=0$ is defined through the limit $x_0-x_0'\to 0-$) and derive bounds for $S(C)$ in terms of norms of $C(k)$.

**Proposition IV.3 (Gram's estimate).**

(i)

$$S(C)\le\sqrt{\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\ |C(k)|}$$

(ii) *Let, for each $s$ in a finite set $\Sigma$, $\chi_s(k)$ be a function on $\mathbb{R}\times\mathbb{R}^d$. Set, for $a\in\{0,1\}$,*

$$\hat\chi_s(x-x',a)=\int e^{(-1)^a\imath\langle k,x-x'\rangle_-}\chi_s(k)\frac{d^{d+1}k}{(2\pi)^{d+1}}$$

*and*

$$\psi_s(x,a)=\int d^{d+1}x'\ \hat\chi_s(x-x',a)\psi(x',a)\,.$$

*Then for all $\xi_1,\ldots,\xi_m\in\mathcal{B}$ and all $s_1,\ldots,s_m\in\Sigma$*

$$\left|\int\psi_{s_1}(\xi_1)\cdots\psi_{s_m}(\xi_m)d\mu_C(\psi)\right|\le\left[\max_{s\in\Sigma}\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\ |C(k)\chi_s(k)^2|\right]^{m/2}\,.$$

**Proof.** Let $\mathcal{H}$ be the Hilbert space $\mathcal{H}=L^2(\mathbb{R}\times\mathbb{R}^d)\otimes\mathbb{C}^2$. For $\sigma\in\{\uparrow,\downarrow\}$ define the element $\omega(\sigma)\in\mathbb{C}^2$ by

$$\omega(\sigma)=\begin{cases}(1,0) & \text{if } \sigma=\uparrow\\ (0,1) & \text{if } \sigma=\downarrow\end{cases}\,.$$

For each $\xi = (x,a) = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B}$ define $w(\xi) \in \mathcal{H}$ by

$$w(\xi) = \begin{cases} \frac{e^{-\imath\langle k,x\rangle_-}}{(2\pi)^{(d+1)/2}} \sqrt{|C(k)|} \otimes \omega(\sigma) & \text{if } a = 0 \\ \frac{e^{-\imath\langle k,x\rangle_-}}{(2\pi)^{(d+1)/2}} \frac{C(k)}{\sqrt{|C(k)|}} \otimes \omega(\sigma) & \text{if } a = 1 \end{cases} .$$

Then

$$\|w(\xi)\|_{\mathcal{H}}^2 = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \; |C(k)| \quad \text{for all } \xi \in \mathcal{B}$$

and

$$C(\xi, \xi') = \langle w(\xi), w(\xi')\rangle_{\mathcal{H}}$$

if $\xi = (x, \sigma, 0), \xi' = (x', \sigma', 1) \in \mathcal{B}$. Part (i) of the proposition now follows from [8, Proposition B.1(i)].

(ii) For each $\xi = (x,a) = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B}$ and $s \in \Sigma$ define $w'(\xi, s) \in \mathcal{H}$ by

$$w'(\xi, s) = \begin{cases} \frac{e^{-\imath\langle k,x\rangle_-}}{(2\pi)^{(d+1)/2}} \sqrt{|C(k)|}\; \overline{\chi_s(k)} \otimes \omega(\sigma) & \text{if } a = 0 \\ \frac{e^{-\imath\langle k,x\rangle_-}}{(2\pi)^{(d+1)/2}} \frac{C(k)}{\sqrt{|C(k)|}} \chi_s(k) \otimes \omega(\sigma) & \text{if } a = 1 \end{cases} .$$

Then

$$\|w'(\xi, s)\|_{\mathcal{H}}^2 = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \; |C(k)||\chi_s(k)|^2$$

and

$$\int \psi_s(\xi)\psi_{s'}(\xi')d\mu_C(\xi) = \langle w(\xi, s), w(\xi', s')\rangle_{\mathcal{H}}$$

if $\xi = (x_0, \mathbf{x}, \sigma, 0), \xi' = (x_0', \mathbf{x}', \sigma', 1) \in \mathcal{B}$. Part (ii) of the proposition now follows from [8, Proposition B.1(i)], applied to the generating system of fields $\psi_s(\xi)$. □

**Lemma IV.4.** *Let* $\Lambda > 0$ *and* $U(\mathbf{k})$ *a function on* $\mathbb{R}^d$. *Assume that*

$$C(k) = \frac{U(\mathbf{k})}{\imath k_0 - \Lambda} .$$

*Then*

$$S(C) \le \sqrt{\int \frac{d^d\mathbf{k}}{(2\pi)^d} \; |U(\mathbf{k})|} .$$

**Proof.** For $a = 0,\ a' = 1$

$$\begin{aligned} &C((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a')) \\ &\quad = \delta_{\sigma,\sigma'} \int \frac{dk_0}{2\pi} \frac{e^{-\imath k_0 (x_0 - x_0')}}{\imath k_0 - \Lambda} \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{\imath\langle \mathbf{k}, \mathbf{x}-\mathbf{x}'\rangle} U(\mathbf{k}) \\ &\quad = -\delta_{\sigma,\sigma'} \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{\imath\langle \mathbf{k}, \mathbf{x}-\mathbf{x}'\rangle} U(\mathbf{k}) \begin{cases} e^{-\Lambda(x_0 - x_0')} & \text{if } x_0 > x_0' \\ 0 & \text{if } x_0 \le x_0' \end{cases} . \end{aligned}$$

Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^2$. For $\sigma \in \{\uparrow, \downarrow\}$ define the element $\omega(\sigma) \in \mathbb{C}^2$ as in the proof of Proposition IV.3, and for each $\xi = (x_0, \mathbf{x}, \sigma, a) \in \mathcal{B}$ define $w(\xi) \in \mathcal{H}$ by

$$w(\xi) = \begin{cases} \dfrac{e^{-\imath\langle \mathbf{k}, \mathbf{x}\rangle}}{(2\pi)^{d/2}} \sqrt{|U(\mathbf{k})|} \otimes \omega(\sigma) & \text{if } a = 0 \\ -\dfrac{e^{-\imath\langle \mathbf{k}, \mathbf{x}\rangle}}{(2\pi)^{d/2}} \dfrac{U(\mathbf{k})}{\sqrt{|U(\mathbf{k})|}} \otimes \omega(\sigma) & \text{if } a = 1 . \end{cases}$$

Again

$$\|w(\xi)\|_{\mathcal{H}}^2 = \frac{1}{(2\pi)^d} \int d^d\mathbf{k}\ |U(\mathbf{k})| \quad \text{for all } \xi \in \mathcal{B} .$$

Furthermore set $\tau(x_0, \mathbf{x}, \sigma, a) = \Lambda x_0$. Then for $\xi = (x_0, \mathbf{x}, \sigma, 0), \xi' = (x_0', \mathbf{x}', \sigma', 1) \in \mathcal{B}$

$$C(\xi, \xi') = \begin{cases} e^{-(\tau(\xi) - \tau(\xi'))} \langle w(\xi), w(\xi')\rangle_{\mathcal{H}} & \text{if } \tau(\xi) > \tau(\xi') \\ 0 & \text{if } \tau(\xi) \le \tau(\xi') \end{cases} .$$

The lemma now follows from [8, Proposition B.1(ii)]. □

**Proposition IV.5.** *Assume that $C$ is of the form*

$$C(k) = \frac{U(\mathbf{k}) - \chi(k)}{\imath k_0 - e(\mathbf{k})}$$

*with real valued measurable functions $U(\mathbf{k}),\ e(\mathbf{k})$ on $\mathbb{R}^d$ and $\chi(k)$ on $\mathbb{R} \times \mathbb{R}^d$ such that $0 \le \chi(k) \le U(\mathbf{k}) \le 1$ for all $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$. Then*

$$S(C)^2 \le 9 \int \frac{d^d\mathbf{k}}{(2\pi)^d}\ U(\mathbf{k}) + \frac{3}{E} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\ \chi(k) + 6 \int_{|k_0| \le E} \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{U(\mathbf{k}) - \chi(k)}{|\imath k_0 - e(\mathbf{k})|}$$

*where $E = \sup_{\mathbf{k} \in \mathrm{supp}\, U} |e(\mathbf{k})|$.*

**Proof.** Write

$$C(k) = \frac{U(\mathbf{k})}{\imath k_0 - E} - \frac{\chi(k)}{\imath k_0 - E} + \frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)} (U(\mathbf{k}) - \chi(k)) .$$

By Remark IV.2, Lemma IV.4 and Proposition IV.3(i)

$$\frac{1}{3}S(C)^2 \le \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, |U(\mathbf{k})| + \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left|\frac{\chi(k)}{\imath k_0 - E}\right| + \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left|\frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)}(U(\mathbf{k}) - \chi(k))\right| .$$

The first two terms are bounded by

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d}\, U(\mathbf{k}) + \frac{1}{E}\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\, \chi(k) .$$

The contribution to the third term having $|k_0| \le E$ is bounded by

$$\int_{|k_0|\le E} \frac{d^{d+1}k}{(2\pi)^{d+1}} \left|\frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)}(U(\mathbf{k}) - \chi(k))\right| \le 2\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{U(\mathbf{k}) - \chi(k)}{|\imath k_0 - e(\mathbf{k})|} .$$

The contribution to the third term having $|k_0| > E$ is bounded by

$$\int_{|k_0|> E} \frac{d^{d+1}k}{(2\pi)^{d+1}} \left|\frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)}(U(\mathbf{k}) - \chi(k))\right| \le 4\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{E}{|\imath k_0 - E|^2} U(\mathbf{k}) = 2\int \frac{d^d\mathbf{k}}{(2\pi)^d} U(\mathbf{k}) .$$

Hence

$$\frac{1}{3}S(C)^2 \le 3\int \frac{d^d\mathbf{k}}{(2\pi)^d} U(\mathbf{k}) + \frac{1}{E}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \chi(k) + 2\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{U(\mathbf{k}) - \chi(k)}{|\imath k_0 - e(\mathbf{k})|} . \qquad \square$$

***Contraction bounds***

We have observed in Example III.4 that the $L_1$–$L_\infty$-norm introduced in Example II.6 has $\max\{\|C\|_{1,\infty}, |||C|||_\infty\}$ as a contraction bound for covariance $C$. For the propagators of the form (IV.1), we estimate these position space quantities by norms of derivatives of $C(k)$ in momentum space.

