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Convergence of Perturbation Expansions in Fermionic Models. Part 1: Nonperturbative Bounds

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Abstract: An estimate on the operator norm of an abstract fermionic renormalization group map is derived. This abstract estimate is applied in another paper to construct the thermodynamic Green's functions of a two dimensional, weakly coupled fermion gas with an asymmetric Fermi curve. The estimate derived here is strong enough to control everything but the sum of all quartic contributions to the Green's functions.

Contents

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I. Introduction

In a Grassmann algebra $\mathbf{A}$ with generators ψ_i, the renormalization group map with respect to a covariance C is the map that associates to each element $W(\psi)$ of the Grassmann algebra the element

$$\Omega_C(W)(\psi) = \log \tfrac{1}{Z(W)} \int e^{W(\psi+\xi)} d\mu_C(\xi), \qquad \text{where} \quad Z(W) = \int e^{W(\xi)} d\mu_C(\xi)$$

whenever $Z(W) \neq 0$. Here, ξ_i is a second set of variables that anticommute amongst themselves and with the ψ_j's. $\int \cdots d\mu_C(\xi)$ is the Grassmann Gaussian integral with respect to these variables (see §II). The Schwinger functional with interaction W is the map that associates to a Grassmann function $f(\xi)$ the complex number

$$\mathcal{S}(f) = \tfrac{1}{Z(W)} \int f(\xi)\, e^{W(\xi)} d\mu_C(\xi).$$

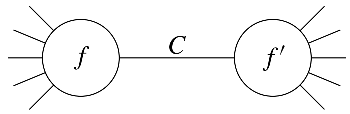
Observe that, if $Z(W) \neq 0$, we may always arrange that $Z(W) = 1$ by adding a constant to W. In this case

$$\Omega_C(W)(\psi) \;=\; \int_0^1 \tfrac{d}{dt} \Omega_C(tW)\, dt \;=\; \int_0^1 \frac{\int W(\psi+\xi)\, e^{tW(\psi+\xi)}\, d\mu_C(\xi)}{\int e^{tW(\psi+\xi)}\, d\mu_C(\xi)}\, dt.$$

The t–integrand of the right-hand side is the Schwinger functional of $W(\psi+\xi)$ with interaction $tW(\psi+\xi)$, where the Schwinger functional is considered in the Grassmann algebra with generators ξ_i and coefficients in $\mathbf{A}$. We exploit this observation and the representation of the Schwinger functional in [FKT1] to develop non–perturbative bounds for the renormalization group map. By "non–perturbative" we mean that we bound the sum of the perturbation expansion, not only individual terms in the expansion.

In a perturbative analysis, one decomposes $W = \sum_n W_n$, where W_n is homogeneous of degree n in ψ. Then $\Omega_C(W)$ is the sum of the values of all connected Feynman diagrams with vertices W_n and propagator C. In most applications, the kernels W_n are translation invariant. To bound the value of a Feynman diagram, one usually (see [FT]) selects a tree in the diagram (to exploit the connectedness of the diagram) and bounds the lines of the tree differently from the other lines.[1] The non–perturbative analysis we present here is close to the diagrammatic analysis (see the introduction to [FKT1]), but it allows implementation of the Pauli exclusion principle for lines not on the tree. The norms we use in applications to many fermion systems are quite complicated. Therefore, here, we axiomatize their relevant properties: We assume that there is a system of norms $\|\cdot\|$ on the homogeneous subspaces of the Grassmann algebra. Furthermore, we assume that there is a "contraction bound" $\mathfrak{c}$ for C, such that for any two homogeneous elements f, f' in the Grassmann algebra, the norm of the diagram

[1] Typically, the lines of the tree contribute a factor of the L_1 norm of the propagator in position space to the bound on the diagram, while the other lines contribute a factor of the L_∞ norm of the propagator to the bound.



that is obtained by joining f and f' by one line is bounded by $\mathfrak{c}\,\|f\|\,\|f'\|$ and we assume that there is an "integral bound" b that controls the effect of integrating out some of the fields attached to a single vertex. See Def. II.25. The contraction bound is analogous to the bound on the tree contribution to a Feynman diagram. The integral bound incorporates the Pauli exclusion principle and is often derived from Gram's estimate for determinants. See Appendix B. It replaces the bound on the non–tree contribution to a Feynman diagram.

When applying a renormalization group transformation, there are often other fields present, that do play no role in the renormalization group transformation. We suppress these fields by allowing a (super)algebra as coefficient ring for the Grassmann algebra on which the renormalization group map is analyzed. See Def. II.1. Also, in the analysis of many fermion systems, we have to control various derivatives (in momentum space) of the effective interactions involved. To get a coherent notation for derivative norms, we allow the norms to take values in a formal power series ring, where the powers code the degree of derivatives. See Def. II.14.

In §II we introduce the concepts discussed above and formulate the main estimate on the renormalization group map in these terms (Theorem II.28). Section III discusses the connection with the Schwinger functional and gives the proof of Theorem II.28[2]. In §IV, further estimates of the renormalization group map are discussed, in particular on its derivative with respect to the interaction W and the covariance C. Part 2 of the paper (§VI–IX) discusses the phenomenon of "overlapping" loops, that is responsible for "improvements over natural power counting" and consequently is important for many fermion systems; see [S] and the introduction to Part 2 of this paper. A notation table is provided at the end of the paper.

II. The Renormalization Group Map

II.1. Superalgebras.

Definition II.1. *(i) A superalgebra is an associative $\mathbb{C}$-algebra A with unit 1, together with a decomposition $A = A_+ \oplus A_-$ such that $1 \in A_+$ and*

$$A_+ \cdot A_+ \subset A_+, \qquad A_- \cdot A_- \subset A_+,$$
$$A_+ \cdot A_- \subset A_-, \qquad A_- \cdot A_+ \subset A_-,$$

and

$$ab = ba, \quad \text{if} \quad a \in A_+ \quad \text{or} \quad b \in A_+,$$
$$ab = -ba, \quad \text{if} \quad a, b \in A_-.$$

The elements of A_+ are called even, the elements of A_- odd.

[2] For other approaches to controlling fermionic renormalization, see [DR] and [SW].

(ii) A graded superalgebra is an associative $\mathbb{C}$-algebra A with unit, together with a decomposition $A = \bigoplus_{m=0}^{\infty} A_m$ such that $A_0 = \mathbb{C}$, $A_m \cdot A_n \subset A_{m+n}$ for all $m, n \geq 0$, and such that the decomposition $A = A_+ \oplus A_-$ with

$$A_+ = \bigoplus_{m \text{ even}} A_m \qquad\qquad A_- = \bigoplus_{m \text{ odd}} A_m$$

gives A the structure of a superalgebra.

(iii) Let A be a graded superalgebra, $f = \sum_m f_m \in A$ with $f_m \in A_m$. Set $\mathcal{Z}(f) = f_0 \in A_0 = \mathbb{C}$. Clearly, if $f_0 \neq 0$,

$$\tfrac{f}{\mathcal{Z}(f)} = 1 + \sum_{m \geq 1} \tfrac{f_m}{f_0}.$$

In this case, set

$$\log \tfrac{f}{\mathcal{Z}(f)} = \sum_{n=1}^{\infty} \tfrac{(-1)^{n-1}}{n} \big(\tfrac{f}{\mathcal{Z}(f)} - 1 \big)^n.$$

(iv) Let A and B be superalgebras. On the tensor product $A \otimes B$ we define multiplication by

$$\begin{aligned}\big[a \otimes (b_+ + b_-)\big]\big[(a_+ + a_-) \otimes b\big] &= a(a_+ + a_-) \otimes (b_+ + b_-)b - 2\, aa_- \otimes b_- b \\ &= aa_+ \otimes b_+ b + aa_+ \otimes b_- b \\ &\quad + aa_- \otimes b_+ b - aa_- \otimes b_- b\end{aligned}$$

for $a \in A,\ b \in B,\ a_\pm \in A_\pm,\ b_\pm \in B_\pm$. This multiplication defines an algebra structure on $A \otimes B$. Setting $(A \otimes B)_+ = (A_+ \otimes B_+) \oplus (A_- \otimes B_-)$, $(A \otimes B)_- = (A_+ \otimes B_-) \oplus (A_- \otimes B_+)$ we get a superalgebra. If A and B are graded superalgebras then the decomposition $A \otimes B = \bigoplus_{m=0}^{\infty} (A \otimes B)_m$ with

$$(A \otimes B)_m = \bigoplus_{m_1 + m_2 = m} A_{m_1} \otimes B_{m_2}$$

gives $A \otimes B$ the structure of a graded superalgebra.

Example II.2. Let V be a complex vector space. The Grassmann algebra $\bigwedge V = \bigoplus_{m \geq 0} \bigwedge^m V$ over V is a graded superalgebra. If A is any superalgebra, the Grassmann algebra over V with coefficients in A is the superalgebra

$$\bigwedge\nolimits_A V = A \otimes \bigwedge V,$$

where the tensor product is taken as in Def. II.1.iv. If A is a graded superalgebra, so is $\bigwedge_A V$.

In fact almost all graded superalgebras A that will be used in this paper will be subalgebras of Grassmann algebras. The one exception is the "enlarged" algebra of Sect. VII.

II.2. Grassmann Gaussian Integrals. Let A be a superalgebra, V be a complex vector space and C an antisymmetric bilinear form (covariance) on V. Then C determines an A–linear map from $\bigwedge_A V$ to A that is called the Grassmann Gaussian integral $d\mu_C$ on $\bigwedge_A V$. Choose a set $\{\xi_i\}$ of generators for V. We write elements of $\bigwedge_A V$ as Grassmann functions $f(\xi)$. A Grassmann function $f(\xi)$ is called even if it is an even element of the algebra $\bigwedge_A V$. The Grassmann Gaussian integral of $f(\xi)$ is denoted $\int f(\xi)\, d\mu_C(\xi)$. Then

$$\int \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}\, d\mu_C(\xi) = \operatorname{Pf}\big[C_{i_k i_\ell}\big]_{1\le k,\ell\le n},$$

where $C_{ij} = C(\xi_i, \xi_j)$ and the Pfaffian of an $n \times n$ matrix M is denoted $\operatorname{Pf} M$. Observe that, for any even Grassmann function $f(\xi)$, the Grassmann Gaussian integral $\int f(\xi)\, d\mu_C(\xi)$ is an even element of the coefficient algebra A.

Let U be another vector space. Using the canonical isomorphism

$$\bigwedge_A(U\oplus V) = \bigoplus_{r,r'} \bigwedge\nolimits_A^{r'} U \bigwedge\nolimits_A^{r} V \cong \bigwedge\nolimits_A U \otimes \bigwedge\nolimits_A V \cong \bigwedge\nolimits_{\bigwedge_A U} V,$$

the Grassmann Gaussian integral defines a map $\int \cdot\, d\mu_C(\xi)$ from $\bigwedge_A(U\oplus V)$ to $\bigwedge_A U$. If $\{\zeta_i\}$ is any set of vectors in U then

$$\int e^{\Sigma\, \xi_i\zeta_i}\, d\mu_C(\xi) = e^{-1/2\, \Sigma\, \zeta_i C_{ij}\zeta_j}.$$

Definition II.3. *Choose a second copy V' of V and denote the element of V' corresponding to the element ξ_i of V by ψ_i. If $\dim V < \infty$, the renormalization group map Ω_C is defined by*

$$\Omega_C(W)(\psi) = \log \tfrac{1}{Z} \int e^{W(\psi+\xi)} d\mu_C(\xi) \qquad \text{where} \quad Z = \mathcal{Z}\Big(\int e^{W(\xi)} d\mu_C(\xi)\Big)$$

for all $W \in \bigwedge_A V'$ for which $\mathcal{Z}\Big(\int e^{W(\xi)} d\mu_C(\xi)\Big) \neq 0$.

Remark II.4. i) If $\dim V < \infty$, then $\Omega_C(W)$ is a rational function of W.
ii) By construction $\mathcal{Z}\big(\Omega_C(W)\big) = 0$ for all W.
iii) In Def. II.27, below, we extend the definition of Ω_C to normed vector spaces.

In this paper we state estimates on the renormalization group map for Wick ordered interactions W. Recall that Wick ordering with respect to a covariance C,

$$f(\xi) \mapsto \;{:}f(\xi){:}_{\xi,C}$$

is the A–linear map on $\bigwedge_A V$ characterized by

$${:}e^{\Sigma\, \xi_i\zeta_i}{:}_{\xi,C} \;=\; e^{1/2\, \Sigma\, \zeta_i C_{ij}\zeta_j}\, e^{\Sigma\, \xi_i\zeta_i}.$$

If the context admits, we delete the Wick ordering covariance C or the variable ξ (or both) from the symbol ${:}\cdot{:}_{\xi,C}$ for Wick ordering.

Also recall the integration by parts formula

$$\int \xi_i\, g(\xi)\, d\mu_C(\xi) = \sum_j C_{ij} \int \tfrac{\delta}{\delta\xi_j} g(\xi)\, d\mu_C(\xi)$$

or more generally

$$\int :\xi_{i_1}\cdots\xi_{i_n}\!: g(\xi)\, d\mu_C(\xi) = \sum_j C_{i_n\, j} \int :\xi_{i_1}\cdots\xi_{i_{n-1}}\!: \tfrac{\delta}{\delta\xi_j} g(\xi)\, d\mu_C(\xi)$$
$$= \tfrac{(-1)^{n-1}}{n} \sum_{i,j} C_{i\, j} \int :\tfrac{\delta}{\delta\xi_i}\xi_{i_1}\cdots\xi_{i_n}\!: \tfrac{\delta}{\delta\xi_j} g(\xi)\, d\mu_C(\xi).$$

II.3. Tensor Algebra and Grassmann Algebras. Again, let V be a complex vector space with a set $\{\xi_i\}$ of generators and A a superalgebra. We denote by $V^{\otimes n}$ the n–fold tensor product (over the complex numbers) of V with itself. The symmetric group S_n of all permutations of $\{1,\cdots,n\}$ acts on $V^{\otimes n}$ (from the right) in such a way that

$$(v_1\otimes\cdots\otimes v_n)^\pi = v_{\pi(1)}\otimes\cdots\otimes v_{\pi(n)}$$

for all $v_1,\cdots,v_n\in V$ and $\pi\in S_n$. The n^{th} exterior power $\bigwedge^n V$ of V can be identified with the set of all antisymmetric elements in $V^{\otimes n}$. We have the canonical projection

$$\mathrm{Ant}_n : \ V^{\otimes n} \longrightarrow \bigwedge^n V \qquad , \qquad f \longmapsto \tfrac{1}{n!}\sum_{\pi\in S_n} \mathrm{sgn}(\pi)\, f^\pi .$$

By A–linearity, Ant_n induces a map from $A\otimes V^{\otimes n}$ to $\bigwedge_A^n V$. The image of $v_1\otimes\cdots\otimes v_n$ under Ant_n is denoted by $v_1\cdots v_n$.

More generally, if $n=n_1+\cdots+n_r$ with nonnegative integers $n_1,\cdots,n_r$ we have the partial antisymmetrization

$$\begin{aligned}\mathrm{Ant}_{n_1,\cdots,n_r} : \ A\otimes V^{\otimes n} &\longrightarrow \bigwedge\nolimits_A^{n_1} V\otimes_A\cdots\otimes_A \bigwedge\nolimits_A^{n_r} V\\ f &\longrightarrow \tfrac{1}{n_1!\cdots n_r!}\sum_{\pi\in S_{n_1}\times\cdots\times S_{n_r}} \mathrm{sgn}(\pi)\, f^\pi \end{aligned} \tag{II.1}$$

Here, $S_{n_1}\times\cdots\times S_{n_r}$ is viewed as a subgroup of S_n, and we view $\bigwedge_A^{n_1} V\otimes_A\cdots\otimes_A \bigwedge_A^{n_r} V$ as a subspace of $A\otimes V^{\otimes n}$. If the context allows, we delete the subscript A in this tensor product. Elements of the r–fold tensor product $\bigwedge_A V\otimes\cdots\otimes\bigwedge_A V$ are written as Grassmann functions $f(\xi^{(1)},\cdots,\xi^{(r)})$, with $\xi^{(\ell)}$ the variable for the ℓ^{th} copy of $\bigwedge_A V$.

Definition II.5. *Let C be an antisymmetric bilinear form on V and let $1\le i,j\le n$. The contraction of the i^{th} variable to the j^{th} variable is the A–linear map*

$$\underset{i\to j}{\mathcal{C}\mathrm{on}_C} : \ A\otimes V^{\otimes n} \longrightarrow A\otimes V^{\otimes(n-2)}$$

characterized by

$$\underset{i\to j}{\mathcal{C}\mathrm{on}_C}(v_1\otimes\cdots\otimes v_n) = \epsilon_{ij}\, C(v_i,v_j)\, v_1\otimes\cdots v_{i-1}\otimes v_{i+1}\otimes\cdots\otimes v_{j-1}\otimes v_{j+1}\otimes\cdots\otimes v_n$$

for all $v_1,\cdots,v_n\in V$. Here

$$\epsilon_{ij} = \begin{cases} (-1)^{j-i+1} & \text{if } j>i\\ 0 & \text{if } j=i\\ (-1)^{i-j} & \text{if } j<i\end{cases}.$$

Remark II.6. Let C_1, C_2 be antisymmetric bilinear forms on $V, \lambda_1, \lambda_2 \in \mathbb{C}, 1 \le i, j \le n$ and $f \in A \otimes V^{\otimes n}$. Then

$$\underset{i\to j}{\mathcal{C}\mathrm{on}}_{(\lambda_1 C_1+\lambda_2 C_2)} f = \lambda_1 \underset{i\to j}{\mathcal{C}\mathrm{on}_{C_1}} f + \lambda_2 \underset{i\to j}{\mathcal{C}\mathrm{on}_{C_2}} f.$$

Remark II.7. Assume that $A = \mathbb{C}$ and that $\{\xi_i\},\ i \in \mathcal{I}$ is a basis of V. Then every element f of $V^{\otimes n}$ can be uniquely written in the form

$$f = \sum_{i_1,\cdots,i_n\in\mathcal{I}} \varphi(i_1, \cdots, i_n)\, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}$$

with a function φ on $\mathcal{I}^n$. Then, for $1 \le \mu < \nu \le n$,

$$\underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f = \sum_{j_1,\cdots,j_{n-2}\in\mathcal{I}} \varphi'(j_1, \cdots, j_{n-2})\, \xi_{j_1} \otimes \cdots \otimes \xi_{j_{n-2}}$$

with

$$\begin{aligned}&\varphi'(j_1, \cdots, j_{n-2})\\ &\quad = (-1)^{\nu-\mu+1} \sum_{i,j\in\mathcal{I}} C_{ij}\, \varphi(j_1, \cdots, j_{\mu-1}, i, j_\mu \cdots, j_{\nu-2}, j, j_{\nu-1} \cdots, j_{n-2}).\end{aligned}$$

Lemma II.8. *Let $r \ge 2$, $n = n_1 + \cdots + n_r$, $1 \le k \ne \ell \le r$ and*

$$\begin{aligned} n_1 + \cdots + n_{k-1} + 1 &\le \mu, \mu' \le n_1 + \cdots + n_k,\\ n_1 + \cdots + n_{\ell-1} + 1 &\le \nu, \nu' \le n_1 + \cdots + n_\ell. \end{aligned}$$

Let C be a covariance (antisymmetric bilinear form) on V and $f \in \bigwedge_A^{n_1} V \otimes \cdots \otimes \bigwedge_A^{n_r} V$.

i) $\underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f$ *is partially antisymmetric, precisely*

$$\underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f \in \bigwedge\nolimits_A^{n_1} V \otimes \cdots \otimes \bigwedge\nolimits_A^{n_k-1} V \otimes \cdots \otimes \bigwedge\nolimits_A^{n_\ell-1} V \otimes \cdots \otimes \bigwedge\nolimits_A^{n_r} V.$$

ii)

$$\underset{\mu'\to\nu'}{\mathcal{C}\mathrm{on}_C} f = \underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f.$$

Proof. We give the proof in the case $r = 2$, $k = 1$. The general case is analogous.
ii) Clearly, it suffices to show that $\underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f = \underset{1\to n_1+1}{\mathcal{C}\mathrm{on}_C} f$ for all $f \in \bigwedge_A^{n_1} V \otimes \bigwedge_A^{n_2} V$.
Let $n = n_1 + n_2$ and

$$\begin{aligned} f &= \sum_{i_1,\cdots,i_n\in\mathcal{I}} \varphi(i_1, \cdots, i_n)\, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n},\\ \underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C} f &= \sum_{j_1,\cdots,j_{n-2}\in\mathcal{I}} \varphi'(j_1, \cdots, j_{n-2})\, \xi_{j_1} \otimes \cdots \otimes \xi_{j_{n-2}},\\ \underset{1\to n_1+1}{\mathcal{C}\mathrm{on}_C} f &= \sum_{j_1,\cdots,j_{n-2}\in\mathcal{I}} \varphi''(j_1, \cdots, j_{n-2})\, \xi_{j_1} \otimes \cdots \otimes \xi_{j_{n-2}}. \end{aligned}$$

As $f \in \bigwedge_A^{n_1} V \otimes \bigwedge_A^{n_2} V$, φ is antisymmetric under permutations of its first n_1 arguments and under permutations of its last n_2 arguments. Consequently,

$$\begin{aligned}
&\varphi'(j_1, \cdots, j_{n-2}) \\
&\quad = (-1)^{\nu-\mu+1} \sum_{i,j\in\mathcal{I}} C_{ij}\ \varphi(j_1, \cdots, j_{\mu-1}, i, j_\mu \cdots, j_{\nu-2}, j, j_{\nu-1} \cdots, j_{n-2}) \\
&\quad = (-1)^{\nu-\mu+1}(-1)^{\mu-1}(-1)^{\nu-n_1-1} \sum_{i,j\in\mathcal{I}} C_{ij}\ \varphi(i, j_1, \cdots, j_{n_1-1}, j, j_{n_1}, \cdots, j_{n-2}) \\
&\quad = (-1)^{n_1+1} \sum_{i,j\in\mathcal{I}} C_{ij}\ \varphi(i, j_1, \cdots, j_{n_1-1}, j, j_{n_1}, \cdots, j_{n-2}) \\
&\quad = \varphi''(j_1, \cdots, j_{n-2}).
\end{aligned}$$

i) By part ii, we may assume that $\mu = 1$ and $\nu = n_1 + 1$. By linearity, we may assume that

$$f = v_1 \cdots v_{n_1} \otimes w_1 \cdots w_{n_2}$$

with $v_1, \cdots, v_{n_1},\ w_1, \cdots, w_{n_2} \in V$. Using

$$v_1 \cdots v_{n_1} = \tfrac{1}{n_1} \sum_{i=1}^{n_1} (-1)^{i-1} v_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_{n_1}$$

and its analog for $w_1 \cdots w_{n_2}$, we have

$$\begin{aligned}
f = \tfrac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} &(-1)^{i-1}(-1)^{j-1} v_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_{n_1} \\
&\otimes w_j \otimes w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n_2}.
\end{aligned}$$

Since

$$\begin{aligned}
&\mathop{\mathcal{C}\mathrm{on}_C}_{1\to n_1+1}\ v_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_{n_1} \otimes w_j \otimes w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n_2} \\
&\quad = (-1)^{n_1+1} C(v_i, w_j) v_1 \cdots v_{i-1} v_{i+1} \cdots v_{n_1} \otimes w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n_2},
\end{aligned}$$

we have

$$\begin{aligned}
\mathop{\mathrm{Con}_C}_{\mu\to\nu} f = \tfrac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} &(-1)^{n_1+j-i+1} C(v_i, w_j) v_1 \cdots v_{i-1} v_{i+1} \cdots v_{n_1} \\
&\otimes w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n_2}.
\end{aligned} \tag{II.2}$$

This shows that $\mathop{\mathrm{Con}_C}_{\mu\to\nu} f \in \bigwedge_A^{n_1-1} V \otimes \bigwedge_A^{n_2-1} V$. □

Definition II.9. *Let C be a covariance on V, $r \ge 1$ and $1 \le k \ne \ell \le r$.*

i) Let $n_1, \cdots, n_r \ge 0$. If $n_k, n_\ell \ge 1$ and $f(\xi^{(1)}, \cdots, \xi^{(r)}) \in \bigwedge_A^{n_1} V \otimes \cdots \otimes \bigwedge_A^{n_r} V$, the contraction of the $\xi^{(k)}$–fields to the $\xi^{(\ell)}$–fields is defined as

$$\mathop{\mathrm{Con}_C}_{\xi^{(k)}\to\xi^{(\ell)}} f = n_\ell \mathop{\mathcal{C}\mathrm{on}_C}_{\mu\to\nu} f,$$

where $n_1 + \cdots + n_{k-1} + 1 \le \mu \le n_1 + \cdots + n_k$ and $n_1 + \cdots + n_{\ell-1} + 1 \le \nu \le n_1 + \cdots + n_\ell$. By Lemma II.8, this definition is independent of the choice of μ, ν. If $n_k = 0$ or $n_\ell = 0$, we set $\mathop{\mathrm{Con}_C}_{\xi^{(k)}\to\xi^{(\ell)}} = 0$. Observe that $\mathop{\mathrm{Con}_C}_{\xi^{(k)}\to\xi^{(\ell)}}$ maps $\bigwedge_A^{n_1} V \otimes \cdots \otimes \bigwedge_A^{n_r} V$ to $\bigwedge_A^{n_1} V \otimes \cdots \bigwedge_A^{n_k-1} V \otimes \cdots \bigwedge_A^{n_\ell-1} V \otimes \bigwedge_A^{n_r} V$.

ii) The maps $\underset{\xi^{(k)}\to\xi^{(\ell)}}{\mathrm{Con}_C}$ *induce an A linear map from the r–fold tensor product*

$$\bigwedge_A V \otimes \cdots \otimes \bigwedge_A V = \bigoplus_{n_1,\cdots,n_r\ge 0} \bigwedge_A^{n_1} V \otimes \cdots \otimes \bigwedge_A^{n_r} V$$

to itself, which is also denoted by $\underset{\xi^{(k)}\to\xi^{(\ell)}}{\mathrm{Con}_C}$.

Lemma II.10. *Assume that* $\{\xi_i\}$ *is a basis of* V. *Then, for every Grassmann function* $f(\xi^{(1)},\cdots,\xi^{(r)})$ *in* $\bigwedge_A^{n_1} V \otimes \cdots \otimes \bigwedge_A^{n_r} V$,

$$\underset{\xi^{(k)}\to\xi^{(\ell)}}{\mathrm{Con}_C}(f) = -\tfrac{1}{n_k}\sum_{i,j} \tfrac{\delta}{\delta\xi_i^{(k)}} C_{ij} \tfrac{\delta}{\delta\xi_j^{(\ell)}}\big(f\big).$$

Proof. We give the proof in the case $r = 2$, $k = 1$, $\ell = 2$. The general case is similar. By linearity we may assume that

$$f = \xi_{i_1}^{(1)}\cdots\xi_{i_{n_1}}^{(1)} \otimes \xi_{j_1}^{(2)}\cdots\xi_{j_{n_2}}^{(2)}.$$

The claim then follows directly from (II.2). □

To simplify notation when $r = 2$, we write $f(\xi,\xi')$ instead of $f(\xi^{(1)},\xi^{(2)})$ for elements of $\bigwedge_A^{n_1} V \otimes \bigwedge_A^{n_2} V$. Similarly, in the case $r = 3$, we write $f(\xi,\xi',\xi'')$ for $f(\xi^{(1)},\xi^{(2)},\xi^{(3)})$.

Example II.11.

$$\underset{\xi\to\xi'}{\mathrm{Con}_C}\big(\xi_k\xi'_\ell\big) = C_{k\ell},$$
$$\underset{\xi\to\xi'}{\mathrm{Con}_C}\big(\xi_k\xi'_\ell\xi'_m\big) = C_{k\ell}\xi'_m - C_{km}\xi'_\ell,$$
$$\underset{\xi\to\xi'}{\mathrm{Con}_C}\big(\xi_j\xi_k\xi'_\ell\big) = \tfrac{1}{2}\big[\xi_j C_{k\ell} - \xi_k C_{j\ell}\big].$$

Remark II.12. Since taking partial derivatives commutes with Wick ordering, for any $f(\xi,\xi') \in \bigwedge_A(V \oplus V')$

$$\underset{\xi\to\xi'}{\mathrm{Con}_C}\big({:}f(\xi,\xi'){:}_{\xi'}\big) = {:}\underset{\xi\to\xi'}{\mathrm{Con}_C}\big(f(\xi,\xi')\big){:}_{\xi'}.$$

The main reason for introducing the contraction operator is the following "integration by parts formula":

Lemma II.13. *Let* $f(\xi,\xi',\xi'')$ *be a Grassmann function of degree at least one in* ξ'. *Set*

$$\tilde f(\xi,\xi',\xi'') = \underset{\xi'\to\xi''}{\mathrm{Con}_C} f(\xi,\xi',\xi''),$$
$$\tilde g(\xi,\xi'') = {:}\tilde f(\xi,\xi,\xi''){:}_\xi,$$
$$g(\xi,\xi'') = {:}f(\xi,\xi,\xi''){:}_\xi.$$

Then

$$\int \tilde g(\xi,\xi)\, d\mu_C(\xi) = \int g(\xi,\xi)\, d\mu_C(\xi).$$

That is,

$$\int \Big[:f(\xi,\xi,\xi''):_\xi \Big]_{\xi''=\xi} d\mu_C(\xi) \;=\; \int \Big[\dot{:}\big[\operatorname*{Con}_{\xi'\to\xi''} {}_C\, f(\xi,\xi',\xi'')\big]_{\xi'=\xi} \dot{:}_\xi \Big]_{\xi''=\xi} d\mu_C(\xi).$$