**Definition IV.6.** (i) For a function $f(k)$ on $\mathbb{R} \times \mathbb{R}^d$ and a multi-index $\delta$ we set

$$\mathrm{D}^\delta f(k) = \frac{\partial^{\delta_0}}{\partial k_0^{\delta_0}} \frac{\partial^{\delta_1}}{\partial k_1^{\delta_1}} \cdots \frac{\partial^{\delta_d}}{\partial k_d^{\delta_d}} f(k)$$

and

$$\|f(k)\|_{\infty}^{\check{}} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\left(\sup_k |\mathrm{D}^\delta f(k)|\right) t^\delta \qquad \in \mathfrak{N}_{d+1}$$

$$\|f(k)\|_{1}^{\check{}} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\left(\int \frac{d^{d+1}k}{(2\pi)^{d+1}} |\mathrm{D}^\delta f(k)|\right) t^\delta \in \mathfrak{N}_{d+1}\,.$$

If $B$ is a measurable subset of $\mathbb{R}\times\mathbb{R}^d$,

$$\|f(k)\|_{\infty,B}^{\check{}} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\left(\sup_{k\in B} |\mathrm{D}^\delta f(k)|\right) t^\delta \qquad \in \mathfrak{N}_{d+1}$$

$$\|f(k)\|_{1,B}^{\check{}} = \sum_{\delta\in\mathbb{N}_0\times\mathbb{N}_0^d} \frac{1}{\delta!}\left(\int_B \frac{d^{d+1}k}{(2\pi)^{d+1}} |\mathrm{D}^\delta f(k)|\right) t^\delta \in \mathfrak{N}_{d+1}\,.$$

(ii) For $\mu>0$ and $X\in\mathfrak{N}_{d+1}$

$$T_\mu X = \frac{1}{\mu^{d+1}}X + \frac{\mu}{d+1}\sum_{j=0}^{d}\left(\frac{\partial}{\partial t_0}\cdots\frac{\partial}{\partial t_d}\right)\frac{\partial}{\partial t_j}X\,.$$

**Remark IV.7.** For functions $f(k)$ and $g(k)$ on $B\subset\mathbb{R}\times\mathbb{R}^d$

$$\|f(k)g(k)\|_{1,B}^{\check{}} \le \|f(k)\|_{1,B}^{\check{}}\|g(k)\|_{\infty,B}^{\check{}}$$

by Leibniz's rule for derivatives. The proof is similar to that of Lemma II.7.

**Proposition IV.8.** *Let $d\ge 1$. Assume that there is a function $C(k)$ such that for $\xi=(x,a)=(x_0,\mathbf{x},\sigma,a),\ \xi'=(x',a')=(x_0',\mathbf{x}',\sigma',a')\in\mathcal{B}$*

$$C(\xi,\xi') = \begin{cases} \delta_{\sigma,\sigma'}\displaystyle\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k,x-x'\rangle_-}C(k) & if\ a=0,\ a'=1\\ 0 & if\ a=a'\\ -C(\xi',\xi) & if\ a=1,\ a'=0\,.\end{cases}$$

*Let $\delta$ be a multi-index and $0<\mu\le 1$.*

(i) $$|||\mathcal{D}^\delta_{1,2}C|||_\infty \le \int \frac{d^{d+1}k}{(2\pi)^{d+1}}|\mathrm{D}^\delta C(k)| \le \frac{\mathrm{vol}}{(2\pi)^{d+1}}\sup_{k\in\mathbb{R}\times\mathbb{R}^d}|\mathrm{D}^\delta C(k)|$$

*and*

$$\|C\|_{1,\infty} \le {\scriptstyle const}\, T_\mu\|C(k)\|_1^{\check{}} \le {\scriptstyle const}\frac{\mathrm{vol}}{(2\pi)^{d+1}}T_\mu\|C(k)\|_\infty^{\check{}}$$

*where* vol *is the volume of the support of $C(k)$ in $\mathbb{R}\times\mathbb{R}^d$ and the constant* const *depends only on the dimension $d$.*

(ii) *Assume that there is an $r$-times differentiable real valued function $e(\mathbf{k})$ on $\mathbb{R}^d$ such that $|e(\mathbf{k})| \ge \mu$ for all $\mathbf{k} \in \mathbb{R}^d$ and a real valued, compactly supported, smooth, non-negative function $U(\mathbf{k})$ on $\mathbb{R}^d$ such that*

$$C(k) = \frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}\,.$$

*Set*

$$g_1 = \int_{\mathrm{supp}\,U} d^d\mathbf{k} \frac{1}{|e(\mathbf{k})|} \qquad g_2 = \int_{\mathrm{supp}\,U} d^d\mathbf{k} \frac{\mu}{|e(\mathbf{k})|^2}\,.$$

*Then there is a constant $\mathsf{const}$ such that, for all multi-indices $\delta$ whose spatial part $|\boldsymbol{\delta}| \le r - d - 1$,*

$$|||C|||_\infty \le \mathsf{const} \qquad \frac{1}{\delta!}|||\mathcal{D}^\delta_{1,2}C|||_{1,\infty} \le \frac{\mathsf{const}}{\mu^{d+|\delta|}} \begin{cases} g_1 & if\ |\delta| = 0 \\ 2^{|\delta|}g_2 & if\ |\delta| \ge 1 \end{cases}.$$

*The constant $\mathsf{const}$ depends only on the dimension $d$, the degree of differentiability $r$, the ultraviolet cutoff $U(\mathbf{k})$ and the quantities $\sup_{\mathbf{k}} |\mathrm{D}^{\boldsymbol{\gamma}} e(\mathbf{k})|$, $\boldsymbol{\gamma} \in \mathbb{N}_0^d,\ |\boldsymbol{\gamma}| \le r$.*

(iii) *Assume that $C$ is of the form*

$$C(k) = \frac{U(\mathbf{k}) - \chi(k)}{\imath k_0 - e(\mathbf{k})}$$

*with real valued functions $U(\mathbf{k})$, $e(\mathbf{k})$ on $\mathbb{R}^d$ and $\chi(k)$ on $\mathbb{R} \times \mathbb{R}^d$ that fulfill the following conditions:*

> *The function $e(\mathbf{k})$ is $r$ times differentiable. $|\imath k_0 - e(\mathbf{k})| \ge \mu$ for all $k = (k_0, \mathbf{k})$ in the support of $U(\mathbf{k}) - \chi(k)$. The function $U(\mathbf{k})$ is smooth and has compact support. The function $\chi(k)$ is smooth and has compact support and $0 \le \chi(k) \le U(\mathbf{k}) \le 1$ for all $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$.*

*There is a constant* const *such that*

$$|||C|||_\infty \le \mathrm{const}\,. \tag{IV.2}$$

*The constant* const *depends on $d$, $\mu$ and the supports of $U(\mathbf{k})$ and $\chi$.*

*Let $r_0 \in \mathbb{N}$. There is a constant* const *such that, for all multi-indices $\delta$ whose spatial part $|\boldsymbol{\delta}| \le r - d - 1$ and whose temporal part $|\delta_0| \le r_0 - 2$,*

$$|||\mathcal{D}^\delta_{1,2}C|||_{1,\infty} \le \mathrm{const}\,. \tag{IV.3}$$

*The constant* const *depends on $d$, $r$, $r_0$, $\mu$, $U(\mathbf{k})$ and the quantities $\sup_{\mathbf{k}} |\mathrm{D}^{\boldsymbol{\gamma}} e(\mathbf{k})|$ with $\boldsymbol{\gamma} \in \mathbb{N}_0^d,\ |\boldsymbol{\gamma}| \le r$ and $\sup_k |\mathrm{D}^\beta \chi(k)|$ with $\beta \in \mathbb{N}_0 \times \mathbb{N}_0^d,\ \beta_0 \le r_0,\ |\boldsymbol{\beta}| \le r$.*

**Proof.** (i) As the Fourier transform of the operator $\mathrm{D}^{\delta'}$ is, up to a sign, multiplication by $[-i(x-x')]^{\delta'}$, we have for $\xi = (x, \sigma, a)$ and $\xi' = (x', \sigma', a')$

$$|(x-x')^{\delta'}|\,|\mathcal{D}^\delta_{1,2}C(\xi,\xi')| \le \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |\mathrm{D}^{\delta+\delta'}C(k)|\,.$$

In particular

$$|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')| \le \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta}C(k)| \tag{IV.4}$$

and, for $j = 0, 1, \ldots, d$,

$$\mu^{d+2}|x_j - x'_j| \prod_{i=0}^{d} |x_i - x'_i|\,|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')|$$

$$\le \mu^{d+2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta+\epsilon+\epsilon_j}C(k)| \tag{IV.5$_j$}$$

where $\epsilon = (1, 1, \ldots, 1)$ and $\epsilon_j$ is the $j$th unit vector. Taking the geometric mean of (IV.5$_0$), …, (IV.5$_d$) on the left-hand side and the arithmetic mean on the right-hand side gives

$$\mu^{d+2} \prod_{i=0}^{d} |x_i - x'_i|^{1+\frac{1}{d+1}}\,|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')|$$

$$\le \frac{\mu^{d+2}}{d+1} \sum_{j=0}^{d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta+\epsilon+\epsilon_j}C(k)|\,. \tag{IV.6}$$

Adding (IV.4) and (IV.6) gives

$$\left(1 + \mu^{d+2} \prod_{i=0}^{d} |x_i - x'_i|^{1+\frac{1}{d+1}}\right)|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')|$$

$$\le \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta}C(k)| + \frac{\mu^{d+2}}{d+1} \sum_{j=0}^{d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta+\epsilon+\epsilon_j}C(k)|\,. \tag{IV.7}$$

Dividing across and using $\int \frac{d^{d+1}x}{1+\mu^{d+2}\prod_{i=0}^{d}|x_i|^{1+\frac{1}{d+1}}} \le \mathit{const}\,\frac{1}{\mu^{d+1}}$ we get

$$|||\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')|||_{1,\infty} \le \mathit{const}\left(\frac{1}{\mu^{d+1}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\,|\mathrm{D}^{\delta}C(k)|\right.$$

$$\left. + \frac{\mu}{d+1} \sum_{j=0}^{d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} |\mathrm{D}^{\delta+\epsilon+\epsilon_j}C(k)|\right).$$

The contents of the bracket on the right-hand side are, up to a factor of $\frac{1}{\delta!}$, the coefficient of $t^{\delta}$ in $T_\mu \|C(k)\|^{\check{}}_1$.