Proof. It suffices to prove the statement in the case that

$$f(\xi,\xi',\xi'') = \xi_{i_1}\cdots\xi_{i_{n-r}}\ \xi'_{i_{n-r+1}}\cdots\xi'_{i_n}\ :\xi''_{j_m}\xi''_{j_{m-1}}\cdots\xi''_{j_1}:_{\xi''}$$

with $1 \le r \le n,\ m \ge 1$. Then

$$\tilde f(\xi,\xi',\xi'') = -\tfrac{1}{r}\,\xi_{i_1}\cdots\xi_{i_{n-r}} \sum_{\substack{k=n-r+1,\cdots n\\ \ell=1,\cdots m}} (-1)^{(n+m+\ell)+(k-1)} C_{i_k j_\ell}$$
$$\times\, \xi'_{i_{n-r+1}}\cdots\xi'_{i_{k-1}}\xi'_{i_{k+1}}\cdots\xi'_{i_n}\ :\xi''_{j_m}\cdots\xi''_{j_{\ell+1}}\xi''_{j_{\ell-1}}\cdots\xi''_{j_1}:$$

and

$$\int \tilde g(\xi,\xi)\,d\mu_C(\xi) = \tfrac{1}{r} \sum_{k=n-r+1,\cdots n}\ \sum_{\ell=1,\cdots m} (-1)^{n+m+k+\ell} C_{i_k j_\ell}$$
$$\times \int (:\xi_{i_1}\cdots\xi_{i_{k-1}}\xi_{i_{k+1}}\cdots\xi_{i_n}:)(:\xi_{j_m}\cdots\xi_{j_{\ell+1}}\xi_{j_{\ell-1}}\cdots\xi_{j_1}:)\,d\mu_C(\xi).$$

If $m \ne n$, both $\int \tilde g(\xi,\xi)\,d\mu_C(\xi)$ and $\int g(\xi,\xi)\,d\mu_C(\xi)$ are zero by A.1. In the case $m = n$, again by Lemma A.1

$$\begin{aligned}
\int \tilde g(\xi,\xi)\,d\mu_C(\xi) &= \tfrac{1}{r} \sum_{k=n-r+1,\cdots n}\ \sum_{\ell=1,\cdots m} (-1)^{k+\ell} C_{i_k j_\ell}\ \det\big(C_{i_{k'} j_{\ell'}}\big)_{\substack{k'\ne k\\ \ell'\ne\ell}} \\
&= \tfrac{1}{r} \sum_{k=n-r+1,\cdots n} \det\big(C_{i_{k'} j_{\ell'}}\big) \\
&= \det\big(C_{i_k j_\ell}\big) \\
&= \int (:\xi_{i_1}\cdots\xi_{i_r}\xi_{i_{r+1}}\cdots\xi_{i_n}:)\,(:\xi_{j_m}\xi_{j_{m-1}}\cdots\xi_{j_1}:)\,d\mu_C(\xi) \\
&= \int g(\xi,\xi)\,d\mu_C(\xi).
\end{aligned}$$

□

II.4. Seminorms. We will formulate an estimate on the renormalization group map in terms of norms for which the contraction maps and Grassmann Gaussian integrals can be controlled. In this subsection we assume that A is a graded superalgebra.

In practice we shall use families of norms on $\bigwedge_A V$ that encode information concerning various derivatives of the coefficient functions. To unify such families in a way that incorporates Leibniz's rule we give

Definition II.14. *i) On* $\mathbb{R}_+ \cup \{\infty\} = \big\{\, x \in \mathbb{R} \,\big|\, x \ge 0 \,\big\} \cup \{+\infty\}$, *addition and the total ordering* $\le$ *are defined in the standard way. With the convention that* $0\cdot\infty = \infty$, *multiplication is also defined in the standard way.*

ii) Let $d \geq 0$. *The d–dimensional norm domain* $\mathfrak{N}_d$ *is the set of all formal power series*

$$X = \sum_{\delta=(\delta_1,\cdots,\delta_d)\in\mathbb{N}_0^d} X_\delta \, t_1^{\delta_1} \cdots t_d^{\delta_d}$$

in the variables $t_1, \cdots, t_d$ *with coefficients* $X_\delta \in \mathbb{R}_+ \cup \{\infty\}$. *Addition and partial ordering on* $\mathfrak{N}_d$ *are defined componentwise: If*

$$X = \sum_{\delta\in\mathbb{N}_0^d} X_\delta \, t_1^{\delta_1} \cdots t_d^{\delta_d}, \qquad X' = \sum_{\delta\in\mathbb{N}_0^d} X'_\delta \, t_1^{\delta_1} \cdots t_d^{\delta_d},$$

then

$$\begin{aligned} X + X' &= \sum_{\delta} (X_\delta + X'_\delta) \, t_1^{\delta_1} \cdots t_d^{\delta_d}, \\ X \leq X' &\iff X_\delta \leq X'_\delta \text{ for all } \delta \in \mathbb{N}_0^d. \end{aligned}$$

Multiplication is defined by

$$(X \cdot X')_\delta = \sum_{\beta+\gamma=\delta} X_\beta X'_\gamma.$$

We identify $\mathbb{R}_+ \cup \{\infty\}$ *with the set of all* $X \in \mathfrak{N}_d$ *with* $X_\delta = 0$ *for all* $\delta \neq \mathbf{0} = (0, \cdots, 0)$. *If* $a > 0$, $X_\mathbf{0} \neq \infty$ *and* $a - X_\mathbf{0} > 0$ *then* $(a - X)^{-1}$ *is defined as*

$$(a - X)^{-1} = \tfrac{1}{a - X_\mathbf{0}} \sum_{n=0}^{\infty} \left(\tfrac{X - X_\mathbf{0}}{a - X_\mathbf{0}}\right)^n.$$

Definition II.15. *Let E be a complex vector space. A (d–dimensional) seminorm on E is a map* $\|\cdot\| : E \to \mathfrak{N}_d$ *such that*

$$\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|, \qquad \|\lambda \, e\| = |\lambda| \, \|e\|$$

for all $e, e_1, e_2 \in E$ *and* $\lambda \in \mathbb{C}$.

Example II.16. Let be the space of all functions $f : \mathbb{R}^d \to \mathbb{C}$. Define

$$\|f\| = \sum_{\delta\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} \left|\partial^\delta f(x)\right| \, t_1^{\delta_1} \cdots t_d^{\delta_d},$$

where $\sup_{x\in\mathbb{R}^d} \left|\partial^\delta f(x)\right| = \infty$ if $\partial^\delta f(x)$ is not everywhere defined.

Remark II.17. i) By convention, $\mathfrak{N}_0 = \mathbb{R}_+ \cup \{\infty\}$.

ii) If E is a complex vector space and $\|\cdot\|$ is a 0–dimensional seminorm on E that obeys $\|e\| < \infty$ for all $e \in E$, then $\|\cdot\|$ is a seminorm on E in the conventional sense.

Definition II.18. *Let* $m, n \geq 0$. *A seminorm* $\|\cdot\|$ *on the space* $A_m \otimes V^{\otimes n}$ *is called symmetric, if for all* $f \in A_m \otimes V^{\otimes n}$ *and all permutations* $\pi \in S_n$,

$$\|f^\pi\| = \|f\|$$

and $\|f\| = 0$ *if* $m = n = 0$.

Remark II.19. Assume, as in Remark II.7, that $A = \mathbb{C}$ and that $\{\xi_i\},\ i \in \mathcal{I}$ is a basis of V. Every element f of $V^{\otimes n}$ can be uniquely written in the form

$$f = \sum_{i_1,\cdots,i_n \in \mathcal{I}} \varphi(i_1,\cdots,i_n)\, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}$$

with a function φ on $\mathcal{I}^n$. Therefore, a symmetric family of seminorms corresponds to a family of seminorms $\|\cdot\|$ on the spaces of functions φ on $\mathcal{I}^n$ such that

$$\|\varphi(i_1,\cdots,i_n)\| = \|\varphi(i_{\pi(1)},\cdots,i_{\pi(n)})\| \qquad \text{for all } \pi \in S_n.$$

Example II.20. Let $A = \mathbb{C}$ and V be a finite dimensional vector space with basis $\xi_1,\cdots,\xi_D$. For a function φ on $\{1,\cdots,D\}^n$ define the L_1–L_∞–norm,

$$\|\varphi\|_{1,\infty} = \max_{1\le k\le n} \sup_{1\le i_k\le D} \sum_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,,i_n=1}^{D} |\varphi(i_1,\cdots,i_n)|.$$

This defines a family of symmetric (0–dimensional) seminorms on the spaces $V^{\otimes n}$.

Definition II.21. *By $A_m[n_1,\cdots,n_r]$ we denote the image of $A_m \otimes V^{\otimes(n_1+\cdots+n_r)}$ under the partial antisymmetrization map $\mathrm{Ant}_{n_1,\cdots,n_r}$ defined in (II.1). It is the $\mathbb{C}$–linear subspace of*

$$A[n_1,\cdots,n_r] = \bigwedge\nolimits_A^{n_1} V^{(1)} \otimes \cdots \otimes \bigwedge\nolimits_A^{n_r} V^{(r)}$$

generated by elements of the form $a_m\ p_1(\xi^{(1)})\cdots p_r(\xi^{(r)})$ with $a_m \in A_m$ and $p_\nu(\xi^{(\nu)}) \in \bigwedge^{n_\nu} V^{(\nu)}$. Elements $f(\xi^{(1)},\cdots,\xi^{(r)})$ of $A_m[n_1,\cdots,n_r]$ are called homogeneous, and n_ν is their degree of homogeneity in the variable ξ_ν. Observe that $A_m[n_1,\cdots,n_r]$ is a subspace of $A_m \otimes V^{\otimes(n_1+\cdots+n_r)}$.

Lemma II.22. *Let $\|\cdot\|$ be a family of symmetric seminorms on the spaces $A_m \otimes V^{\otimes n}$. Then for all $f(\xi^{(1)},\cdots,\xi^{(r)}) \in A_m[n_1,\cdots,n_r]$,*

(i) for all permutations $\pi \in S_r$,

$$\|f(\xi^{(1)},\cdots,\xi^{(r)})\| = \|f(\xi^{(\pi(1))},\cdots,\xi^{(\pi(r))})\|.$$

(ii)

$$\|f(\xi^{(1)},\cdots,\xi^{(r-2)},\xi^{(r-1)},\xi^{(r-1)})\| \le \|f(\xi^{(1)},\cdots,\xi^{(r-2)},\xi^{(r-1)},\xi^{(r)})\|.$$

Here, $f(\xi^{(1)},\cdots,\xi^{(r-2)},\xi^{(r-1)},\xi^{(r-1)})$ is an element of $A_m[n_1,\cdots,n_{r-2},n_{r-1}+n_r]$.

(iii) If $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$ and

$$f(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)}+\epsilon\xi^{(r+1)}) = \sum_{k=0}^{n_r} f_k(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)},\xi^{(r+1)})$$

with $f_k \in A_m[n_1,\cdots,n_{r-1},n_r-k,k]$ then

$$\|f_k\| \le \binom{n_r}{k}\|f\|.$$

(iv) $\|f\| = 0$ if $f \in A_0[0,0,\cdots,0]$.

Proof. Parts (i) and (iv) are trivial. To prove part (ii) set

$$f'(\xi^{(1)},\cdots,\xi^{(r-2)},\xi^{(r-1)}) = f(\xi^{(1)},\cdots,\xi^{(r-2)},\xi^{(r-1)},\xi^{(r-1)}) \\ = \mathrm{Ant}_{n_1,\cdots,n_{r-2},n_{r-1}+n_r} f.$$

Then, by Def. II.18

$$\|f'\| \le \tfrac{1}{n_1!\cdots n_{r-2}!\,(n_{r-1}+n_r)!} \sum_{\pi\in S_{n_1}\times\cdots\times S_{n_{r-2}}\times S_{n_{r-1}+n_r}} \|f^\pi\| \le \|f\|.$$

We now prove part (iii). For any subset I of $\{n_1+\cdots+n_{r-1}+1,\cdots,n_1+\cdots+n_r\}$ let π_I^{-1} be the permutation that brings the sequence $I,\ \{n_1+\cdots+n_{r-1}+1,\cdots,n_1+\cdots+n_r\}\setminus I$ into standard order. Then

$$f_k = \sum_{\substack{I\subset\{n_1+\cdots+n_{r-1}+1,\cdots,n_1+\cdots+n_r\}\\ |I|=k}} \epsilon^{n_r-k}\,\mathrm{sgn}(\pi_I)\ f^{\pi_I}.$$

Again, by Def. II.18,

$$\|f_k\| \le \sum_{|I|=k} \|f\| = \binom{n_r}{k}\|f\|.$$

□

Definition II.23. *Let $\|\cdot\|$ be a family of symmetric d–dimensional seminorms and let $f(\xi^{(1)},\cdots,\xi^{(r)}) \in \bigwedge_A V\otimes\cdots\otimes\bigwedge_A V \cong \bigwedge_A(V\oplus\cdots\oplus V)$. Write*

$$f = \sum_{m,n_1,\cdots,n_r\ge 0} f_{m;n_1,\cdots,n_r}$$

with $f_{m;n_1,\cdots,n_r}\in A_m[n_1,\cdots,n_r]$. For any real $\alpha\ge 1$, $\mathrm{b}>0$ and $\mathfrak{c}\in\mathfrak{N}_d$ set

$$N(f;\alpha) = \tfrac{1}{\mathrm{b}^2}\,\mathfrak{c}\sum_{m,n_1,\cdots,n_r\ge 0} \alpha^{|n|}\,\mathrm{b}^{|n|}\,\|f_{m;n_1,\cdots,n_r}\|.$$

We omit the dependence on b and $\mathfrak{c}$ from the symbol $N(f;\alpha)$. If the context allows, we will also delete the reference to α.

In the applications we have in mind (see [FKTo1, Theorem V.2], [FKTo2, Theorem VIII.6], [FKTo3, Lemma XV.5], [FKTf1, Subsect. 7 of §II]), $\mathfrak{c}$ is a bound for various weighted L^1–norms of the propagator C (in position space), while b^2 is a bound for its L^∞–norm. $\frac{1}{\alpha}$ is a (possibly) fractional power of the coupling constant.

Remark II.24. Let $f(\xi^{(1)},\cdots,\xi^{(r)}) \in \bigwedge_A V\otimes\cdots\otimes\bigwedge_A V \cong \bigwedge_A(V\oplus\cdots\oplus V)$. Then

(i) for $1\le s\le r$,

$$N\big(f(\xi^{(1)}+\xi^{(r+1)},\cdots,\xi^{(s)}+\xi^{(r+s)},\xi^{(s+1)},\cdots,\xi^{(r)});\alpha\big) \\ \le N\big(f(\xi^{(1)},\cdots,\xi^{(r)});2\alpha\big),$$

(ii) for all $a\ge 1$,

$$N(f;\alpha)\le N(f;a\alpha).$$

Proof. (i) Since

$$f(\xi^{(1)}+\xi^{(r+1)},\cdots,\xi^{(s)}+\xi^{(r+s)},\cdots,\xi^{(r)})$$
$$= f(\xi^{(1)}+\xi^{(r+1)},\cdots,\xi^{(r)}+\xi^{(2r)})\Big|_{\xi^{(r+1)}=\cdots=\xi^{(2r)}=0},$$

it suffices to prove the claim in the case $s=r$. Write

$$f=\sum_{m,n_1,\cdots,n_r\ge 0} f_{m;n_1,\cdots,n_r}$$

with $f_{m;n_1,\cdots,n_r}\in A_m[n_1,\cdots,n_r]$. By part (iii) of Lemma II.22,

$$f_{m;n_1,\cdots,n_r}(\xi^{(1)}+\xi^{(r+1)},\cdots,\xi^{(r)}+\xi^{(2r)})$$
$$=\sum_{\substack{k_i=0,\cdots,n_i\\ \text{for } i=1,\cdots,r}} f_{m;\,n_1-k_1,\cdots,n_r-k_r,k_1,\cdots,k_r}(\xi^{(1)},\cdots,\xi^{(r)},\xi^{(r+1)},\cdots,\xi^{(2r)})$$

with $f_{m;\,n_1-k_1,\cdots,n_r-k_r,k_1,\cdots,k_r}\in A_m[n_1-k_1,\cdots,n_r-k_r,k_1,\cdots,k_r]$ and

$$\left\| f_{m;\,n_1-k_1,\cdots,n_r-k_r,k_1,\cdots,k_r}\right\| \le \|f_{m;n_1,\cdots,n_r}\| \prod_{i=1}^r \binom{n_i}{k_i}.$$

Consequently

$$N\big(f(\xi^{(1)}+\xi^{(r+1)},\cdots,\xi^{(r)}+\xi^{(2r)})\big)$$
$$=\tfrac{1}{\mathfrak{b}^2}\,\mathfrak{c}\sum_{m,n_1,\cdots,n_r\ge 0}\ \sum_{\substack{k_i=0,\cdots,n_i\\ \text{for } i=1,\cdots,r}} \alpha^{|n|}\,\mathfrak{b}^{|n|}\left\|f_{m;\,n_1-k_1,\cdots,n_r-k_r,k_1,\cdots,k_r}\right\|$$
$$\le \tfrac{1}{\mathfrak{b}^2}\,\mathfrak{c}\sum_{m;\,n_1,\cdots n_r}\alpha^{|n|}\,\mathfrak{b}^{|n|}\,\|f_{m;n_1,\cdots,n_r}\| \sum_{\substack{k_i=0,\cdots,n_i\\ \text{for } i=1,\cdots,r}}\ \prod_{i=1}^r\binom{n_i}{k_i}$$
$$=\tfrac{1}{\mathfrak{b}^2}\,\mathfrak{c}\sum_{m;\,n_1,\cdots n_r}\alpha^{|n|}\,\mathfrak{b}^{|n|}\,2^{n_1+\cdots+n_r}\,\|f_{m;n_1,\cdots,n_r}\|$$
$$=N\big(f(\xi^{(1)},\cdots,\xi^{(r)});2\alpha\big).$$

(ii) is trivial □

II.5. An Estimate of the Renormalization Group Map. Let A be a graded superalgebra, $\|\cdot\|$ be a family of symmetric d–dimensional seminorms on the spaces $A_m\otimes V^{\otimes n}$ and C be a covariance on V.

Definition II.25. *(i) We say that $\mathfrak{c}\in\mathfrak{N}_d$, with $\mathfrak{c}_0\ne 0,\infty$, is a contraction bound for C with respect to these seminorms if for all $f\in A_m\otimes V^{\otimes n}$, $f'\in A_{m'}\otimes V^{\otimes n'}$ and all $1\le i\le n,\ 1\le j\le n'$,*

$$\|\underset{i\to n+j}{\mathcal{C}\mathrm{on}_C}(f\otimes f')\|\le\mathfrak{c}\,\|f\|\,\|f'\|.$$

Observe that $\underset{i\to n+j}{\mathcal{C}\mathrm{on}_C}(f\otimes f')\in A_{m+m'}\otimes V^{\otimes(n+n'-2)}$.

(ii) For $n' \leq n$ define the partial antisymmetrization

$$Ant_{n'} = \mathrm{Ant}_{n',1,1,\cdots,1}.$$

It is characterized by

$$\begin{aligned} &Ant_{n'}(v_1 \otimes \cdots \otimes v_{n'} \ \otimes \ w_1 \otimes \cdots \otimes w_{n-n'}) \\ &\quad = v_1 \cdots v_{n'} \ \otimes \ w_1 \otimes \cdots \otimes w_{n-n'} \ \in \ \bigwedge\nolimits_A^{n'} V \otimes_A V^{\otimes(n-n')} \end{aligned}$$

for all $v_1, \cdots, v_{n'}, w_1 \cdots, w_{n-n'} \in V$. We say that the real number $\mathrm{b} \geq 0$ is a (Grassmann) integral bound for C with respect to the family of seminorms if for every $f \in A_m \otimes V^{\otimes n}$ and every $n' \leq n$,

$$\left\| \int Ant_{n'}(f) d\mu_C \right\| \leq (\mathrm{b}/2)^{n'} \|f\|.$$

Here, the Grassmann Gaussian integral maps $\bigwedge_A^{n'} V \otimes_A V^{\otimes(n-n')}$ to $A \otimes V^{\otimes(n-n')}$.

Example II.26. In Example II.20, a contraction bound for a covariance $C = \big(C_{ij}\big)_{1\leq i,j\leq D}$ with respect to the L_1–L_∞–norm is given by

$$\mathfrak{c} = \|C\|_{1,\infty} = \max_{1\leq i\leq D} \sum_{1\leq j\leq D} \big|C_{ij}\big|.$$

Let $\mathrm{b} > 0$ be such that

$$\left| \int \xi_{i_1} \cdots \xi_{i_n} \, d\mu_C(\xi) \right| \leq (\mathrm{b}/2)^n \tag{II.3}$$

for all $n \geq 0$ and all $i_1, \cdots, i_n$. Then b is an integral bound for the covariance C. In Appendix B we give criteria under which (II.3) is fulfilled.

When $\dim V = \infty$, it is not a priori clear whether or not the renormalization group map

$$\Omega_C(W)(\psi) = \log \tfrac{1}{Z} \int e^{W(\psi+\xi)} d\mu_C(\xi)$$

of Def. II.3 makes sense. However, in the case of interest, we can define it as a formal power series in W. The Taylor expansion of $\int e^{W(\psi+\xi)} d\mu_C(\xi)$ is $\sum_{n=1}^{\infty} \mathcal{G}_n(W, \cdots, W)$, where the n^{th} term is the n–linear map

$$\mathcal{G}_n(W_1, \cdots, W_n) = \tfrac{1}{n!} \int W_1(\psi+\xi) \cdots W_n(\psi+\xi) \, d\mu_C(\xi)$$

from $\bigwedge_A V' \times \cdots \times \bigwedge_A V'$ to $\bigwedge_A V'$, restricted to the diagonal. Explicit evaluation of the Grassmann integral expresses $\mathcal{G}_n$ as the sum of all graphs with vertices $W_1, \cdots, W_n$ and lines C. The (formal) Taylor coefficient $\frac{d}{dt_1} \cdots \frac{d}{dt_n} \Omega_C(t_1 W_1 + \cdots + t_n W_n)\Big|_{t_1=\cdots=t_n=0}$ of $\Omega_C(W)$ is similar, but with only connected graphs.

Definition II.27. *Let $\|\cdot\|$ be a family of symmetric seminorms and let C be a covariance on V with a finite integral bound and a contraction bound $\mathfrak{c}$ obeying $\mathfrak{c}_0 < \infty$. Then*

$$\tfrac{d}{dt_1}\cdots\tfrac{d}{dt_n}\Omega_C(t_1W_1+\cdots+t_nW_n)\Big|_{t_1=\cdots=t_n=0}$$

interpreted as a sum of graphs, as above, is a bounded n–linear map . Then Ω_C is defined to be the formal Taylor series associated to the sequence of these multilinear maps.

Theorem II.28. *Let $\|\cdot\|$ be a family of symmetric seminorms and let C be a covariance on V with contraction bound $\mathfrak{c}$ and integral bound b. Then the formal Taylor series $\Omega_C(:W:)$ converges to an analytic map[3] on $\left\{ W\in\bigwedge_A V' \,\middle|\, W \text{ even},\ N\big(W;8\alpha\big)_{\mathbf{0}} < \frac{\alpha^2}{4}\right\}$. Furthermore, if $W(\psi)\in\bigwedge_A V'$ is an even Grassmann function such that*

$$N\big(W;8\alpha\big)_{\mathbf{0}} < \tfrac{\alpha^2}{4},$$

then

$$N\big(\Omega_C(:W:)-W;\alpha\big) \;\le\; \tfrac{2}{\alpha^2}\,\frac{N(W;8\alpha)^2}{1-\frac{4}{\alpha^2}N(W;8\alpha)}.$$

This theorem is proven in §III.3.

II.6. Contraction and Integral Bounds. In this subsection we investigate properties of contraction and integral bounds as introduced in Def. II.25. These properties will be used in §III to prove Theorem II.28.

Lemma II.29. *Assume that $\mathfrak{c}$ is a contraction bound and b an integral bound for C.*

(i) If $n_r, n'_{r+1}\ge 1$, then for

$$f_1(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)})\in A_m[n_1,\cdots,n_{r-1},n_r,0],$$
$$f_2(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r+1)})\in A_{m'}[n'_1,\cdots n'_{r-1},0,n'_{r+1}],$$

one has

$$\Big\|\operatorname*{Con}_{C \atop \xi^{(r)}\to\xi^{(r+1)}} f_1(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)})\,f_2(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r+1)})\Big\|$$
$$\le n'_{r+1}\mathfrak{c}\,\|f_1(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)})\|\,\|f_2(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r+1)})\|.$$

Observe that the contraction $\operatorname*{Con}_{C \atop \xi^{(r)}\to\xi^{(r+1)}} f_1(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)})\,f_2(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r+1)})$ lies in $A_{m+m'}[n_1+n'_1,\cdots,n_{r-1}+n'_{r-1},n_r-1,n'_{r+1}-1]$.

(ii) Let $0\le s\le t\le r$ and let

$$f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\xi^{(t+1)},\cdots,\xi^{(r)})$$
$$\in A_m[n_1,\cdots,n_s,n_{s+1},\cdots,n_t,n_{t+1},\cdots,n_r].$$

[3] For an elementary discussion of analytic maps between Banach spaces see, for example, Appendix A of [PT].

Set

$$\tilde f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)})$$
$$= {:}f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)}){:}_{\xi^{(1)},\cdots,\xi^{(s)}},$$

where $:\cdot:_{\xi^{(1)},\cdots,\xi^{(s)}}$ *denotes Wick ordering, with respect to the covariance C, in the variables* $\xi^{(1)},\cdots,\xi^{(s)}$ *separately. Then*

$$\left\| \int \tilde f(\xi,\cdots,\xi,\xi^{(t+1)}\cdots,\xi^{(r)})\,d\mu_C(\xi)\right\| \le \mathrm{b}^{n_1+\cdots+n_t}\,\|f(\xi^{(1)},\cdots,\xi^{(r)})\|.$$

Observe that $\int\tilde f(\xi,\cdots,\xi,\xi^{(t+1)}\cdots,\xi^{(r)})\,d\mu_C(\xi)\in A_m[0,\cdots,0,n_{t+1},\cdots,n_r]$.

Proof. i) By definition

$$\underset{\xi^{(r)}\to\xi^{(r+1)}}{\mathrm{Con}_C}\; f_1(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r)})\; f_2(\xi^{(1)},\cdots,\xi^{(r-1)},\xi^{(r+1)})$$
$$= \pm\, n'_{r+1}\,\mathrm{Ant}_{n_1+n'_1,\cdots,n_{r-1}+n'_{r-1},n_r-1,n_{r+1}-1}\Big(\underset{\mu\to\nu}{\mathcal{C}\mathrm{on}_C}\; f_1\otimes f_2\Big)^{\pi},$$

where $1+\sum_{i=1}^{r-1} n_i \le \mu \le \sum_{i=1}^{r} n_i$, $1+\sum_{i=1}^{r} n_i + \sum_{i=1}^{r-1} n'_i \le \nu \le n'_{r+1} + \sum_{i=1}^{r} n_i + \sum_{i=1}^{r-1} n'_i$ and π is the permutation that maps the sequence

$$1,\cdots,n_1,\cdots,n_1+\cdots+n_r-1,\cdots,n_1+\cdots+n_r+n'_1+\cdots+n'_{r+1}-2$$

to the sequence

$$1,\cdots,n_1,n_1+\cdots+n_r-1+1,\cdots,n_1+\cdots+n_r-1+n'_1,n_1+1,\cdots.$$

The desired estimate now follows from Defs. II.25, II.18 and II.9.

ii) Set $g(\xi,\xi^{(t+1)}\cdots,\xi^{(r)}) = \tilde f(\xi,\cdots,\xi,\xi^{(t+1)}\cdots,\xi^{(r)})$. In the case $s=0$ one has $\tilde f = f$ and hence $\|g\| \le \|\tilde f\| \le \|f\|$ by part (ii) of Lemma II.22. Therefore, by Def. II.25,

$$\left\| \int \tilde f(\xi,\cdots,\xi,\xi^{(t+1)}\cdots,\xi^{(r)})\,d\mu_C(\xi)\right\|$$
$$\le (\mathrm{b}/2)^{n_1+\cdots+n_t}\,\|g\| \le (\mathrm{b}/2)^{n_1+\cdots+n_t}\,\|f\|.$$

In the general case, we have by part (i) of Prop. A.2 and part (iii) of Lemma II.22,

$$\tilde f = \int f(\xi^{(1)}+\imath\zeta^{(1)},\cdots,\xi^{(s)}$$
$$+\imath\zeta^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)})\,d\mu_C(\zeta^{(1)},\cdots,\zeta^{(s)})$$
$$= \sum_{\substack{k_i=1,\cdots,n_i\\ i=1,\cdots,s}} \int f_{k_1,\cdots,k_s}(\xi^{(1)},\zeta^{(1)},\cdots,\xi^{(s)},\zeta^{(s)},\xi^{(s+1)},\cdots,$$
$$\xi^{(t)},\cdots,\xi^{(r)})\,d\mu_C(\zeta^{(1)},\cdots,\zeta^{(s)})$$

with

$$f_{k_1,\cdots,k_s}\in A_m[n_1-k_1,k_1,\cdots,n_s-k_s,k_s,\,n_{s+1},\cdots,n_r]$$

fulfilling

$$\| f_{k_1,\cdots,k_s} \| \le \| f \| \prod_{i=1}^{s} \tbinom{n_i}{k_i}.$$

By the special case discussed above

$$\begin{aligned}\Big\| \int & f_{k_1,\cdots,k_s}(\xi^{(1)}, \zeta^{(1)}, \cdots, \xi^{(s)}, \zeta^{(s)}, \xi^{(s+1)}, \cdots, \\ & \xi^{(t)}, \xi^{(t+1)}, \cdots, \xi^{(r)})\, d\mu_C(\zeta^{(1)}, \cdots, \zeta^{(s)}) \Big\| \\ & \le (\mathrm{b}/2)^{k_1+\cdots+k_s} \| f_{k_1,\cdots,k_s} \|.\end{aligned}$$

Therefore, again by the special case discussed above,

$$\begin{aligned}\Big\| \int \Big[\int & f_{k_1,\cdots,k_s}(\xi, \zeta^{(1)}, \cdots, \xi, \zeta^{(s)}, \xi, \cdots, \\ & \xi, \xi^{(t+1)}, \cdots, \xi^{(r)})\, d\mu_C(\zeta^{(1)}, \cdots, \zeta^{(s)}) \Big] d\mu_C(\xi) \Big\| \\ & \le (\mathrm{b}/2)^{n_1+\cdots+n_t} \| f_{k_1,\cdots,k_s} \|.\end{aligned}$$

Consequently

$$\begin{aligned}\Big\| \int \tilde{f}(\xi, \cdots, \xi, \xi^{(t+1)} \cdots, \xi^{(r)})\, d\mu_C(\xi) \Big\| & \le \mathrm{b}^{n_1+\cdots+n_t} \| f \| \sum_{\substack{k_i=1,\cdots,n_i \\ i=1,\cdots,s}} \prod_{i=1}^{s} \tfrac{1}{2^{n_i}} \tbinom{n_i}{k_i} \\ & \le \mathrm{b}^{n_1+\cdots+n_t} \| f \|.\end{aligned}$$

□

Remark II.30. Let C_1, C_2 be covariances on V and $\lambda_1, \lambda_2 \in \mathbb{C}$.

i) If $\mathfrak{c}_1$ is a contraction bound for C_1 and $\mathfrak{c}_2$ is a contraction bound for C_2 then $|\lambda_1|\mathfrak{c}_1 + |\lambda_2|\mathfrak{c}_2$ is a contraction bound for $\lambda_1 C_1 + \lambda_2 C_2$.
ii) If b_1 and b_2 are integral bounds for C_1 and C_2, then $\sqrt{|\lambda_1|}\,\mathrm{b}_1 + \sqrt{|\lambda_2|}\,\mathrm{b}_2$ is an integral bound for $\lambda_1 C_1 + \lambda_2 C_2$.