(ii) Denote by

$$C(t,\mathbf{k}) = \int \frac{dk_0}{2\pi} e^{-\imath k_0 t} \frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}$$

$$= U(\mathbf{k}) e^{-e(\mathbf{k})t} \begin{cases} -\chi(e(\mathbf{k}) > 0) & \text{if } t > 0 \\ \chi(e(\mathbf{k}) < 0) & \text{if } t \le 0 \end{cases}$$

the partial Fourier transform of $C(k)$ in the $k_0$ direction. (As usual, the case $t = 0$ is defined through the limit $t \to 0-$.) Then, for $|\boldsymbol{\delta}| + |\boldsymbol{\delta}'| \le r$,

$$\begin{aligned}
&|(\mathbf{x}-\mathbf{x}')^{\boldsymbol{\delta}'}|\,|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')| \\
&\quad \le \int \frac{d^d\mathbf{k}}{(2\pi)^d}|\mathcal{D}^{\delta_0}_{1,2}\mathrm{D}^{\boldsymbol{\delta}+\boldsymbol{\delta}'}C(t-t',\mathbf{k})| \\
&\quad \le \mathit{const}\int_{\mathrm{supp}\,U} d^d\mathbf{k}(|t-t'|^{\delta_0+|\boldsymbol{\delta}|+|\boldsymbol{\delta}'|}+|t-t'|^{\delta_0})e^{-|e(\mathbf{k})(t-t')|} \\
&\quad \le \mathit{const}\int_{\mathrm{supp}\,U} d^d\mathbf{k}\left[\frac{(\frac{3}{2})^{\delta_0+|\boldsymbol{\delta}|+|\boldsymbol{\delta}'|}(\delta_0+|\boldsymbol{\delta}|+|\boldsymbol{\delta}'|)!}{|e(\mathbf{k})|^{\delta_0+|\boldsymbol{\delta}|+|\boldsymbol{\delta}'|}}+\frac{(\frac{3}{2})^{\delta_0}\delta_0!}{|e(\mathbf{k})|^{\delta_0}}\right]e^{-|e(\mathbf{k})(t-t')/3|} \\
&\quad \le \mathit{const}\;2^{\delta_0}\delta_0!\int_{\mathrm{supp}\,U} d^d\mathbf{k}\frac{1}{|e(\mathbf{k})|^{|\delta|+|\boldsymbol{\delta}'|}}e^{-|e(\mathbf{k})(t-t')/3|}\,.
\end{aligned}$$

In particular, $|||C|||_\infty \le \mathit{const}$ and

$$\begin{aligned}
|(\mathbf{x}-\mathbf{x}')^{\boldsymbol{\delta}'}|\int dt'\,|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')| &\le \mathit{const}\;2^{\delta_0}\delta_0!\int_{\mathrm{supp}\,U} d^d\mathbf{k}\frac{1}{|e(\mathbf{k})|^{|\delta|+|\boldsymbol{\delta}'|+1}} \\
&\le \mathit{const}\;2^{\delta_0}\delta_0!\begin{cases} g_1 & \text{if } |\delta|+|\boldsymbol{\delta}'|=0 \\ \dfrac{g_2}{\mu^{|\delta|+|\boldsymbol{\delta}'|}} & \text{if } |\delta|+|\boldsymbol{\delta}'|>0\end{cases} \\
&\le \mathit{const}\frac{2^{\delta_0}\delta_0!}{\mu^{|\boldsymbol{\delta}'|}}\begin{cases} g_1 & \text{if } |\delta|=0 \\ \dfrac{g_2}{\mu^{|\delta|}} & \text{if } |\delta|>0\end{cases}
\end{aligned}$$

since $g_1 \ge g_2$. As in Eqs. (IV.4)–(IV.7), choosing various $\boldsymbol{\delta}'$'s with $|\boldsymbol{\delta}'| = d+1$,

$$\begin{aligned}
\int dt'\,|\mathcal{D}^{\delta}_{1,2}C(\xi,\xi')| &\le \mathit{const}\;2^{\delta_0}\delta_0!\frac{1}{1+\mu^{d+1}\prod_{i=1}^d|x_i-x_i'|^{1+\frac{1}{d}}} \\
&\quad\times\begin{cases} g_1 & \text{if } |\delta|=0 \\ \dfrac{g_2}{\mu^{|\delta|}} & \text{if } |\delta|>0\end{cases}\,.
\end{aligned}$$

Integrating $\mathbf{x}'$ gives the desired bound on $|||\mathcal{D}^{\delta}_{1,2}C|||_{1,\infty}$.

(iii) Write

$$C(k) = C_1(k) - C_2(k) + C_3(k)$$

with

$$\begin{aligned}
C_1(k) &= \frac{U(\mathbf{k})}{\imath k_0 - E} \\
C_2(k) &= \frac{\chi(k)}{\imath k_0 - E}
\end{aligned}$$

$$C_3(k) = \frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)}(U(\mathbf{k}) - \chi(k))$$

and define the covariances $C_j$ by

$$C_j(\xi, \xi') = \begin{cases} \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k, x-x'\rangle_-} C_j(k) & \text{if } a = 0,\ a' = 1 \\ 0 & \text{if } a = a' \\ -C_j(\xi', \xi) & \text{if } a = 1,\ a' = 0 \end{cases}$$

for $j = 1, 2, 3$. For $a = 0,\ a' = 1$

$$C_1((x_0, \mathbf{x}, \sigma, a), (x_0', \mathbf{x}', \sigma', a'))$$
$$= -\delta_{\sigma,\sigma'} \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{\imath\langle \mathbf{k}, \mathbf{x}-\mathbf{x}'\rangle} U(\mathbf{k}) \begin{cases} e^{-E(x_0 - x_0')} & \text{if } x_0 > x_0' \\ 0 & \text{if } x_0 \le x_0' \end{cases}$$

and, for $|\boldsymbol{\delta}| \le r,\ |\delta_0| \le r_0$,

$$|||\mathcal{D}_{1,2}^{\delta} C_1|||_\infty \le \frac{const}{E^{\delta_0}} \delta_0! \le const$$

$$|||\mathcal{D}_{1,2}^{\delta} C_1|||_{1,\infty} \le \frac{const}{E^{\delta_0+1}} \delta_0! \le const\,.$$

By Remark IV.7

$$\|C_2(k)\|_1^{\check{}} \le \|\chi(k)\|_1^{\check{}} \left\| \frac{1}{\imath k_0 - E} \right\|_\infty^{\check{}} \le \|\chi(k)\|_1^{\check{}} \left( \sum_{n=0}^{\infty} \frac{1}{E^{n+1}} t_0^n \right)$$

so that, for $|\delta_0| \le r_0 - 2$ and $|\boldsymbol{\delta}| \le r - d - 1$,

$$|||\mathcal{D}_{1,2}^{\delta} C_2|||_\infty \le const \qquad |||\mathcal{D}_{1,2}^{\delta} C_2|||_{1,\infty} \le const$$

by part (i).

We now bound $C_3$. Let $B$ be the support of $U(\mathbf{k}) - \chi(k)$. On $B$, $|\imath k_0 - e(\mathbf{k})| \ge \mu > 0$ and $|e(\mathbf{k})| \le E$, so we have, for $\delta = (\delta_0, \boldsymbol{\delta}) \ne 0$ with $|\boldsymbol{\delta}| \le r$ and $\delta_0 \le r_0$,

$$\left| \mathrm{D}^\delta \frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)} \right| \le const\, \frac{E}{|\imath k_0 - E|} \left( \frac{1}{|\imath k_0 - e(\mathbf{k})|^{|\delta|+1}} + \frac{1}{|\imath k_0 - e(\mathbf{k})|} \right)$$
$$\le const\, \frac{1}{\mu^{|\delta|}} \frac{E}{|\imath k_0 - E|\,|\imath k_0 - \mu|}\,.$$

Integrating

$$\frac{1}{\delta!} \int_B \frac{d^{d+1}k}{(2\pi)^{d+1}} \left| \mathrm{D}^\delta \frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)} \right| \le \text{const}\,.$$

It follows that

$$\left\| \frac{e(\mathbf{k}) - E}{(\imath k_0 - e(\mathbf{k}))(\imath k_0 - E)} \right\|_{1,B}^{\check{}} \le \text{const} \sum_{\substack{|\boldsymbol{\delta}| \le r \\ |\delta_0| \le r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}| > r \\ \text{or } |\delta_0| > r_0}} \infty\, t^\delta$$

and, by Remark IV.7, that

$$\|C_3(k)\|_1^{\check{}} \le \text{const}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }|\delta_0|>r_0}} \infty t^\delta\right)(\|U(\mathbf{k})\|_\infty^{\check{}} + \|\chi(k)\|_\infty^{\check{}})\,.$$