Proof. Part (i) follows from Remark II.6. To prove part (ii), let $n' \le n$. For $I \subset \{1, \cdots, n'\}$ let Ant_I be the map from $A \otimes V^{\otimes n}$ to $\bigwedge_A^{|I|} V \otimes_A \bigwedge_A^{n'-|I|} V \otimes_A V^{\otimes(n-n')}$ characterized by

$$\begin{aligned}& Ant_I(v_1 \otimes \cdots \otimes v_{n'} \otimes w_1 \otimes \cdots \otimes w_{n-n'}) \\ & \quad = \epsilon_I \Big(\prod_{i \in I} v_i \Big) \otimes \Big(\prod_{j \in \{1,\cdots,n'\} \setminus I} v_i \Big) \otimes w_1 \otimes \cdots \otimes w_{n-n'},\end{aligned}$$

where ϵ_I is the sign of the permutation that puts the sequence $I,\ \{1, \cdots, n'\} \setminus I$ back in increasing order. Then

$$\int Ant_{n'}(f)\, d\mu_{\lambda_1 C_1 + \lambda_2 C_2} = \sum_{\substack{I \subset \{1,\cdots,n'\} \\ |I| \text{ even}}} \int Ant_I(f)(\xi, \xi')\, d\mu_{\lambda_1 C_1}(\xi)\, d\mu_{\lambda_2 C_2}(\xi'),$$

and consequently

$$\Big\| \int Ant_{n'}(f) d\mu_C \Big\| \le \sum_{r=0}^{n'} \sum_{\substack{I\subset\{1,\cdots,n'\}\\ |I|=r}} \Big\| \int Ant_I(f)(\xi,\xi')\, d\mu_{\lambda_1 C_1}(\xi)\, d\mu_{\lambda_2 C_2}(\xi') \Big\|$$

$$\le \sum_{r=0}^{n'} \binom{n'}{r} \left(\tfrac{\sqrt{|\lambda_1|}\ \mathrm{b}_1}{2}\right)^r \left(\tfrac{\sqrt{|\lambda_2|}\ \mathrm{b}_2}{2}\right)^{n'-r} \|f\|$$

$$\le \left(\tfrac{1}{2}\right)^{n'} \left(\sqrt{|\lambda_1|}\ \mathrm{b}_1 + \sqrt{|\lambda_2|}\ \mathrm{b}_2\right)^{n'} \|f\|.$$

□

Lemma II.31. *Let $f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\xi^{(t+1)},\cdots,\xi^{(r)})$ be a Grassmann function in $\bigwedge_A V\otimes\cdots\otimes\bigwedge_A V \cong \bigwedge_A(V\oplus\cdots\oplus V)$. Set*

$$\tilde f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)}) = \,:f(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)}):_{\xi^{(1)},\cdots,\xi^{(s)}}.$$

Let $0<\varepsilon\le\alpha$. If $\varepsilon\mathrm{b}$ is is an integral bound for the covariance C, then

$$N\Big(\int \tilde f(\xi,\cdots,\xi,\xi^{(t+1)},\cdots,\xi^{(r)})\, d\mu_C(\xi)\ ;\ \alpha\Big) \le N(f;\alpha).$$

If $f(0,\cdots,0,\xi^{(t+1)},\cdots,\xi^{(r)})=0$ then

$$N\Big(\int \tilde f(\xi,\cdots,\xi,\xi^{(t+1)},\cdots,\xi^{(r)})\, d\mu_C(\xi)\ ;\ \alpha\Big) \le \tfrac{\varepsilon^2}{\alpha^2}\, N(f;\alpha).$$

Proof. Set

$$g(\xi^{(t+1)},\cdots,\xi^{(r)}) = \int \tilde f(\xi,\cdots,\xi,\xi^{(t+1)},\cdots,\xi^{(r)})\, d\mu_C(\xi).$$

Write

$$f = \sum_{m,n_1,\cdots,n_r\ge 0} f_{m;n_1,\cdots,n_r}$$

with $f_{m;n_1,\cdots,n_r}\in A_m[n_1,\cdots,n_r]$ and set

$$\tilde f_{m;n_1,\cdots,n_r}(\xi^{(1)},\cdots,\xi^{(t)},\cdots,\xi^{(r)}) = \,:f_{m;n_1,\cdots,n_r}(\xi^{(1)},\cdots,\xi^{(s)},\cdots,\xi^{(t)},\cdots,\xi^{(r)}):_{\xi^{(1)},\cdots,\xi^{(s)}}$$

$$g_{m;n_1,\cdots,n_r}(\xi^{(t+1)},\cdots,\xi^{(r)}) = \int \tilde f_{m;n_1,\cdots,n_r}(\xi,\cdots,\xi,\xi^{(t+1)},\cdots,\xi^{(r)})\, d\mu_C(\xi).$$

Then $g=\sum_{m;n_1,\cdots,n_r\ge 0} g_{m;n_1,\cdots,n_r}$ and therefore, by part (ii) of Lemma II.29,

$$N(g;\alpha) \le \tfrac{1}{\mathrm{b}^2}\,\mathfrak{c} \sum_{m;n_{t+1},\cdots,n_r} \big(\alpha\mathrm{b}\big)^{n_{t+1}+\cdots+n_r} \sum_{n_1,\cdots,n_t} (\varepsilon\mathrm{b})^{n_1+\cdots+n_t}\, \|f_{m;n_1,\cdots,n_r}\|$$

$$= \tfrac{1}{\mathrm{b}^2}\,\mathfrak{c} \sum_{m;n_1,\cdots,n_{t+1},\cdots,n_r} \left(\tfrac{\varepsilon}{\alpha}\right)^{n_1+\cdots+n_t} \alpha^{|n|}\mathrm{b}^{|n|}\, \|f_{m;n_1,\cdots,n_r}\|.$$

Since $\frac{\varepsilon}{\alpha} \le 1$, this implies that $N(g;\alpha) \le N(f;\alpha)$. If $f(0,\cdots,0,\xi^{(t+1)},\cdots,\xi^{(r)}) = 0$ then $f_{m;n_1,\cdots,n_r} = 0$ for $n_1 = \cdots = n_t = 0$ and $g_{m;n_1,\cdots,n_r} = 0$ for $n_1 + \cdots + n_t \le 1$, since $\int \xi_i \, d\mu_C(\xi) = 0$. Consequently

$$N(g;\alpha) \ \le\ \tfrac{1}{\mathrm{b}^2}\,\mathfrak{c} \sum_{\substack{m;n_1+\cdots+n_t\ge 2\\ n_{t+1},\cdots,n_r}} \Big(\tfrac{\varepsilon}{\alpha}\Big)^{n_1+\cdots+n_t} \alpha^{|n|}\mathrm{b}^{|n|} \|f_{m;n_1,\cdots,n_r}\| \ \le\ \tfrac{\varepsilon^2}{\alpha^2}\, N(f;\alpha).$$

□

Corollary II.32. *Let $f(\xi) \in \bigwedge_A V$.*

(i) If $\alpha\mathrm{b}$ is an integral bound for the covariance C, then

$$N(:f:_C\,;\alpha) \le N(f;2\alpha),$$
$$N(f;\alpha) \le N(:f:_C\,;2\alpha).$$

(ii) Let C_1, C_2 be two covariances and $0 < \varepsilon \le \frac{\alpha}{\sqrt{2}}$. Assume that $\varepsilon\mathrm{b}$ is an integral bound for $C_1 - C_2$. Set

$$:f(\xi):_{C_2} \ =\ :f'(\xi):_{C_1}.$$

Then

$$N(f'-f;\alpha) \le \tfrac{2\varepsilon^2}{\alpha^2} N(f;2\alpha).$$

(iii) Let, for κ in a neighbourhood of zero, C_κ be a covariance on V. Assume that C_0 and $\frac{d}{d\kappa}C_\kappa\big|_{\kappa=0}$ have integral bounds b and b', respectively. Define f_κ by $\,\vdots\, f_\kappa \,\vdots_{\,C_\kappa} = f$. Then

$$N\big(\tfrac{d}{d\kappa} f_\kappa\big|_{\kappa=0}\,;\alpha\big) \le \tfrac{1}{(\alpha-1)^2}\big(\tfrac{\mathrm{b}'}{\mathrm{b}}\big)^2 N(f;2\alpha).$$

Proof. (i) By part (i) of Prop. A.2, Remark II.30, the lemma above with $s = 0$ and $\varepsilon = \alpha$, and part (i) of Remark II.24,

$$\begin{aligned} N(:f:_C\,;\alpha) &= N\Big(\int f(\xi+\xi')\, d\mu_{-C}(\xi');\alpha\Big)\\ &\le N\big(f(\xi+\xi');\alpha\big)\\ &\le N(f;2\alpha). \end{aligned}$$

The proof of the other bound is similar.

(ii) Define f_z by $:f(\xi):_{C_2} \ =\ :f_z(\xi):_{C_2+z(C_1-C_2)}$. By part (i) of Lemma A.4,

$$f_z(\xi) \ =\ \vdots\, f(\xi)\,\vdots_{-z(C_1-C_2)}.$$

By Remark II.30.ii, $-z(C_1-C_2)$ has integral bound $\sqrt{|z|}\,\varepsilon\,\mathrm{b}$, which is bounded by $\alpha\mathrm{b}$ for all $|z| \le \big(\frac{\alpha}{\varepsilon}\big)^2$. Hence, by part (i),

$$N(f_z;\alpha) \ \le\ N(f;2\alpha)$$

for all $|z| \le \big(\frac{\alpha}{\varepsilon}\big)^2$. Let $\mathcal{C}$ be the contour $|\zeta| = \big(\frac{\alpha}{\varepsilon}\big)^2$ in the complex plane, with standard orientation. Then

$$f'-f = f_1 - f_0 = \tfrac{1}{2\pi\imath}\int_{\mathcal{C}} \big(\tfrac{1}{\zeta-1} - \tfrac{1}{\zeta}\big)\, f_\zeta \, d\zeta = \tfrac{1}{2\pi\imath}\int_{\mathcal{C}} \tfrac{1}{\zeta(\zeta-1)}\, f_\zeta\, d\zeta.$$

Hence,

$$N\big(f'-f;\alpha\big) \ \le\ \tfrac{1}{(\frac{\alpha}{\varepsilon})^2-1} \sup_{|\zeta|=(\frac{\alpha}{\varepsilon})^2} N\big(f_\zeta;\alpha\big) \ \le\ 2\tfrac{\varepsilon^2}{\alpha^2}\, N(f;2\alpha).$$

(iii) Set $D_z = C_0 + z\frac{d}{d\kappa}C_\kappa\big|_{\kappa=0}$ and define g_z by $:g_z:_{D_z} = f$. Since C_κ and D_κ agree to order κ, $\frac{d}{d\kappa}f_\kappa\big|_{\kappa=0} = \frac{d}{dz}g_z\big|_{z=0}$. By Remark II.30.ii, D_z has integral bound $\mathrm{b} + \sqrt{|z|}\,\mathrm{b}'$. For all $|z| \le (\alpha-1)^2\big(\frac{\mathrm{b}}{\mathrm{b}'}\big)^2$ this is bounded by $\alpha\mathrm{b}$. By part (i),

$$N(g_z;\alpha) \ \le\ N(f;2\alpha)$$

for all $|z| \le (\alpha-1)^2\big(\frac{\mathrm{b}}{\mathrm{b}'}\big)^2$. Hence, by the Cauchy integral formula

$$\begin{aligned} N\big(\tfrac{d}{d\kappa}f_\kappa\big|_{\kappa=0};\alpha\big) = N\big(\tfrac{d}{dz}g_z\big|_{z=0};\alpha\big) &\le \tfrac{1}{(\alpha-1)^2}\big(\tfrac{\mathrm{b}'}{\mathrm{b}}\big)^2 \sup_{|z|=(\alpha-1)^2(\frac{\mathrm{b}}{\mathrm{b}'})^2} N\big(g_z;\alpha\big) \\ &\le \tfrac{1}{(\alpha-1)^2}\big(\tfrac{\mathrm{b}'}{\mathrm{b}}\big)^2 N(f;2\alpha) \end{aligned}$$

□

The following proposition is the key to our estimate on the renormalization group map and will be used in the proof of Prop. III.7.

Proposition II.33. *Assume that* $\mathfrak{c}$ *is a contraction bound and* b *an integral bound for* C. *Let* $\ell \ge 1$ *and* $r \ge s \ge 1$. *Let* $f_i(\xi^{(1)},\cdots,\xi^{(s)},\xi^{(s+1)},\cdots,\xi^{(r)})$, $1 \le i \le \ell$ *and* $f(\xi^{(1)},\cdots,\xi^{(r)})$ *be Grassmann functions. Assume that for each* $1 \le i \le \ell$ *there is a* $1 \le j_i \le s$ *such that* f_i *vanishes when* $\xi^{(j_i)} = 0$. *Set*

$$\begin{aligned} g(\xi^{(s+1)},\cdots,\xi^{(r)}) = \int :\prod_{i=1}^{\ell} f_i(\xi^{(1)},\cdots,\xi^{(s)},\cdots,\xi^{(r)}):_{\xi^{(1)},\cdots,\xi^{(s)}} \\ :f(\xi^{(1)},\cdots,\xi^{(s)},\cdots,\xi^{(r)}):_{\xi^{(1)},\cdots,\xi^{(s)}} \prod_{i=1}^{s} d\mu_C(\xi^{(i)}). \end{aligned}$$

Then, for $\alpha \ge 2$, *we have the two bounds*

$$\begin{aligned} N(g;\alpha) \ &\le \tfrac{\ell!}{\alpha^\ell} N(f;\alpha) \prod_{i=1}^{\ell} N(f_i;\alpha), \\ N(g;\alpha) \ &\le \tfrac{\ell^\ell}{\alpha^{2\ell}} N(f;\alpha) \prod_{i=1}^{\ell} N(f_i;\alpha). \end{aligned}$$

Proof. We first prove the statement in the case that f and all the f_i's are homogeneous. That is

$$\begin{aligned} f &\in A_m[n^{(1)},\cdots,n^{(s)},\cdots,n^{(r)}], \\ f_i &\in A_{m_i}[n_i^{(1)},\cdots,n_i^{(r)}] \qquad 1 \le i \le \ell. \end{aligned}$$

By hypothesis $n_i^{(j_i)} \ge 1$ for each $1 \le i \le \ell$. Set, for each $1 \le i \le \ell$, $\mathrm{Con}_i = \underset{\xi_i^{(j_i)} \to \zeta^{(j_i)}}{\mathrm{Con}_C}$ and

$$\begin{aligned} &g'(\xi^{(1)},\cdots,\xi^{(r)};\zeta^{(1)},\cdots,\zeta^{(r)}) \\ &\quad= \prod_{i=1}^{\ell}\mathrm{Con}_i \prod_{i=1}^{\ell} f_i(\xi_i^{(1)},\cdots,\xi_i^{(r)})\ f(\zeta^{(1)},\cdots,\zeta^{(r)})\Big|_{\xi_i^{(j)}=\xi^{(j)}}, \\ &g'_w(\xi^{(1)},\cdots,\xi^{(r)};\zeta^{(1)},\cdots,\zeta^{(r)}) = :g'(\xi^{(1)},\cdots,\xi^{(r)};\zeta^{(1)},\cdots,\zeta^{(r)}):_{\substack{\xi^{(1)},\cdots,\xi^{(s)} \\ \zeta^{(1)},\cdots,\zeta^{(s)}}}. \end{aligned}$$

The $\Big|_{\xi_i^{(j)}=\xi^{(j)}}$ signifies that $\xi_i^{(j)}$ is to be evaluated at $\xi^{(j)}$ for each $1 \le j \le r$ and $1 \le i \le \ell$. By Lemma II.13, ℓ times,

$$g = \int g'_w(\xi^{(1)}, \cdots, \xi^{(r)}; \xi^{(1)}, \cdots, \xi^{(r)}) \prod_{i=1}^{s} d\mu_C(\xi^{(i)}).$$

Observe that $g = 0$ unless $n^{(j)} = \sum_{i=1}^{\ell} n_i^{(j)}$ for all $1 \le j \le s$ and then g' is of degree

$$\sum_{j=1}^{s} \Big[n^{(j)} + \sum_{i=1}^{\ell} n_i^{(j)} \Big] - 2\ell = \sum_{j=1}^{s} 2n^{(j)} - 2\ell$$

in $\xi^{(1)}, \cdots, \xi^{(s)}$. Hence

$$\begin{aligned} \|g\| &\le \mathsf{b}^{2(n^{(1)}+\cdots+n^{(s)}-\ell)} \, \|g'(\xi^{(1)}, \cdots, \xi^{(r)}; \xi^{(1)}, \cdots, \xi^{(r)})\| \\ &\qquad\qquad \text{by Lemma II.29.ii} \\ &\le \mathsf{b}^{2(n^{(1)}+\cdots+n^{(s)}-\ell)} \, \| \prod_{i=1}^{\ell} \mathrm{Con}_i \prod_{i=1}^{\ell} f_i(\xi_i^{(1)}, \cdots, \xi_i^{(r)}) \; f(\zeta^{(1)}, \cdots, \zeta^{(r)}) \| \\ &\qquad\qquad \text{by Lemma II.22.ii.} \end{aligned}$$

Set, for each $1 \le j \le s$,

$$p_j = \#\big\{ \, i \; \big| \; 1 \le i \le \ell, \; j_i = j \, \big\}.$$

Note that $p_j \le n^{(j)}$ and $\sum_{j=1}^{s} p_j = \ell$. Therefore, by Lemma II.29.i,

$$\begin{aligned} \|g\| &\le \prod_{j=1}^{s} p_j! \tbinom{n^{(j)}}{p_j} \, \mathsf{c}^{\ell} \mathsf{b}^{2(n^{(1)}+\cdots+n^{(s)}-\ell)} \, \|f\| \prod_{i=1}^{\ell} \|f_i\| \\ &= \prod_{j=1}^{s} \Big[p_j! \tbinom{n^{(j)}}{p_j} \tfrac{1}{\alpha^{2n^{(j)}}} \Big] \big(\alpha \mathsf{b}\big)^{2(n^{(1)}+\cdots+n^{(s)})} \, \|f\| \prod_{i=1}^{\ell} \mathsf{b}^{-2} \mathsf{c} \|f_i\| \\ &= \prod_{j=1}^{s} \Big[p_j! \tbinom{n^{(j)}}{p_j} \tfrac{1}{\alpha^{2n^{(j)}}} \Big] \big(\alpha \mathsf{b}\big)^{n^{(1)}+\cdots+n^{(s)}} \, \|f\| \prod_{i=1}^{\ell} \mathsf{b}^{-2} \mathsf{c} \big(\alpha \mathsf{b}\big)^{n_i^{(1)}+\cdots+n_i^{(s)}} \|f_i\|. \end{aligned} \tag{II.4}$$

As g is of degree $\sum_{j=s+1}^{r} \big[n^{(j)} + \sum_{i=1}^{\ell} n_i^{(j)} \big]$ in $\xi^{(s+1)}, \cdots, \xi^{(r)}$, Def. II.23 implies that

$$\begin{aligned} N(g; \alpha) &= \tfrac{1}{\mathsf{b}^2} \, \mathsf{c} \, \big(\alpha \mathsf{b}\big)^{\Sigma_j (n^{(j)} + \Sigma_i n_i^{(j)})} \, \|g\| \\ &\le \prod_{j=1}^{s} \Big[p_j! \tbinom{n^{(j)}}{p_j} \tfrac{1}{\alpha^{2n^{(j)}}} \Big] N(f; \alpha) \prod_{i=1}^{\ell} N(f_i; \alpha). \end{aligned} \tag{II.5}$$

To complete the proof in the homogeneous case, we derive two bounds on $p! \binom{n}{p} \frac{1}{\alpha^{2n}}$ for all $p \le n$. The first is

$$p! \tbinom{n}{p} \tfrac{1}{\alpha^{2n}} \le p! \tfrac{2^n}{\alpha^{2n}} \le p! \tfrac{1}{\alpha^n} \le p! \tfrac{1}{\alpha^p}$$

since $\alpha \ge 2$. It yields the first bound of the proposition, since

$$\prod_{j=1}^{s} \Big[p_j! \frac{1}{\alpha^{p_j}} \Big] \le \big(\Sigma_j p_j\big)! \, \frac{1}{\alpha^{\Sigma_j p_j}} = \ell! \frac{1}{\alpha^{\ell}}.$$

The second is, setting $n = p + m$,

$$p!\binom{p+m}{p}\frac{1}{\alpha^{2(p+m)}} = \frac{1}{\alpha^{2p}}\frac{(p+m)!}{m!}\frac{1}{\alpha^{2m}} \le \frac{1}{\alpha^{2p}}(p+m)^p\frac{1}{\alpha^{2m}} = \frac{p^p}{\alpha^{2p}}\big(1+\tfrac{m}{p}\big)^p\frac{1}{\alpha^{2m}}$$
$$\le \frac{p^p}{\alpha^{2p}}\big(e^{m/p}\big)^p\frac{1}{\alpha^{2m}} = \frac{p^p}{\alpha^{2p}}\big(\tfrac{e}{\alpha^2}\big)^m \le \frac{p^p}{\alpha^{2p}},$$

since $\alpha^2 > e$. It yields the second bound of the proposition, since

$$\prod_{j=1}^{s}\left[\frac{p_j^{p_j}}{\alpha^{2p_j}}\right] \le \big(\Sigma_j p_j\big)^{\Sigma_j p_j}\,\frac{1}{\alpha^{2\Sigma_j p_j}} = \ell^\ell\frac{1}{\alpha^{2\ell}}.$$

The general case now follows by decomposing f and the f_i's into homogeneous pieces. □

III. The Schwinger Functional

III.1. Description of the Schwinger Functional. Let A be a superalgebra and V be a complex vector space with generators $\{\xi_i\}$. Furthermore let C be an antisymmetric bilinear form (covariance) on V. First, suppose that V is finite dimensional. Let $U(\xi) \in \bigwedge_A V$ be an even Grassmann function such that

$$Z(U,C) = \int e^{U(\xi)}\,d\mu_C(\xi)$$

is invertible in A. For any $f(\xi) \in \bigwedge_A V$, set

$$\mathcal{S}(f) = \mathcal{S}_{U,C}(f) = \frac{1}{Z(U,C)}\int f(\xi)\,e^{U(\xi)}\,d\mu_C(\xi).$$

$\mathcal{S}$ is called the Schwinger functional. If V is infinite dimensional, we define $\mathcal{S}$ as a formal power series in U, as in Def. II.27. The coefficient that is of order n in U is a sum of connected graphs that has n vertices U, one vertex f and lines C.

Remark III.1. (i) The renormalization group map can be expressed in terms of the Schwinger functional. Recall that

$$\Omega_C(W)(\psi) = \log\frac{1}{Z}\int e^{W(\psi+\xi)}d\mu_C(\xi), \quad \text{where} \quad Z = \mathcal{Z}\Big(\int e^{W(\xi)}d\mu_C(\xi)\Big),$$

where the ψ_i are the generators of a vector space V', which is a second copy of V. As $\Omega_C(0) = 0$, for even Grassmann functions $W(\psi)$,

$$\begin{aligned}\Omega_C(W)(\psi) &= \int_0^1 \tfrac{d}{dt}\Omega_C(tW)\,dt\\ &= \int_0^1 \frac{\int W(\psi+\xi)\,e^{tW(\psi+\xi)}\,d\mu_C(\xi)}{\int e^{tW(\psi+\xi)}\,d\mu_C(\xi)}\,dt - \log Z\\ &= \int_0^1 \mathcal{S}_{tU,C}(U)\,dt - \log Z,\end{aligned}$$

where in the integral $U(\psi;\xi) = W(\psi+\xi) \in \bigwedge_A(V'\oplus V) \cong \bigwedge_{\bigwedge_A V'} V$ and the Schwinger functional is taken in the Grassmann algebra over V with coefficients in the algebra $\bigwedge_A V'$.

(ii) More generally, if W_1 and W_2 are even Grassmann functions and $W_2 = W_1 + W$, then

$$\begin{aligned}\Omega_C(W_2)(\psi) - \Omega_C(W_1)(\psi) &= \int_0^1 \tfrac{d}{dt}\Omega_C(W_1 + tW)\,dt \\ &= \int_0^1 \mathcal{S}_{U_1+tU,C}(U)\,dt - \log\tfrac{Z_1}{Z_2},\end{aligned}$$

where $U_1(\psi;\xi) = W_1(\psi+\xi)$ and $U(\psi;\xi) = W(\psi+\xi)$.

In [FKT1] we gave a representation for Schwinger functionals which we repeat in the present context. Choose an additional copy V'' of the vector space V. We denote the canonical isomorphism from V to V'' by σ and set $\eta_i = \sigma(\xi_i)$. The antisymmetric bilinear form C'' on V'' corresponding to C is given by $C''(v,w) = C(\sigma^{-1}v, \sigma^{-1}w)$. Using the canonical isomorphisms

$$\bigwedge\nolimits_A(V\oplus V'') = \bigoplus_{r,r'} \bigwedge\nolimits_A^r V \otimes \bigwedge\nolimits_A^{r'} V'' \cong \bigwedge\nolimits_A V \otimes \bigwedge\nolimits_A V'' \cong \bigwedge\nolimits_{\bigwedge_A V} V'',$$

C'' defines the Grassmann Gaussian integral $d\mu_{C''}(\eta)$ as a map from $\bigwedge_A(V\oplus V'')$ to $\bigwedge_A V$. The diagonal embedding $\substack{V\to V\oplus V' \\ v\mapsto v\oplus\sigma(v)}$ induces an embedding $\substack{\bigwedge_A V\to\bigwedge_A(V\oplus V'') \\ f(\xi)\mapsto f(\xi+\eta)}$ and the isomorphism $\substack{V\to V'' \\ v\mapsto\sigma(v)}$ induces an isomorphism $\substack{\bigwedge_A V\to\bigwedge_A V'' \\ f(\xi)\mapsto f(\eta)}$. With this notation we define the map

$$R = R_{U,C} : \bigwedge\nolimits_A V \longrightarrow \bigwedge\nolimits_A V$$

by

$$f \longmapsto \int :e^{U(\xi+\eta)-U(\xi)} - 1:_\eta f(\eta)\,d\mu_C(\eta).$$

Once again, if $\dim V = \infty$, R, is, a priori, defined as a formal power series in U, i.e. as a sequence of multilinear maps. In this case, it is easy to explicitly find maps. The n^{th} map is

$$(U_1, U_2, \cdots, U_n, f) \mapsto \tfrac{1}{n!}\int :\prod_{i=1}^n [U_i(\xi+\eta) - U_i(\xi)]:_\eta f(\eta)\,d\mu_C(\eta).$$

For all Grassmann functions As in [FKT1] we have

Theorem III.2. *$f \in \bigwedge_A V$,*

$$\mathcal{S}_{U,C}(f) = \int \big(\mathbb{1} - R_{U,C}\big)^{-1}(f)\,d\mu_C.$$

Proof. We first prove

$$\begin{aligned}\int f(\xi)\,e^{U(\xi)}\,d\mu_C(\xi) &= \int f(\eta)\,d\mu_C(\eta)\int e^{U(\xi)}\,d\mu_C(\xi) \\ &\quad + \int R_{U,C}(f)(\xi)\,e^{U(\xi)}\,d\mu_C(\xi). \qquad \text{(III.1)}\end{aligned}$$

Inserting the definition of $R_{U,C}(f)$ into the right-hand side,

$$\begin{aligned}\int f(\eta)\,d\mu_C(\eta)\int e^{U(\xi)}\,d\mu_C(\xi)&+\int R_{U,C}(f)(\xi)\,e^{U(\xi)}\,d\mu_C(\xi)\\&=\int\Big[\int :e^{U(\xi+\eta)-U(\xi)}:_\eta f(\eta)\,d\mu_C(\eta)\Big]e^{U(\xi)}\,d\mu_C(\xi)\\&=\iint :e^{U(\xi+\eta)}:_\eta f(\eta)\;d\mu_C(\eta)\;d\mu_C(\xi),\end{aligned}$$

since $:e^{U(\xi+\eta)-U(\xi)}:_\eta = :e^{U(\xi+\eta)}:_\eta e^{-U(\xi)}$. Continuing,

$$\begin{aligned}\int f(\eta)\,d\mu_C(\eta)\int e^{U(\xi)}\,d\mu_C(\xi)\;&+\;\int R_{U,C}(f)(\xi)\,e^{U(\xi)}\,d\mu_C(\xi)\\&=\iint :e^{U(\xi+\eta)}:_\xi f(\eta)\;d\mu_C(\eta)\;d\mu_C(\xi)\text{ by Prop. A.2.ii}\\&=\int f(\eta)\,e^{U(\eta)}\;d\mu_C(\eta)\text{ by Prop. A.2.i}\\&=\int f(\xi)\,e^{U(\xi)}\;d\mu_C(\xi).\end{aligned}$$

This completes the proof of (III.1). Now we prove the theorem itself. For all $g(\xi)\in\bigwedge_A V$,

$$\int(\mathbb{1}-R_{U,C})(g)\,e^{U(\xi)}\;d\mu_C(\xi)\;=\;Z(U,C)\int g(\xi)\;d\mu_C(\xi)$$

by (III.1). Since $R_{0,C}=0$, the map $\mathbb{1}-R_{U,C}$ trivially has a formal power series inverse and we may choose $g=(\mathbb{1}-R_{U,C})^{-1}(f)$. So

$$\int f(\xi)\,e^{U(\xi)}\;d\mu_C(\xi)\;=\;Z(U,C)\int(\mathbb{1}-R_{U,C})^{-1}(f)(\xi)\;d\mu_C(\xi).$$

The left-hand side does not vanish for all $f\in\bigwedge_A V$ (for example, for $f=e^{-U}$) so $Z(U,C)$ is nonzero and

$$\tfrac{1}{Z(U,C)}\int f(\xi)\,e^{U(\xi)}\;d\mu_C(\xi)\;=\;\int(\mathbb{1}-R_{U,C})^{-1}(f)(\xi)\;d\mu_C(\xi).$$

□

This construction is functorial in the following sense:

Remark III.3. Let $\pi_A:\tilde{A}\to A$ be a homomorphism of superalgebras, and $\pi_V:\tilde{V}\to V$ a linear map of complex vector spaces. Define the antisymmetric bilinear $\tilde{C}$ form on $\tilde{V}$ by $\tilde{C}(v,w)=C(\pi_V(v),\pi_V(w))$. π_A and π_V induce an algebra homomorphism $\pi_*:\bigwedge_{\tilde{A}}\tilde{V}\to\bigwedge_A V$. Let $\tilde{U}\in\bigwedge_{\tilde{A}}\tilde{V}$ with $\pi_*(\tilde{U})=U$. Then for all even $\tilde{f}\in\bigwedge_{\tilde{A}}\tilde{V}$,

$$\pi_A\big(\mathcal{S}_{\tilde{U},\tilde{C}}(\tilde{f})\big)=\mathcal{S}_{U,C}(\pi_* f)$$

and

$$\pi_*\,R_{\tilde{U},\tilde{C}}(\tilde{f})=R_{U,C}(\pi_*\tilde{f}).$$

In our context the Grassmann functions will all be Wick ordered with respect to the covariance C. We give a description of the map R of Theorem III.3 adapted to Wick ordering. We use a further copy of the vector space V with generators $\{\xi_i'\}$ corresponding to the generators $\{\xi_i\}$ of V.