By part (i) of this proposition and the previous bounds on $C_1$ and $C_2$, this concludes the proof of part (iii). □

**Corollary IV.9.** *Under the hypotheses of Proposition* IV.8(ii), *the* $(d+1)$-*dimensional norm*

$$\|C\|_{1,\infty} \le \frac{const}{\mu^d}\left(g_1 + g_2 \sum_{\substack{|\delta|\ge 1\\ |\boldsymbol{\delta}|\le r-d-1}} \left(\frac{2}{\mu}\right)^{|\delta|} t^\delta + \sum_{|\boldsymbol{\delta}|\ge r-d} \infty t^\delta\right)$$

$$\le \frac{const\ g_1}{\mu^d}\left(\sum_{|\boldsymbol{\delta}|\le r-d-1} \left(\frac{2}{\mu}\right)^{|\delta|} t^\delta + \sum_{|\boldsymbol{\delta}|\ge r-d} \infty t^\delta\right).$$

*Under the hypotheses of Proposition* IV.8(iii)

$$\|C\|_{1,\infty} \le \text{const}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r-d-1\\ |\delta_0|\le r_0-2}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r-d-1\\ \text{or }|\delta_0|>r_0-2}} \infty t^\delta\right).$$

In the renormalization group analysis we shall add a counterterm $\delta e(\mathbf{k})$ to the dispersion relation $e(\mathbf{k})$. For such a counterterm, we define the Fourier transform[e]

$$\delta\hat{e}(\xi,\xi') = \delta_{\sigma,\sigma'}\delta_{a,a'}\delta(x_0 - x_0')\int e^{(-1)^a \imath\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\delta e(\mathbf{k})\frac{d^d\mathbf{k}}{(2\pi)^d}$$

for $\xi = (x,a) = (x_0,\mathbf{x},\sigma,a),\ \xi' = (x',a') = (x_0',\mathbf{x}',\sigma',a') \in \mathcal{B}$.

**Definition IV.10.** Fix $r_0$ and $r$. Let

$$\mathfrak{c}_0 = \sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }|\delta_0|>r_0}} \infty t^\delta \in \mathfrak{N}_{d+1}\,.$$

The map $\mathfrak{e}_0(X) = \frac{\mathfrak{c}_0}{1-X}$ from $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < 1$ to $\mathfrak{N}_{d+1}$ is used to implement the differentiability properties of various kernels depending on a counterterm whose norm is bounded by $X$.

**Proposition IV.11.** *Let*

$$C(k) = \frac{U(\mathbf{k}) - \chi(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})} \qquad C_0(k) = \frac{U(\mathbf{k}) - \chi(k)}{\imath k_0 - e(\mathbf{k})}$$

[e]A comprehensive set of Fourier transform conventions are formulated in Sec. IX.

*with real valued functions* $U(\mathbf{k})$, $e(\mathbf{k})$, $\delta e(\mathbf{k})$ *on* $\mathbb{R}^d$ *and* $\chi(k)$ *on* $\mathbb{R}\times\mathbb{R}^d$ *that fulfill the following conditions*:

> *The function* $e(\mathbf{k})$ *is* $r+d+1$ *times differentiable.* $|\imath k_0 - e(\mathbf{k})| \ge \mu_e > 0$ *for all* $k=(k_0,\mathbf{k})$ *in the support of* $U(\mathbf{k})-\chi(k)$. *The function* $U(\mathbf{k})$ *is smooth and has compact support. The function* $\chi(k)$ *is smooth and has compact support and* $0 \le \chi(k) \le U(\mathbf{k}) \le 1$ *for all* $k=(k_0,\mathbf{k}) \in \mathbb{R}\times\mathbb{R}^d$. *The function* $\delta e(\mathbf{k})$ *obeys*

$$\|\delta\hat{e}\|_{1,\infty} < \mu + \sum_{\delta\neq\mathbf{0}} \infty t^{\delta}\,.$$

*Then, there is a constant* $\mu_1 > 0$ *such that if* $\mu < \mu_1$, *the following hold*

(i) $C$ *is an analytic function of* $\delta e$ *and*

$$|||C|||_\infty \le \text{const} \qquad |||C - C_0|||_\infty \le \text{const}|||\delta\hat{e}|||_{1,\infty}$$

*and*

$$\|C\|_{1,\infty} \le \text{const}\ \mathfrak{e}_0(\|\delta\hat{e}\|_{1,\infty}) \qquad \|C-C_0\|_{1,\infty} \le \text{const}\ \mathfrak{e}_0(\|\delta\hat{e}\|_{1,\infty})\|\delta\hat{e}\|_{1,\infty}\,.$$

(ii) *Let*

$$C_s(k) = \frac{U(\mathbf{k})-\chi(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k}) + s\delta e'(\mathbf{k})}\,.$$

*Then*

$$\left|\left|\left| \frac{d}{ds} C_s\Big|_{s=0}\right|\right|\right|_\infty \le \text{const}|||\delta\hat{e}'|||_{1,\infty}$$

$$\left\| \frac{d}{ds} C_s\Big|_{s=0}\right\|_{1,\infty} \le \text{const}\ \mathfrak{e}_0(\|\delta\hat{e}\|_{1,\infty})\,\|\delta\hat{e}'\|_{1,\infty}\,.$$

**Proof.** (i) The first bound follows from (IV.2), by replacing $e$ by $e-\delta e$.

Select a smooth, compactly supported function $\tilde{U}(\mathbf{k})$ and a smooth compactly supported function $\tilde{\chi}(k)$ such that $0 \le \tilde{\chi}(k) \le \tilde{U}(\mathbf{k}) \le 1$ for all $k=(k_0,\mathbf{k}) \in \mathbb{R}\times\mathbb{R}^d$, $\tilde{U}(\mathbf{k})-\tilde{\chi}(k)$ is identically 1 on the support of $U(\mathbf{k})-\chi(k)$ and $|\imath k_0 - e(\mathbf{k})| \ge \frac{1}{2}\mu_e$ for all $k=(k_0,\mathbf{k})$ in the support of $\tilde{U}(\mathbf{k})-\tilde{\chi}(k)$. Let

$$\tilde{C}_0(k) = \frac{\tilde{U}(\mathbf{k})-\tilde{\chi}(k)}{\imath k_0 - e(\mathbf{k})}\,.$$

Then

$$C(k) = \frac{C_0(k)}{1+\frac{\delta e(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}} = \frac{C_0(k)}{1+\frac{\delta e(\mathbf{k})\,(\tilde{U}(\mathbf{k})-\tilde{\chi}(k))}{\imath k_0 - e(\mathbf{k})}} = \frac{C_0(k)}{1+\delta e(\mathbf{k})\,\tilde{C}_0(k)}$$

$$= C_0(k)\sum_{n=0}^{\infty}(-\delta e(\mathbf{k})\,\tilde{C}_0(k))^n\,.$$

Then, by iterated application of Lemma II.7 and the second part of Corollary IV.9, with $r$ replaced by $r+d+1$ and $r_0$ replaced by $r_0+2$,

$$\begin{aligned}\|C\|_{1,\infty} &\le \|C_0\|_{1,\infty} \sum_{n=0}^{\infty}(\|\delta\hat{e}\|_{1,\infty}\,\|\tilde{C}_0\|_{1,\infty})^n \\ &\le \text{const}\ \mathfrak{c}_0 \sum_{n=0}^{\infty}(\text{const}'\ \mathfrak{c}_0\|\delta\hat{e}\|_{1,\infty})^n \\ &= \text{const}\,\frac{\mathfrak{c}_0}{1-\text{const}'\ \mathfrak{c}_0\|\delta\hat{e}\|_{1,\infty}}\,.\end{aligned}$$

If $\mu_1 < \min\{\frac{1}{2\,\text{const}'},1\}$, then, by Corollary A.5(i), with $\Delta = \{\delta \in \mathfrak{N}_{d+1}\,|\,|\boldsymbol{\delta}| \le r, |\delta_0| \le r_0\}$, $\mu = \text{const}'$, $\Lambda = 1$ and $X = \|\delta\hat{e}\|_{1,\infty}$,

$$\|C\|_{1,\infty} \le \text{const}\,\frac{\mathfrak{c}_0}{1-\|\delta\hat{e}\|_{1,\infty}}\,.$$

Similarly

$$\begin{aligned}\|C-C_0\|_{1,\infty} &\le \|C_0\|_{1,\infty} \sum_{n=1}^{\infty}(\|\delta\hat{e}\|_{1,\infty}\,\|\tilde{C}_0\|_{1,\infty})^n \\ &\le \text{const}\ \mathfrak{c}_0 \sum_{n=1}^{\infty}(\text{const}'\ \mathfrak{c}_0\|\delta\hat{e}\|_{1,\infty})^n \\ &\le \text{const}\,\frac{\mathfrak{c}_0^2\|\delta\hat{e}\|_{1,\infty}}{1-\text{const}'\ \mathfrak{c}_0\|\delta\hat{e}\|_{1,\infty}} \\ &\le \text{const}\,\frac{\mathfrak{c}_0\|\delta\hat{e}\|_{1,\infty}}{1-\|\delta\hat{e}\|_{1,\infty}}\end{aligned}$$

and

$$\begin{aligned}|||C-C_0|||_{\infty} &\le |||C_0|||_{\infty} \sum_{n=1}^{\infty}(|||\delta\hat{e}|||_{1,\infty}\ |||\tilde{C}_0|||_{1,\infty})^n \\ &\le \text{const} \sum_{n=1}^{\infty}(\text{const}'|||\delta\hat{e}|||_{1,\infty})^n \\ &\le \text{const}\,\frac{|||\delta\hat{e}|||_{1,\infty}}{1-\text{const}'\mu} \\ &\le \text{const}|||\delta\hat{e}|||_{1,\infty}\,.\end{aligned}$$

(ii) As

$$\frac{d}{ds}C_s(k)\bigg|_{s=0} = -\frac{U(\mathbf{k})-\chi(k)}{[\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})]^2}\delta e'(\mathbf{k})$$

the first bound is a consequence of Proposition IV.8(i).

Let $\tilde U(\mathbf{k})$ and $\tilde\chi(k)$ be as in part (i) and set

$$\tilde C(k) = \frac{\tilde U(\mathbf{k}) - \tilde\chi(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})} .$$

Then

$$\frac{d}{ds} C_s(k)\bigg|_{s=0} = -C(k)\tilde C(k)\delta e'(\mathbf{k})$$

and

$$\begin{aligned}\left\| \frac{d}{ds} C_s(k)\bigg|_{s=0} \right\|_{1,\infty} &\le \|C\|_{1,\infty}\|\tilde C\|_{1,\infty}\|\delta\hat e'\|_{1,\infty} \\ &\le \text{const }\mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})^2\|\delta\hat e'\|_{1,\infty} \\ &\le \text{const }\mathfrak{e}_0(\|\delta\hat e\|_{1,\infty})\|\delta\hat e'\|_{1,\infty}\end{aligned}$$

by Corollary A.5(ii). □

## V. Insulators

An insulator is a many fermion system as described in the introduction, for which the dispersion relation $e(\mathbf{k})$ does not have a zero on the support of the ultraviolet cutoff $U(\mathbf{k})$. We may assume that there is a constant $\mu > 0$ such that $e(\mathbf{k}) \ge \mu$ for all $\mathbf{k} \in \mathbb{R}^d$. We shall show in Theorem V.2 that for a sufficiently small coupling constant the Green's functions for the interacting system exist and differ by very little from the Green's functions of the noninteracting system in the supremum norm.

**Lemma V.1.** *Let $\rho_{m;n}$ be a sequence of nonnegative real numbers such that $\rho_{m;n'} \le \rho_{m;n}$ for $n' \le n$. Define for $f \in \mathcal{F}_m(n)$*

$$\|f\| = \rho_{m;n}\,\|f\|_{1,\infty}$$

*where $\|f\|_{1,\infty}$ is the $L_1$–$L_\infty$-norm introduced in Example* II.6.

(i) *The seminorms $\|\cdot\|$ are symmetric.*
(ii) *For a covariance $C$, let $S(C)$ be the quantity introduced in Definition* IV.1. *Then $2S(C)$ is an integral bound for the covariance $C$ with respect to the family of seminorms $\|\cdot\|$.*
(iii) *Let $C$ be a covariance. Assume that for all $m \ge 0$ and $n, n' \ge 1$*

$$\rho_{m;\,n+n'-2} \le \rho_{m;n}\rho_{0;n'} \,.$$

*Let $\mathfrak{c} \in \mathfrak{N}_{d+1}$ obey*

$$\begin{aligned}\mathfrak{c} &\ge \|C\|_{1,\infty} \\ \mathfrak{c}_0 &\ge \frac{\rho_{m+m';\,n+n'-2}}{\rho_{m;n}\rho_{m';n'}}\,|||C|||_\infty \quad \textit{for all } m, m', n, n' \ge 1\end{aligned}$$

*where* $\mathfrak{c}_0$ *is the constant coefficient of the formal power series* $\mathfrak{c}$*. Then* $\mathfrak{c}$ *is a contraction bound for the covariance* $C$ *with respect to the family of seminorms* $\|\cdot\|$.

**Proof.** Parts (i) and (ii) are trivial. To prove part (iii), let $f \in \mathcal{F}_m(n),\ f' \in \mathcal{F}_{m'}(n')$ and $1 \le i \le n,\ 1 \le j \le n'$. Set

$$g(\eta_1,\dots,\eta_{m+m'};\,\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_n,\xi_{n+1},\dots,\xi_{n+j-1},\xi_{n+j+1},\dots,\xi_{n+n'})$$
$$= \int d\zeta d\zeta' f(\eta_1,\dots,\eta_m;\,\xi_1,\dots,\xi_{i-1},\zeta,\xi_{i+1},\dots,\xi_n) C(\zeta,\zeta')$$
$$\times f'(\eta_{m+1},\dots,\eta_{m+m'};\,\xi_{n+1},\dots,\xi_{n+j-1},\zeta',\xi_{n+j+1},\dots,\xi_{n+n'})\,.$$

Then

$$\mathop{\mathcal{C}\mathrm{on}_C}_{i\to n+j}\ \mathrm{Ant}_{\mathrm{ext}}(f\otimes f') = \mathrm{Ant}_{\mathrm{ext}} g$$

and therefore

$$\left\|\mathop{\mathcal{C}\mathrm{on}_C}_{i\to n+j}\ \mathrm{Ant}_{\mathrm{ext}}(f\otimes f')\right\| \le \|g\|\,.$$

If $m, m' \ge 1$

$$\|g\|_{1,\infty} \le \|f\|_{1,\infty} |||C|||_\infty \|f'\|_{1,\infty}$$

and consequently

$$\begin{aligned}\left\|\mathop{\mathcal{C}\mathrm{on}_C}_{i\to n+j}\ \mathrm{Ant}_{\mathrm{ext}}(f\otimes f')\right\| &\le \rho_{m+m';\,n+n'-2} |||C|||_\infty \|f\|_{1,\infty} \|f'\|_{1,\infty}\\ &\le \mathfrak{c}_0 \rho_{m;n}\|f\|_{1,\infty}\ \rho_{m';n'}\|f'\|_{1,\infty}\\ &\le \mathfrak{c}\,\|f\|\,\|f'\|\,.\end{aligned}$$

If $m = 0$ or $m' = 0$, by iterated application of Lemma II.7

$$\begin{aligned}\|g\|_{1,\infty} &\le \left\|\int_{\mathcal{B}} d\zeta f(\xi_1,\dots,\xi_m;\,\xi_1,\dots,\xi_{i-1},\zeta,\xi_{i+1},\dots,\xi_n)\, C(\zeta,\zeta')\right\|_{1,\infty} \|f'\|_{1,\infty}\\ &\le \|f\|_{1,\infty}\, \|C\|_{1,\infty}\, \|f'\|_{1,\infty}\end{aligned}$$

and again

$$\left\|\mathop{\mathcal{C}\mathrm{on}_C}_{i\to n+j}\ \mathrm{Ant}_{\mathrm{ext}}(f\otimes f')\right\| \le \mathfrak{c}\,\|f\|\,\|f'\|\,. \qquad \square$$

To formulate the result about insulators, we define for a function $f(x_1,\dots,x_n)$, on $(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^n$, the $L_1$–$L_\infty$-norm as in Example II.6 to be

$$|||f|||_{1,\infty} = \max_{1\le j\le n}\ \sup_{x_j} \int \prod_{\substack{i=1\\ i\ne j}}^n dx_i\, |f(x_1,\dots,x_n)|\,.$$

**Theorem V.2 (Insulators).** *Let $r$ and $r_0$ be natural numbers. Let $e(\mathbf{k})$ be a dispersion relation on $\mathbb{R}^d$ that is at least $r+d+1$ times differentiable, and let $U(\mathbf{k})$ be a compactly supported, smooth ultraviolet cutoff on $\mathbb{R}^d$. Assume that there is a constant $0<\mu<\frac{1}{2}$ such that*

$$e(\mathbf{k}) \geq \mu \quad \textit{for all } \mathbf{k} \textit{ in the support of } U\,.$$

*Set*

$$g = \int_{\mathrm{supp}\ U} d^d\mathbf{k} \frac{1}{|e(\mathbf{k})|} \qquad \gamma = \max\left\{1, \sqrt{\int d^d\mathbf{k}\ U(\mathbf{k}) \log \frac{E}{|e(\mathbf{k})|}}\right\}$$

*where $E = \max\{1, \sup_{\mathbf{k}\in \mathrm{supp}\ U} |e(\mathbf{k})|\}$. Let, for $x=(x_0,\mathbf{x},\sigma)$, $x'=(x_0',\mathbf{x}',\sigma') \in \mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}$*

$$C(x,x') = \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath\langle k, x-x'\rangle_-} \frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k})}$$

*and set, for $\xi=(x,a)$, $\xi'=(x',a')\in\mathcal{B}$*

$$C(\xi,\xi') = C(x,x')\delta_{a,0}\delta_{a',1} - C(x',x)\delta_{a,1}\delta_{a',0}\,.$$

*Furthermore let*

$$\mathcal{V}(\psi,\bar\psi) = \int_{(\mathbb{R}\times\mathbb{R}^2\times\{\uparrow,\downarrow\})^4} dx_1dy_1dx_2dy_2\ V_0(x_1,y_1,x_2,y_2)\bar\psi(x_1)\psi(y_1)\bar\psi(x_2)\psi(y_2)$$

*be a two particle interaction with a kernel $V_0$ that is antisymmetric in the variables $x_1,x_2$ and $y_1,y_2$ separately. Set*

$$\upsilon = \sup_{\substack{\mathcal{D}\ \text{decay operator}\\ \text{with } \delta_0\le r_0,\ |\boldsymbol{\delta}|\le r}} \mu^{|\delta(\mathcal{D})|} |||\mathcal{D}V_0|||_{1,\infty}\,.$$