Definition III.4. *For any Grassmann function $K(\xi,\xi',\eta)$ define the map $\mathcal{R}_{K,C} : \bigwedge_A V \to \bigwedge_A V$ by*

$$\mathcal{R}_{K,C}(f) = \mathopen{:}\int\int \mathopen{:}e^{:K(\xi,\xi',\eta):_{\xi'}} - 1\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\xi')\, d\mu_C(\eta)\mathclose{:}_\xi .$$

Yet again, when $\dim V = \infty$, *$\mathcal{R}_{K,C}$ is a formal power series in K.*

Proposition III.5. *Assume that $U(\xi) = :\hat{U}(\xi):$. Set $K(\xi,\xi',\eta) = \hat{U}(\xi+\xi'+\eta) - \hat{U}(\xi+\xi')$. Then*

$$R_{U,C} = \mathcal{R}_{K,C}.$$

Proof. By part (ii) of Prop. A.2,

$$U(\xi+\eta) - U(\xi) = :\hat{U}(\xi+\eta) - \hat{U}(\xi):_\xi .$$

Hence by part (iv) of Prop. A.2,

$$e^{U(\xi+\eta)-U(\xi)} = \mathopen{:}\int e^{:\hat{U}(\xi+\xi'+\eta)-\hat{U}(\xi+\xi'):_{\xi'}} d\mu_C(\xi')\mathclose{:}_\xi .$$

Consequently

$$\begin{aligned} R_{U,C}(f) &= \int \mathopen{:}e^{U(\xi+\eta)-U(\xi)} - 1\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\eta) \\ &= \int \mathopen{:}\left(\mathopen{:}\int e^{:\hat{U}(\xi+\xi'+\eta)-\hat{U}(\xi+\xi'):_{\xi'}} d\mu_C(\xi')\mathclose{:}_\xi - 1\right)\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\eta) \\ &= \int \mathopen{:}\mathopen{:}\int \left(e^{:K(\xi,\xi',\eta):_{\xi'}} - 1\right) d\mu_C(\xi')\mathclose{:}_\xi\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\eta) \\ &= \mathopen{:}\int\int \mathopen{:}e^{:K(\xi,\xi',\eta):_{\xi'}} - 1\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\xi')\, d\mu_C(\eta)\mathclose{:}_\xi . \end{aligned}$$

□

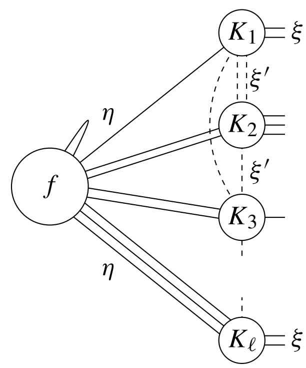
To perform estimates we expand the exponential on the right-hand side of Def. III.4. For even Grassmann functions $K_1(\xi,\xi',\eta),\cdots,K_\ell(\xi,\xi',\eta)$ define the map

$$R_C(K_1,\cdots,K_\ell) : \bigwedge_A V \longrightarrow \bigwedge_A V$$

by

$$f \longmapsto \mathopen{:}\int\int \mathopen{:}\Big(\prod_{i=1}^{\ell} :K_i(\xi,\xi',\eta):_{\xi'}\Big)\mathclose{:}_\eta\, f(\eta)\, d\mu_C(\xi')\, d\mu_C(\eta)\mathclose{:}_\xi . \qquad \text{(III.2)}$$

Observe that $R_C(K_1,\cdots,K_\ell)$ is multilinear and symmetric in $K_1,\cdots,K_\ell$.



Expanding the exponential gives

Remark III.6. For any even Grassmann function $K(\xi, \xi', \eta)$,

$$\mathcal{R}_{K,C} = \sum_{\ell=1}^{\infty} \tfrac{1}{\ell!} R_C(K, \cdots, K).$$

That is, the ℓ^{th} order term in the formal Taylor expansion of $\mathcal{R}_{K,C}$ is the ℓ–linear map $\frac{1}{\ell!} R_C(K_1, \cdots, K_\ell)$.

III.2. Norm Estimates on the Schwinger Functional. Again we assume that A is a graded superalgebra and that we are given a family of symmetric seminorms on the spaces $A_m \otimes V^{\otimes n}$. Assume that $\mathfrak{c}$ is a contraction bound and b an integral bound for the covariance C. (See Def. II.25.) Fix $\alpha > 1$. We write $N(f)$ for the $N(f; \alpha)$ of Def. II.23.

Proposition III.7. *Let $K^{(1)}(\xi, \xi', \eta), \cdots, K^{(\ell)}(\xi, \xi', \eta)$ be even Grassmann functions that obey $K^{(i)}(\xi, \xi', 0) = 0$. Furthermore let $f(\xi)$ be a Grassmann function and $\ell \geq 1$. Set*

$$\tfrac{1}{\ell!} R_C(K^{(1)}, \cdots, K^{(\ell)})(:f:)(\xi) = :f'(\xi):.$$

(i) Assume that $f \in A_m[n]$ for some index m and some $n \geq 0$. Then $f' = 0$ if $\ell > n$, and

$$N(f') \leq \tfrac{1}{\alpha^{2n}} \tbinom{n}{\ell} N(f) \prod_{i=1}^{\ell} N(K^{(i)}).$$

(ii) In general, if $\alpha \geq 2$, then

$$N(f') \leq \tfrac{1}{\alpha^{\ell}} N(f) \prod_{i=1}^{\ell} N(K^{(i)}).$$

Proof. Set

$$G_1(\xi; \xi'^{(1)}, \cdots, \xi'^{(\ell)}; \eta) = :\prod_{i=1}^{\ell} K^{(i)}(\xi, \xi'^{(i)}, \eta):_\eta :f(\eta):_\eta,$$

$$G_2(\xi; \xi'^{(1)}, \cdots, \xi'^{(\ell)}) = \int G_1(\xi; \xi'^{(1)}, \cdots, \xi'^{(\ell)}; \eta)\, d\mu_C(\eta),$$

$$G_3(\xi; \xi'^{(1)}, \cdots, \xi'^{(\ell)}) = :G_2(\xi; \xi'^{(1)}, \cdots, \xi'^{(\ell)}):_{\xi'^{(1)}, \cdots, \xi'^{(\ell)}}.$$

Then

$$f'(\xi) = \tfrac{1}{\ell!} \int G_3(\xi; \xi', \cdots, \xi')\, d\mu_C(\xi').$$

By Lemma II.31,

$$N(f') \le \tfrac{1}{\ell!} N(G_2).$$

By Prop. II.33, with $s = 1$ and $r = \ell + 2$,

$$N(G_2) \le \tfrac{\ell!}{\alpha^\ell}\, N(f) \prod_{i=1}^{\ell} N(K^{(i)}).$$

This proves part ii. Part i follows from (II.5) with $s = 1$, $p_1 = \ell$ and $n^{(1)} = n$. $\square$

Lemma III.8. *Let $f(\xi)$ be a Grassmann function over A. The formal Taylor series $\mathcal{R}_{K,C}(:f:)$ converges to an analytic map on $\big\{ K(\xi, \xi', \eta) \,\big|\, K$ even, $K(\xi, \xi', 0) = 0,\ N(K)_{\mathbf{0}} < \frac{\alpha^2}{2} \big\}$. Furthermore, if $K(\xi, \xi', \eta)$ is an even Grassmann function over A with $K(\xi, \xi', 0) = 0$ and $N(K)_{\mathbf{0}} < \frac{\alpha^2}{2}$ and if $:f': \ = \ \mathcal{R}_{K,C}(:f:)$ then*

$$N(f') \le \ \tfrac{2}{\alpha^2}\, \tfrac{N(K)}{1-\frac{2}{\alpha^2}N(K)}\, N(f).$$

Proof. Write

$$f(\xi) = \sum_{m,n\ge 0} f_{m;n}(\xi)$$

with $f_{m;n} \in A_m[n]$ and set

$$:f'_{m;n}: \ = \sum_{\ell=1}^{\infty} \tfrac{1}{\ell!}\, R_C(K, \cdots, K)(:f_{m;n}:).$$

By part (i) of Prop. III.7, $\frac{1}{\ell!} R_C(K, \cdots, K)(:f_{m;n}:) = 0$ for $\ell > n$ and

$$\begin{aligned} N\big(f'_{m;n}\big) \ &\le \tfrac{1}{\alpha^{2n}}\, N(f_{m;n}) \sum_{\ell=1}^{n} \tbinom{n}{\ell} N(K)^\ell \\ &\le N(f_{m;n}) \sum_{\ell=1}^{n} \tfrac{2^\ell}{\alpha^{2\ell}}\, N(K)^\ell \\ &\le \tfrac{2}{\alpha^2}\, N(f_{m;n})\, \tfrac{N(K)}{1-\frac{2}{\alpha^2}N(K)}. \end{aligned}$$

Consequently

$$\begin{aligned} N(f') &\le \sum_{m\ge 0,\ n\ge 1} N\big(f'_{m;n}\big) \\ &\le \tfrac{2}{\alpha^2}\, \tfrac{N(K)}{1-\frac{2}{\alpha^2}N(K)} \sum_{m\ge 0,\ n\ge 1} N(f_{m;n}) \\ &\le \tfrac{2}{\alpha^2}\, \tfrac{N(K)}{1-\frac{2}{\alpha^2}N(K)}\, N(f). \end{aligned}$$

This also proves that the formal Taylor series expansion of $\mathcal{R}_{K,C}$ converges to an analytic function. $\square$

Corollary III.9. *Let $f(\xi)$ and $K(\xi,\xi',\eta)$ be Grassmann functions over A with K even and $K(\xi,\xi',0)=0$. Assume that $N(K)_{\mathbf{0}} < \frac{\alpha^2}{4}$. If*

$$:f': \;=\; \frac{1}{\mathbb{1}-\mathcal{R}_{K,C}}(:f:) - :f:,$$

then

$$N(f') \le \tfrac{2}{\alpha^2}\,\tfrac{N(K)}{1-\frac{4}{\alpha^2}N(K)}\,N(f).$$

Proof. Set $\beta = \frac{2}{\alpha^2}\,\frac{N(K)}{1-\frac{2}{\alpha^2}N(K)}$. Observe that $\beta_{\mathbf{0}} = \frac{2}{\alpha^2}\,\frac{N(K)_{\mathbf{0}}}{1-\frac{2}{\alpha^2}N(K)_{\mathbf{0}}} < \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. For $\ell \ge 0$ set

$$\mathcal{R}^{\ell}_{K,C}(:f:) \;=\; :f'_\ell:.$$

By Lemma III.8 above

$$N(f'_\ell) \le \beta^\ell\, N(f).$$

Since $f' = \sum_{\ell=1}^\infty f'_\ell$,

$$N(f') \;\le\; \sum_{\ell=1}^\infty N(f'_\ell) \;\le\; N(f)\sum_{\ell=1}^\infty \beta^\ell \;=\; \tfrac{\beta}{1-\beta}\,N(f) \;=\; \tfrac{2}{\alpha^2}\,\tfrac{N(K)}{1-\frac{4}{\alpha^2}N(K)}\,N(f).$$

□

Proposition III.10. *Let $f(\xi)\in\bigwedge_A V$. The formal Taylor series $\mathcal{S}_{:U:,C}(:f:)$ converges to an analytic map on $\left\{\, U(\xi)\in\bigwedge_A V \;\middle|\; U \text{ even},\; N(U;4\alpha)_{\mathbf{0}} < \frac{\alpha^2}{4}\,\right\}$. Furthermore, if $U(\xi)$ is an even Grassmann function with coefficients in A and $N(U;4\alpha)_{\mathbf{0}} < \frac{\alpha^2}{4}$, then*

$$N\Big(\mathcal{S}_{:U:,C}(:f:) - f(0)\,;\alpha\Big) \;\le\; \tfrac{2}{\alpha^2}\,\tfrac{N(U;4\alpha)}{1-\frac{4}{\alpha^2}N(U;4\alpha)}\,N(f).$$

Proof. Set

$$K(\xi,\xi',\eta) = U(\xi+\xi'+\eta) - U(\xi+\xi').$$

By Remark II.24,

$$N(K;\alpha) \le N\big(U(\xi+\xi'+\eta);\alpha\big) \le N(U;4\alpha).$$

By Prop. III.5, $R_{:U:,C} = \mathcal{R}_{K,C}$. Therefore, by Theorem III.3 and part (i) of Prop. A.2,

$$\mathcal{S}_{:U:,C}(:f:) - f(0) = \int \Big(\tfrac{1}{\mathbb{1}-\mathcal{R}_{K,C}}(:f:)(\xi) - :f(\xi):\Big)d\mu_C(\xi).$$

Consequently, by Lemma II.31 and Cor. III.9

$$N\big(\mathcal{S}_{:U:,C}(:f:) - f(0)\big) \;\le\; \tfrac{2}{\alpha^2}\,\tfrac{N(K)}{1-\frac{4}{\alpha^2}N(K)}\,N(f) \;\le\; \tfrac{2}{\alpha^2}\,\tfrac{N(U;4\alpha)}{1-\frac{4}{\alpha^2}N(U;4\alpha)}\,N(f).$$

□

III.3. Proof of Theorem II.28. Recall that the renormalization group map Ω_C is defined by

$$\Omega_C(:W:)(\psi) = \log \tfrac{1}{Z} \int e^{:W:(\psi+\xi)} d\mu_C(\xi), \quad \text{where} \quad Z = \mathcal{Z}\Big(\int e^{:W:(\xi)} d\mu_C(\xi)\Big).$$

Let $A' = \bigwedge_A V'$ be the Grassmann algebra in the variables ψ with coefficients in A. As $W(\psi+\xi) \in \bigwedge_A (V' \oplus V) \cong \bigwedge_{A'} V$ and $\int \cdot\, d\mu_C(\xi)$ maps $\bigwedge_{A'} V$ to A', Ω_C is map from (a subset of) $\bigwedge_A V$ to $A' = \bigwedge_A V' \cong \bigwedge_A V$ (since V' is a copy of V).