*Then there exists $\varepsilon>0$ and a constant $\mathsf{const}$ such that*

(i) *If $|||V_0|||_{1,\infty} \leq \frac{\varepsilon\mu^d}{g\gamma^2}$, the connected amputated Green's functions $G_{2n}^{\mathrm{amp}}(x_1,y_1,\ldots,x_n,y_n)$ exist in the space of all functions on $(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^{2n}$ with finite $|||\cdot|||_{1,\infty}$ norms. They are analytic functions of $V_0$.*

(ii) *Suppose that $\upsilon \leq \frac{\varepsilon\mu^d}{g\gamma^2}$. For all decay operators $\mathcal{D}$ with $\delta_0(\mathcal{D})\leq r_0$ and $|\boldsymbol{\delta}(\mathcal{D})|\leq r$*

$$|||\mathcal{D}G_{2n}^{\mathrm{amp}}|||_{1,\infty} \leq \frac{\mathsf{const}^n\ g\gamma^{6-2n}}{\mu^{d+|\delta(\mathcal{D})|}}\ \upsilon^2 \quad \textit{if } n\geq 3$$

$$|||\mathcal{D}(G_4^{\mathrm{amp}} - V_0)|||_{1,\infty} \leq \frac{\mathsf{const}^2\ g\gamma^2}{\mu^{d+|\delta(\mathcal{D})|}}\ \upsilon^2$$

$$|||\mathcal{D}(G_2^{\mathrm{amp}} - K)|||_{1,\infty} \leq \frac{\mathsf{const}\ g\gamma^4}{\mu^{d+|\delta(\mathcal{D})|}}\ \upsilon^2$$

*where*

$$K(x,y) = 4\int dx'dy'\ V_0(x,y,x',y')C(x',y')\,.$$

*The constants $\varepsilon$ and $\mathit{const}$ depend on $r$, $r_0$, $U$, and the suprema of the* $\mathbf{k}$*-derivatives of the dispersion relation $e(\mathbf{k})$ up to order $r+d+1$, but not on $\mu$ or $V_0$.*

**Proof.** By (I.6), the generating functional for the connected amputated Green's functions is

$$\mathcal{G}^{\rm amp}_{\rm gen}(\phi) = \Omega_C(\mathcal{V})(0,\phi)\,.$$

To estimate it, we use the norms $\|\cdot\|$ of Lemma V.1 with $\rho_{m;n}=1$. By Lemma V.1(ii) and Proposition IV.5, there is a constant $\mathit{const}_0$ such that $\mathrm{b} = \mathit{const}_0\,\gamma$ is an integral bound for the covariance $C$ with respect to these norms. By Lemma V.1(iii), Corollary IV.9 and Proposition IV.8(ii), there is a constant $\mathit{const}_1$ such that

$$\mathfrak{c} = \frac{\mathit{const}_1\, g}{\mu^d}\left(\sum_{|\boldsymbol{\delta}|\le r}\left(\frac{2}{\mu}\right)^{|\delta|} t^\delta + \sum_{|\boldsymbol{\delta}|>r}\infty\, t^\delta\right)$$

is a contraction bound for $C$ with respect to these norms. Here we used that $\frac{g}{\mu^d}$ is bounded below by a nonzero constant. As in Definition III.9, we set for any Grassmann function $\mathcal{W}(\phi,\psi)$ and any $\alpha>0$

$$N(\mathcal{W};\,\mathfrak{c},\mathrm{b},\alpha) = \frac{1}{\mathrm{b}^2}\,\mathfrak{c}\sum_{m,n\ge 0}\alpha^n\,\mathrm{b}^n\|\mathcal{W}_{m,n}\|\,.$$

In particular

$$N(\mathcal{V};\,\mathfrak{c},\mathrm{b},\alpha) = \alpha^4\,\mathrm{b}^2\mathfrak{c}\,\|V_0\|_{1,\infty}$$

and

$$N(\mathcal{V};\,\mathfrak{c},\mathrm{b},8\alpha)_{\mathbf{0}} \le \mathit{const}_3\frac{8^4\,\alpha^4\gamma^2 g}{\mu^d}|||V_0|||_{1,\infty}\,. \tag{V.1}$$

Observe that

$$\begin{aligned}\mathfrak{c}\|V_0\|_{1,\infty} &\le \frac{\mathit{const}_1\ g}{\mu^d}\left(\sum_{|\boldsymbol{\delta}|\le r}\left(\frac{2}{\mu}\right)^{|\delta|} t^\delta + \sum_{|\boldsymbol{\delta}|>r}\infty t^\delta\right)\\ &\quad\times\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}}\frac{1}{\delta!}\frac{\upsilon}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or } |\delta_0|>r_0}}\infty t^\delta\right)\\ &\le \mathit{const}_2\,\frac{g\upsilon}{\mu^d}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}}\frac{1}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or } |\delta_0|>r_0}}\infty t^\delta\right).\end{aligned}$$

Write $\mathcal{V} = {:}\mathcal{V}'{:}_C$ . By [8, Proposition A.2(i)],

$$\mathcal{V}' = \mathcal{V} + \int dxdyK(x,y)\bar\psi(x)\psi(y) + \mathrm{const}$$

and by [8, Corollary II.32(i)]

$$N(\mathcal{V}';\,\mathfrak{c},\mathrm{b},\alpha) \le N(\mathcal{V};\,\mathfrak{c},\mathrm{b},2\alpha) = 16\alpha^4\mathrm{b}^2\,\mathfrak{c}\|V_0\|_{1,\infty}$$

$$\le {\scriptstyle const_3}\ \alpha^4\gamma^2\frac{g\upsilon}{\mu^d}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}}\frac{1}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right).$$

We set $\alpha = 2$ and $\varepsilon = \frac{1}{2^{17}\,const_3}$. Then

$$N(\mathcal{V}';\,\mathfrak{c},\mathrm{b},16) \le \frac{g\gamma^2\upsilon}{2\varepsilon\mu^d}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}}\frac{1}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right)$$

and, by (V.1)

$$N(\mathcal{V}';\,\mathfrak{c},\mathrm{b},16)_{\mathbf{0}} \le \frac{g\gamma^2}{2\varepsilon\mu^d}|||V_0|||_{1,\infty}\,.$$

Therefore, whenever $|||V_0|||_{1,\infty} \le \frac{\varepsilon\mu^d}{g\gamma^2}$, $\mathcal{V}'$ fulfills the hypotheses of Theorem III.10 and $\mathcal{G}^{\mathrm{amp}}_{\mathrm{gen}}(\psi) = \Omega_C(:\mathcal{V}':)(0,\psi)$ exists. Part (i) follows.

If, in addition, $\upsilon \le \frac{\varepsilon\mu^d}{g\gamma^2}$, then

$$N(\mathcal{G}^{\mathrm{amp}}_{\mathrm{gen}} - \mathcal{V}';\,\mathfrak{c},\mathrm{b},2) \le \frac{1}{2}\frac{N(\mathcal{V}';\,\mathfrak{c},\mathrm{b},16)^2}{1-N(\mathcal{V}';\,\mathfrak{c},\mathrm{b},16)} \le \frac{1}{2}\left(\frac{g\gamma^2\upsilon}{2\varepsilon\mu^d}\right)^2 f\left(\frac{t}{\mu}\right)$$

where

$$f(t) = \frac{\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right)^2}{1-\frac{1}{2}\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right)}$$

$$= \sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} F_\delta t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta$$

with $F_\delta$ finite for all $|\boldsymbol{\delta}| \le r,\ |\delta_0| \le r_0$. Hence

$$N(\mathcal{G}^{\mathrm{amp}}_{\mathrm{gen}} - \mathcal{V}';\,\mathfrak{c},\mathrm{b},2) \le {\scriptstyle const_4}\left(\frac{g\gamma^2\upsilon}{\varepsilon\mu^d}\right)^2\left(\sum_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}}\frac{1}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right)$$

with $const_4 = \frac{1}{8}\max_{\substack{|\boldsymbol{\delta}|\le r\\ |\delta_0|\le r_0}} F_\delta$. As

$$N(\mathcal{G}^{\mathrm{amp}}_{\mathrm{gen}} - \mathcal{V}';\,\mathfrak{c},\mathrm{b},2)$$

$$= \mathfrak{c}\left(4\|G^{\mathrm{amp}}_2 - K\| + 16\mathrm{b}^2\|G^{\mathrm{amp}}_4 - V_0\| + \sum_{n=3}^{\infty}4(2\mathrm{b})^{2n-2}\|G^{\mathrm{amp}}_{2n}\|\right)$$

$$\geq \frac{4\,const_1\,g}{\mu^d}\Bigg(\|G_2^{\rm amp} - K\|_{1,\infty} + 4\,const_0^2\,\gamma^2\|G_4^{\rm amp} - V_0\|_{1,\infty} + \sum_{n=3}^{\infty}(const_0\,\gamma)^{2n-2}\|G_{2n}^{\rm amp}\|_{1,\infty}\Bigg)$$

we have

$$\|G_2^{\rm amp} - K\|_{1,\infty} + 4\,const_0^2\,\gamma^2\|G_4^{\rm amp} - V_0\|_{1,\infty} + \sum_{n=3}^{\infty}(const_0\,\gamma)^{2n-2}\|G_{2n}^{\rm amp}\|_{1,\infty}$$
$$\leq const_4\,\frac{g\gamma^4 \mathfrak{v}^2}{4\,const_1\,\varepsilon^2\mu^d}\left(\sum_{\substack{|\boldsymbol{\delta}|\leq r\\ |\delta_0|\leq r_0}}\frac{1}{\mu^{|\delta|}}t^\delta + \sum_{\substack{|\boldsymbol{\delta}|>r\\ \text{or }\ \delta_0|>r_0}}\infty t^\delta\right).$$

The estimates on the amputated Green's functions follow. □

**Remark V.3.** (i) In reasonable situations, for example if the gradient of $e(\mathbf{k})$ is bounded below, the constants $\gamma$ and $g$ in Theorem V.2 are of order one and $\log\mu$ respectively.

(ii) Using Example A.3, one may prove an analog of Theorem V.2 with the constants $\varepsilon$ and $const$ independent of $r_0$ and

$$|||\mathcal{D}G_{2n}^{\rm amp}|||_{1,\infty} \leq const^n\ \delta(\mathcal{D})!g\gamma^{6-2n}\left(\frac{8(d+1)}{\mu}\right)^{d+|\delta(\mathcal{D})|}\mathfrak{v}^2 \quad \text{if } n\geq 3$$

$$|||\mathcal{D}(G_4^{\rm amp} - V_0)|||_{1,\infty} \leq const^2\ \delta(\mathcal{D})!g\gamma^2\left(\frac{8(d+1)}{\mu}\right)^{d+|\delta(\mathcal{D})|}\mathfrak{v}^2$$

$$|||\mathcal{D}(G_2^{\rm amp} - K)|||_{1,\infty} \leq const\ \delta(\mathcal{D})!g\gamma^4\left(\frac{8(d+1)}{\mu}\right)^{d+|\delta(\mathcal{D})|}\mathfrak{v}^2\,.$$

(iii) Roughly speaking, the connected Green's function are constructed from the connected amputated Green's functions by appending propagators $C$. The details are given in Sec. VI in Part 2. Using Proposition IV.8(ii), one sees that, under the hypotheses of Theorem V.2(i), the connected Green's functions exist in the space of all functions on $(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^{2n}$ with finite $|||\cdot|||_{1,\infty}$ and $|||\cdot|||_\infty$ norms.

## Appendices

### A. *Calculations in the norm domain*

Recall from Definition II.4 that the $(d+1)$-dimensional norm domain $\mathfrak{N}_{d+1}$ is the set of all formal power series

$$X = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta \, t_0^{\delta_0} t_1^{\delta_1} \cdots t_d^{\delta_d} = \sum_{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d} X_\delta \, t^\delta$$

in the variables $t_0, t_1, \ldots, t_d$ with coefficients $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$.

**Definition A.1.** A nonempty subset $\Delta$ of $\mathbb{N}_0 \times \mathbb{N}_0^d$ is called saturated if, for every $\delta \in \Delta$ and every multi-index $\delta'$ with $\delta' \le \delta$, the multi-index $\delta'$ also lies in $\Delta$. If $\Delta$ is a finite set, then

$$N(\Delta) = \min\{n \in \mathbb{N} | n\delta \notin \Delta \text{ for all } \mathbf{0} \neq \delta \in \Delta\}$$

is finite.

For example, if $r, r_0 \in \mathbb{N}$ then the set $\{\delta \in \mathbb{N}_0 \times \mathbb{N}_0^d | \delta_0 \le r_0,\ |\boldsymbol{\delta}| \le r\}$ is saturated and $N(\Delta) = \max\{r_0 + 1, r + 1\}$.

**Lemma A.2.** *Let $\Delta$ be a saturated set of multi-indices and $X, Y \in \mathfrak{N}_{d+1}$. Furthermore, let $f(t_0, \ldots, t_d)$ and $g(t_0, \ldots, t_d)$ be analytic functions in a neighborhood of the origin in $\mathbb{C}^{d+1}$ such that, for all $\delta \in \Delta$, the $\delta$th Taylor coefficients of $f$ and $g$ at the origin are real and nonnegative. Assume that $g(0) < 1$ and that, for all $\delta \in \Delta$,*

$$X_\delta \le \frac{1}{\delta!} \left( \prod_{i=0}^{d} \frac{\partial^{\delta_i}}{\partial t_i^{\delta_i}} \right) f(t_0, \ldots, t_d) \Bigg|_{t_0 = \cdots = t_d = 0}$$

$$Y_\delta \le \frac{1}{\delta!} \left( \prod_{i=0}^{d} \frac{\partial^{\delta_i}}{\partial t_i^{\delta_i}} \right) g(t_0, \ldots, t_d) \Bigg|_{t_0 = \cdots = t_d = 0} .$$

*Set $Z = \frac{X}{1-Y}$ and $h(t) = \frac{f(t)}{1-g(t)}$. Then, for all $\delta \in \Delta$,*

$$Z_\delta \le \frac{1}{\delta!} \left( \prod_{i=0}^{d} \frac{\partial^{\delta_i}}{\partial t_i^{\delta_i}} \right) h(t_0, \ldots, t_d) \Bigg|_{t_0 = \cdots = t_d = 0} .$$

**Proof.** Trivial. □

**Example A.3.** Let $\Delta$ be a saturated set and $a \ge 0$, $0 \le \lambda \le \frac{1}{2}$. Then

$$\frac{\left( \sum_{\delta \in \Delta} a^{|\delta|} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \right)^2}{1 - \lambda \left( \sum_{\delta \in \Delta} a^{|\delta|} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \right)} \le \frac{16}{3} \sum_{\delta \in \Delta} (4(d+1)\, a)^{|\delta|} \, t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta .$$

**Proof.** Set

$$X = \left( \sum_{\delta \in \Delta} a^{|\delta|} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \right)^2 \qquad f(t) = \left( \sum_{\delta} a^{|\delta|} t^\delta \right)^2 = \prod_{i=0}^{d} \frac{1}{(1 - a t_i)^2}$$

$$Y = \lambda \left( \sum_{\delta \in \Delta} a^{|\delta|} t^\delta + \sum_{\delta \notin \Delta} \infty t^\delta \right) \qquad g(t) = \lambda \left( \sum_{\delta} a^{|\delta|} t^\delta \right) = \lambda \prod_{i=0}^{d} \frac{1}{1 - a t_i} .$$

Set

$$h(t) = \frac{f(t)}{1-g(t)} = \frac{1}{\Pi(1-at_i)}\frac{1}{\Pi(1-at_i)-\lambda}\,.$$

By the Cauchy integral formula, with $\rho = \frac{1}{a}(1 - \sqrt[d+1]{\frac{3}{4}})$

$$\begin{aligned}
&\frac{1}{\delta!}\left(\prod_{j=0}^{d}\frac{\partial^{\delta_j}}{\partial t_j^{\delta_j}}\right)h(t_0,\ldots,t_d)\bigg|_{t_0=\cdots=t_d=0} \\
&\quad = \int_{|z_0|=\rho}\cdots\int_{|z_d|=\rho} h(z)\prod_{j=0}^{d}\left(\frac{1}{z_j^{\delta_j+1}}\frac{dz_j}{2\pi\imath}\right) \\
&\quad \le \frac{1}{\rho^{|\delta|}}\sup_{|z_0|=\cdots=|z_d|=\rho}|h(z)| \\
&\quad \le \frac{1}{\rho^{|\delta|}}\frac{1}{(1-a\rho)^{d+1}}\frac{1}{(1-a\rho)^{d+1}-\lambda} \\
&\quad \le \frac{a^{|\delta|}}{(1-(3/4)^{1/(d+1)})^{|\delta|}}\frac{4}{3}\frac{1}{3/4-1/2} \\
&\quad \le \frac{16}{3}(4(d+1)a)^{|\delta|}\,.
\end{aligned}$$

□

**Lemma A.4.** (i) *Let $X, Y \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} + Y_{\mathbf{0}} < 1$*

$$\frac{1}{1-X}\frac{1}{1-Y} \le \frac{1}{1-(X+Y)}\,.$$

(ii) *Let $\Delta$ be a finite saturated set and $X, Y \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} + Y_{\mathbf{0}} < \frac{1}{2}$. There is a constant, $_{const}$ depending only on $\Delta$, such that*

$$\frac{1}{1-(X+Y)} \le {}_{const}\ \frac{1}{1-X}\frac{1}{1-Y} + \sum_{\delta\notin\Delta}\infty t^{\delta} \in \mathfrak{N}_{d+1}\,.$$

**Proof.** (i)

$$\begin{aligned}
\frac{1}{1-X}\frac{1}{1-Y} &= \sum_{m,n=0}^{\infty}X^mY^n = \sum_{p=0}^{\infty}\sum_{m=0}^{p}X^mY^{p-m} \\
&\le \sum_{p=0}^{\infty}\sum_{m=0}^{p}\binom{p}{m}X^mY^{p-m} \\
&= \sum_{p=0}^{\infty}(X+Y)^p = \frac{1}{1-(X+Y)}\,.
\end{aligned}$$

(ii) Set $\hat{X} = X - X_{\mathbf{0}}$ and $\hat{Y} = Y - Y_{\mathbf{0}}$. Then

$$\begin{aligned}
\frac{1}{1-(X+Y)} &= \frac{1}{1-(X_{\mathbf{0}}+Y_{\mathbf{0}})-(\hat{X}+\hat{Y})} \le \frac{1}{\frac{1}{2}-(\hat{X}+\hat{Y})} \\
&\le 2\sum_{n=0}^{N(\Delta)-1}(2\hat{X}+2\hat{Y})^n + \sum_{\delta\notin\Delta}\infty t^\delta \\
&= 2\sum_{n=0}^{N(\Delta)-1}\sum_{m=0}^{n} 2^n \binom{n}{m} \hat{X}^m\hat{Y}^{n-m} + \sum_{\delta\notin\Delta}\infty t^\delta \\
&\le 2^{2N(\Delta)-1}\sum_{n=0}^{N(\Delta)-1}\sum_{m=0}^{n} \hat{X}^m\hat{Y}^{n-m} + \sum_{\delta\notin\Delta}\infty t^\delta \\
&\le 2^{2N(\Delta)-1}\frac{1}{1-\hat{X}}\frac{1}{1-\hat{Y}} + \sum_{\delta\notin\Delta}\infty t^\delta \\
&\le 2^{2N(\Delta)-1}\frac{1}{1-X}\frac{1}{1-Y} + \sum_{\delta\notin\Delta}\infty t^\delta\,.
\end{aligned}$$

□

**Corollary A.5.** *Let $\Delta$ be a finite saturated set, $\mu, \Lambda > 0$. Set $\mathfrak{c} = \sum_{\delta\in\Delta}\Lambda^{|\delta|}t^\delta + \sum_{\delta\notin\Delta}\infty t^\delta$. There is a constant, $_{const}$ depending only on $\Delta$ and $\mu$, such that the following hold.*

(i) *For all $X \in \mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \min\{\frac{1}{2\mu},\ 1\}$.*

$$\frac{\mathfrak{c}}{1-\mu\mathfrak{c}X} \le {}_{const}\ \frac{\mathfrak{c}}{1-X}\,.$$

(ii) *Set, for $X \in \mathfrak{N}_{d+1}$, $\mathfrak{e}(X) = \frac{\mathfrak{c}}{1-\Lambda X}$. If $\mu + \Lambda X_{\mathbf{0}} < \frac{1}{2}$, then*

$$\mathfrak{e}(X)^2 \le {}_{const}\ \mathfrak{e}(X) \qquad \frac{\mathfrak{e}(X)}{1-\mu\mathfrak{e}(X)} \le {}_{const}\,\mathfrak{e}(X)\,.$$

**Proof.** (i) Decompose $X = X_{\mathbf{0}} + \hat{X}$. Then, by Example A.3 and Lemma A.4,

$$\begin{aligned}
\frac{\mathfrak{c}}{1-\mu\mathfrak{c}X} &= \frac{\mathfrak{c}}{1-\mu X_{\mathbf{0}}\mathfrak{c}-\mu\mathfrak{c}\hat{X}} \\
&\le {}_{const}\,\frac{\mathfrak{c}}{1-\mu X_{\mathbf{0}}\mathfrak{c}}\frac{1}{1-\mu\mathfrak{c}\hat{X}} \\
&\le {}_{const}\,\frac{\mathfrak{c}}{1-\mathfrak{c}/2}\frac{1}{1-\mu\mathfrak{c}\hat{X}} \\
&\le {}_{const}\,\frac{\mathfrak{c}}{1-\mu\mathfrak{c}\hat{X}}\,.
\end{aligned}$$

Expanding in a geometric series

$$\begin{aligned}\frac{\mathfrak{c}}{1-\mu\mathfrak{c}X} &\le \textit{const}\ \mathfrak{c}\sum_{n=0}^{N(\Delta)-1}(\mu\mathfrak{c}\hat{X})^n\\ &\le \textit{const}\ \mathfrak{c}^{N(\Delta)}(1+\mu^{N(\Delta)})\sum_{n=0}^{N(\Delta)-1}\hat{X}^n\\ &\le \textit{const}\ \frac{\mathfrak{c}}{1-\hat{X}}\\ &\le \textit{const}\ \frac{\mathfrak{c}}{1-X}\,.\end{aligned}$$

(ii) The first claim follows from the second, by expanding the geometric series. By Lemma A.4(ii) and part (i),

$$\begin{aligned}\frac{\mathfrak{e}(X)}{1-\mu\mathfrak{e}(X)} &= \frac{\frac{\mathfrak{c}}{1-\Lambda X}}{1-\mu\frac{\mathfrak{c}}{1-\Lambda X}} = \frac{\mathfrak{c}}{1-\Lambda X-\mu\mathfrak{c}}\\ &\le \textit{const}\ \frac{\mathfrak{c}}{1-\mu\mathfrak{c}}\frac{1}{1-\Lambda X} \le \textit{const}\ \frac{\mathfrak{c}}{1-\Lambda X} = \textit{const}\ \mathfrak{e}(X)\,.\end{aligned}$$ □

**Remark A.6.** The following generalization of Corollary A.5 is proven in the same way. Let $\Delta$ be a finite saturated set, $\mu,\ \lambda,\ \Lambda > 0$. Set $\mathfrak{c} = \sum_{\delta\in\Delta}\lambda^{\delta_0}\Lambda^{|\boldsymbol{\delta}|}t^\delta + \sum_{\delta\notin\Delta}\infty t^\delta$. There is a constant, *const* depending only on $\Delta$ and $\mu$, such that the following hold.

(i) For all $X\in\mathfrak{N}_{d+1}$ with $X_{\mathbf{0}} < \min\{\frac{1}{2\mu}, 1\}$.

$$\frac{\mathfrak{c}}{1-\mu\mathfrak{c}X} \le \textit{const}\ \frac{\mathfrak{c}}{1-X}\,.$$

(ii) Set, for $X\in\mathfrak{N}_{d+1}$, $\mathfrak{e}(X) = \frac{\mathfrak{c}}{1-\Lambda X}$. If $\mu+\Lambda X_{\mathbf{0}} < \frac{1}{2}$, then

$$\mathfrak{e}(X)^2 \le \textit{const}\ \mathfrak{e}(X) \qquad \frac{\mathfrak{e}(X)}{1-\mu\mathfrak{e}(X)} \le \textit{const}\ \mathfrak{e}(X)\,.$$

**Lemma A.7.** *Let $\Delta$ be a finite saturated set and*

$$X = \sum_{\delta\in\Delta}X_\delta t^\delta + \sum_{\delta\notin\Delta}\infty t^\delta \in \mathfrak{N}_{d+1}\,.$$

*Let $f(z)$ be analytic at $X_{\mathbf{0}}$, with $f^{(n)}(X_{\mathbf{0}})\ge 0$ for all $n$, whose radius of convergence at $X_{\mathbf{0}}$ is at least $r>0$. Let $0<\beta<\frac{1}{X_{\mathbf{0}}}$. Then there exists a constant $C$, depending only on $\Delta$, $\beta$, $r$ and $\max_{|z-X_{\mathbf{0}}|=r}|f(z)|$ such that*

$$f(X) \le C\frac{1}{1-\beta X}\,.$$

**Proof.** Set $\alpha = \frac{\beta}{1-\beta X_{\mathbf{0}}}$ and $\hat{X} = X - X_{\mathbf{0}}$. Then

$$\begin{aligned} f(X) &= \sum_n \frac{1}{n!} f^{(n)}(X_{\mathbf{0}}) \hat{X}^n \\ &\le \sum_{n<N(\Delta)} \frac{1}{n!} f^{(n)}(X_{\mathbf{0}}) \hat{X}^n + \sum_{\delta \notin \Delta} \infty t^\delta \\ &\le C \sum_{n<N(\Delta)} \alpha^n \hat{X}^n + \sum_{\delta \notin \Delta} \infty t^\delta \end{aligned}$$

where

$$C = \max_{n<N(\Delta)} \frac{f^{(n)}(X_{\mathbf{0}})}{n!\, \beta^n} > \max_{n<N(\Delta)} \frac{f^{(n)}(X_{\mathbf{0}})}{n!\, \alpha^n}\,.$$

Hence

$$\begin{aligned} f(X) &\le \frac{C}{1-\alpha \hat{X}} + \sum_{\delta \notin \Delta} \infty t^\delta \\ &= \frac{C}{1-\alpha(X - X_{\mathbf{0}})} \\ &= \frac{C(1-\beta X_{\mathbf{0}})}{1-\beta X} \\ &\le \frac{C}{1-\beta X}\,. \end{aligned}$$

□

## *Notation*

*Norms*

| Norm | Characteristics | Reference |
|---|---|---|
| $\vert\vert\vert \cdot \vert\vert\vert_{1,\infty}$ | no derivatives, external positions, acts on functions | Example II.6 |
| $\Vert \cdot \Vert_{1,\infty}$ | derivatives, external positions, acts on functions | Example II.6 |
| $\Vert \cdot \Vert_{\infty}^{\check{}}$ | derivatives, external momenta, acts on functions | Definition IV.6 |
| $\vert\vert\vert \cdot \vert\vert\vert_{\infty}$ | no derivatives, external positions, acts on functions | Example III.4 |
| $\Vert \cdot \Vert_{1}^{\check{}}$ | derivatives, external momenta, acts on functions | Definition IV.6 |
| $\Vert \cdot \Vert_{\infty,B}^{\check{}}$ | derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$ | Definition IV.6 |
| $\Vert \cdot \Vert_{1,B}^{\check{}}$ | derivatives, external momenta, $B \subset \mathbb{R} \times \mathbb{R}^d$ | Definition IV.6 |
| $\Vert \cdot \Vert$ | $\rho_{m;n} \Vert \cdot \Vert_{1,\infty}$ | Lemma V.1 |
| $N(\mathcal{W};\, \mathfrak{c}, \mathrm{b}, \alpha)$ | $\frac{1}{\mathrm{b}^2}\, \mathfrak{c} \sum_{m,n\ge 0} \alpha^n \mathrm{b}^n \Vert \mathcal{W}_{m,n} \Vert$ | Definition III.9 |
| | | Theorem V.2 |

*Other notation*

| Notation | Description | Reference |
| --- | --- | --- |
| $\Omega_S(\mathcal{W})(\phi, \psi)$ | $\log \frac{1}{Z} \int e^{\mathcal{W}(\phi, \psi+\zeta)} d\mu_S(\zeta)$ | before (I.6) |
| $S(C)$ | $\sup_m \sup_{\xi_1,\ldots,\xi_m \in \mathcal{B}} \left( \left| \int \psi(\xi_1) \cdots \psi(\xi_m) d\mu_C(\psi) \right| \right)^{1/m}$ | Definition IV.1 |
| $\mathcal{B}$ | $\mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ viewed as position space | beginning of Sec. II |
| $\mathcal{F}_m(n)$ | functions on $\mathcal{B}^m \times \mathcal{B}^n$, antisymmetric in $\mathcal{B}^m$ arguments | Definition II.9 |

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