Set

$$U(\psi;\xi) = W(\psi+\xi) \in \bigwedge\nolimits_{A'} V.$$

Then $U(\psi, 0) = W(\psi)$. By part (ii) of Prop. A.2,

$$:U(\psi,\xi):_\xi \;=\; :W:(\psi+\xi),$$

and by Remark III.1.i,

$$\Omega_C(:W:)(\psi) - W(\psi) = \int_0^1 \big(\mathcal{S}_{:tU:,C}(:U:) - U(\psi;0)\big)\, dt - \log Z.$$

We now wish to apply Prop. III.10, with (U, f, A) replaced by (tU, U, A'), to bound $\Omega_C(:W:)(\psi) - W$. We have been given, in the statement of Theorem II.28, a system $\|\cdot\|$ of symmetric seminorms on the spaces $A_m \otimes V^{\otimes n}$. Any $f \in A'_m \otimes V^{\otimes n}$ may be uniquely expressed as

$$f = \sum_{m'+m''=m} f_{m',m''}$$

with $f_{m',m''} \in A_{m'} \otimes \bigwedge^{m''} V \otimes V^{\otimes n}$. We define

$$\|f\|' = \sum_{m'+m''=m} \alpha^{m''} \mathrm{b}^{m''}\, \|f_{m',m''}\|.$$

Then $\|\cdot\|'$ is a system of symmetric seminorms on the spaces $A'_m \otimes V^{\otimes n}$ and the covariance C has contraction bound $\mathfrak{c}$ and integral bound b with respect to these norms. For $f \in \bigwedge_{A'} V \cong \bigwedge_A (V' \oplus V)$ and $\alpha > 1$, let $N(f;\alpha)$ be the quantity of Def. II.23, considering f as an element of $\bigwedge_A(V' \oplus V)$ and using the seminorms $\|\cdot\|$; and let $N'(f;\alpha)$ be defined viewing f as an element of $\bigwedge_{A'} V$ and using the seminorms $\|\cdot\|'$. Then $N'(f;\alpha) = N(f;\alpha)$, while for $\alpha' > \alpha$, $N'(f;\alpha') \le N(f;\alpha')$.

By Remark II.24

$$N\big(U;\alpha\big) \le N\big(W;2\alpha\big).$$

Prop. III.10 now implies that

$$\begin{aligned}
N\big(\Omega_C(:W:)(\psi) - W\,;\alpha\big) &= N'\big(\Omega_C(:W:)(\psi) - W\,;\alpha\big) \\
&\le \sup_{0\le t\le 1} N'\big(\mathcal{S}_{:tU:,C}(:U:) - U(0)\big) \\
&\le \tfrac{2}{\alpha^2}\, \frac{N'(U;4\alpha)}{1-\frac{4}{\alpha^2}N'(U;4\alpha)}\, N'(U;\alpha) \le \tfrac{2}{\alpha^2}\, \frac{N(U;4\alpha)}{1-\frac{4}{\alpha^2}N(U;4\alpha)}\, N(U;\alpha) \\
&\le \tfrac{2}{\alpha^2}\, \frac{N(W;8\alpha)}{1-\frac{4}{\alpha^2}N(W;8\alpha)}\, N(W;2\alpha) \le \tfrac{2}{\alpha^2}\, \frac{N(W;8\alpha)^2}{1-\frac{4}{\alpha^2}N(W;8\alpha)}.
\end{aligned}$$

In the last step we used Remark II.24, part (ii). The hypotheses of Prop. III.10 are fulfilled, since, by hypothesis

$$N\big(U;4\alpha\big)_{\mathbf{0}} \le\ N\big(W;8\alpha\big)_{\mathbf{0}} < \tfrac{\alpha^2}{4}.$$

□

Remark III.11. Suppose that $\|\ \cdot\ \|_{\mathfrak{bc}}$ is a norm on the space of antisymmetric bilinear forms on V and that there is a $\kappa > 0$ such that every C with $\|C\|_{\mathfrak{bc}} < \kappa$ has integral bound b and contraction bound $\mathfrak{c}_{\mathbf{0}} + \sum_{\delta\neq 0}\infty t^\delta$. Then $\Omega_C(:W:)$ is jointly analytic in C and W on $\big\{\ (W,C)\ \big|\ W \text{ even},\ N\big(W;8\alpha\big)_{\mathbf{0}} < \frac{\alpha^2}{4},\ \|C\|_{\mathfrak{bc}} < \kappa\ \big\}$.

IV. More Estimates on the Renormalization Group Map

In the situation of [FKTf1, FKTf2, FKTf3], the effective interaction is Wick ordered both with respect to the covariance that is integrated out at the renormalization group step and a covariance that is approximately the sum of the covariances for all future renormalization group steps. In this section we modify the construction of the previous two sections to accommodate the second "output" Wick ordering. Furthermore, we estimate the derivative of $\Omega_C\big(:W:_{\psi,C+D}\big)$ with respect to the effective interaction W and the covariances C and D.

Let again V be a complex vector space with generators $\{\xi_i\}$, let A be a graded superalgebra, and let $\|\cdot\|$ be a system of symmetric seminorms. Furthermore let C and D be two covariances on V.

Theorem IV.1. *Let $W(\psi)$ be an even Grassmann function with coefficients in A. Let*

$$:W'(\psi):_{\psi,D}\ =\ \Omega_C(:W:_{\psi,C+D}).$$

Let $\alpha \ge 1$ and assume that $\mathfrak{c}$ is a contraction bound for the covariance C and b *is an integral bound for C and for D. If $N(W;32\alpha)_{\mathbf{0}} < \alpha^2$, then*

$$N\big(W'-W;\alpha\big) \le \tfrac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}.$$

Remark IV.2. By Remark II.4.ii, $\mathcal{Z}\big(:W'(\psi):_{\psi,D}\big) = 0$. In general $\mathcal{Z}\big(W'\big) \neq 0$. If one defines

$$\begin{aligned}\Omega_{C,D}(W)(\psi) &= \log \tfrac{1}{Z_{C,D}} \int e^{W(\psi+\xi)} d\mu_C(\xi) \text{ where } Z_{C,D}\\ &= \mathcal{Z}\Big(\iint e^{W(\psi+\xi)} d\mu_C(\xi) d\mu_D(\psi)\Big),\end{aligned}$$

then $:W''(\psi):_{\psi,D}\ =\ \Omega_{C,D}(:W:_{\psi,C+D})$ obeys the normalization condition $\mathcal{Z}(W'') = 0$. Furthermore, W' and W'' differ only by a constant, so that

$$N\big(W''-W;\alpha\big) \le \tfrac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}.$$

Proof of Theorem IV.1. Define U and U' by

$$U(\psi) = {:}W(\psi){:}_{\psi,D}, \qquad U'(\psi) = {:}W'(\psi){:}_{\psi,D}.$$

Then, by Lemma A.4.i,

$$\begin{aligned} &{:}U{:}_{\psi,C} \;=\; {:}W{:}_{\psi,C+D}, \quad U' \;=\; \Omega_C({:}W{:}_{\psi,C+D}) = \;\Omega_C({:}U{:}_{\psi,C}),\\ &U' - U = {:}W' - W{:}_{\psi,D} \end{aligned}$$

By Cor. II.32

$$N\big(U;16\alpha\big)_{\mathbf{0}} = N\big({:}W{:}_{\psi,D};16\alpha\big)_{\mathbf{0}} \le N\big(W;32\alpha\big)_{\mathbf{0}} < \alpha^2,$$

so that by Cor. II.32, followed by Theorem II.28 (with $W = U$ and 2α replacing α)

$$N\big(W' - W;\alpha\big) \le N\big(U' - U;2\alpha\big) \le \tfrac{1}{2\alpha^2}\,\frac{N(U;16\alpha)^2}{1-\frac{1}{\alpha^2}N(U;16\alpha)} \le \tfrac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}$$

□

Remark IV.3. Suppose that $\|\cdot\|_{\mathrm{b}}$ and $\|\cdot\|_{\mathrm{b}\mathfrak{c}}$ are norms on the space of antisymmetric bilinear forms on V and that there are $\kappa,\kappa' > 0$ such that every C with $\|C\|_{\mathrm{b}\mathfrak{c}} < \kappa$ has integral bound b and contraction bound $\mathfrak{c}_{\mathbf{0}} + \sum_{\delta\neq 0}\infty t^\delta$ and every D with $\|D\|_{\mathrm{b}} < \kappa'$ has integral bound b. Then W' is jointly analytic in C, D and W on

$$\Big\{\, (W,C,D) \;\Big|\; W \text{ even},\; N\big(W;32\alpha\big)_{\mathbf{0}} < \alpha^2,\; \|C\|_{\mathrm{b}\mathfrak{c}} < \kappa,\; \|D\|_{\mathrm{b}} < \kappa' \,\Big\}.$$

The derivatives of $\Omega_C({:}W{:}_{\psi,C+D})$ with respect to W, C and D are bounded in the following theorem, which is an amalgam of Lemmas IV.5, IV.7 and IV.8 below.

Theorem IV.4. *Let, for κ in a neighbourhood of* 0*, $W_\kappa(\psi)$ be an even Grassmann function and C_κ, D_κ be covariances on V. Assume that $\alpha \ge 1$ and*

$$N(W_0;32\alpha)_{\mathbf{0}} < \alpha^2,$$

and that

C_0 has contraction bound $\mathfrak{c}$, $\quad\frac{1}{2}$b is an integral bound for C_0, D_0,
$\frac{d}{d\kappa}C_\kappa\big|_{\kappa=0}$ has contraction bound $\mathfrak{c}'$, $\quad\frac{1}{2}$b$'$ is an integral bound for $\frac{d}{d\kappa}D_\kappa\big|_{\kappa=0}$,

and that $\mathfrak{c} \le \frac{1}{\mu}\mathfrak{c}^2$. Set

$${:}\tilde{W}_\kappa(\psi){:}_{\psi,D_\kappa} \;=\; \Omega_{C_\kappa}({:}W_\kappa{:}_{\psi,C_\kappa+D_\kappa}).$$

Then

$$\begin{aligned} N\Big(\tfrac{d}{d\kappa}[\tilde{W}_\kappa - W_\kappa]_{\kappa=0};\alpha\Big) \le{}& \tfrac{1}{2\alpha^2}\,\frac{N(W_0;32\alpha)}{1-\frac{1}{\alpha^2}N(W_0;32\alpha)}\,N\Big(\tfrac{d}{d\kappa}W_\kappa\big|_{\kappa=0};8\alpha\Big)\\ &+\tfrac{1}{2\alpha^2}\,\frac{N(W_0;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W_0;32\alpha)}\Big\{\tfrac{1}{4\mu}\mathfrak{c}' + \big(\tfrac{\mathrm{b}'}{\mathrm{b}}\big)^2\Big\}. \end{aligned}$$

Lemma IV.5. *Let C be a covariance on V with contraction bound $\mathfrak{c}$ and integral bound* b. *Let, for κ in a neighbourhood of* 0*, $W_\kappa(\psi) \in \bigwedge_A V'$ be an even Grassmann function.*

i) Set

$$\tilde{W}_\kappa(\psi) \;=\; \Omega_C(:W_\kappa:_{\psi,C}).$$

If $N\big(W_0; 8\alpha\big)_{\mathbf{0}} < \frac{\alpha^2}{4}$ *, then*

$$N\Big(\tfrac{d}{d\kappa}[\tilde{W}_\kappa - W_\kappa]_{\kappa=0}\,;\alpha\Big) \le \tfrac{2}{\alpha^2}\,\frac{N(W_0;8\alpha)}{1-\frac{4}{\alpha^2}N(W_0;8\alpha)}N\Big(\tfrac{d}{d\kappa}W_\kappa\big|_{\kappa=0}\,;2\alpha\Big).$$

ii) Let D be a covariance on V with integral bound b. *Set*

$$:\tilde{W}_\kappa(\psi):_{\psi,D} \;=\; \Omega_C(:W_\kappa:_{\psi,C+D}).$$

If $N\big(W_0; 32\alpha\big)_{\mathbf{0}} < \alpha^2$ *, then*

$$N\Big(\tfrac{d}{d\kappa}[\tilde{W}_\kappa - W_\kappa]_{\kappa=0}\,;\alpha\Big) \le \tfrac{1}{2\alpha^2}\,\frac{N(W_0;32\alpha)}{1-\frac{1}{\alpha^2}N(W_0;32\alpha)}N\Big(\tfrac{d}{d\kappa}W_\kappa\big|_{\kappa=0}\,;8\alpha\Big).$$

Proof. Set

$$U_\kappa(\psi,\xi) = W_\kappa(\psi+\xi), \quad U'_\kappa(\psi,\xi) = \tfrac{d}{d\kappa}W_\kappa(\psi+\xi).$$

By Prop. A.2.ii,

$$:U_\kappa(\psi,\xi):_\xi \;=\; :W_\kappa:(\psi+\xi), \quad :U'_\kappa(\psi,\xi):_\xi \;=\; :\tfrac{d}{d\kappa}W_\kappa:(\psi+\xi).$$

By Remark II.24,

$$N\big(U_\kappa;\alpha\big) \le N\big(W_\kappa;2\alpha\big), \quad N\big(U'_\kappa;\alpha\big) \le N\big(\tfrac{d}{d\kappa}W_\kappa;2\alpha\big).$$

Differentiating Def. II.3,

$$\tfrac{d}{d\kappa}\Omega_C\big(:W_\kappa:\big) = \frac{\int :\frac{d}{d\kappa}W_\kappa:(\psi+\xi)\; e^{:W_\kappa:(\psi+\xi)}d\mu_C(\xi)}{\int e^{:W_\kappa:(\psi+\xi)}d\mu_C(\xi)} = \mathcal{S}_{:U_\kappa:,C}(:U'_\kappa:) \quad \mathrm{mod}\ A_0$$

so that

$$\tfrac{d}{d\kappa}\big[\Omega_C\big(:W_\kappa:\big) - W_\kappa\big] = \mathcal{S}_{:U_\kappa:,C}(:U'_\kappa:) - U'_\kappa(\psi,0) \quad \mathrm{mod}\ A_0.$$

Define the system of symmetric seminorms $\|\cdot\|'$ and the norm $N'\big(f;\alpha\big)$ as in the proof of Theorem II.28. Prop. III.10 now implies that

$$\begin{aligned}
N\big(\tfrac{d}{d\kappa}[\tilde{W}_\kappa - W_\kappa];\alpha\big) &= N'\big(\tfrac{d}{d\kappa}[\tilde{W}_\kappa - W_\kappa];\alpha\big)\\
&= N'\big(\mathcal{S}_{:U_\kappa:,C}(:U'_\kappa:) - U'_\kappa(\psi,0)\,;\alpha\big)\\
&\le \tfrac{2}{\alpha^2}\,\frac{N'(U_\kappa;4\alpha)}{1-\frac{4}{\alpha^2}N'(U_\kappa;4\alpha)}\,N'(U'_\kappa;\alpha)\\
&\le \tfrac{2}{\alpha^2}\,\frac{N(U_\kappa;4\alpha)}{1-\frac{4}{\alpha^2}N(U_\kappa;4\alpha)}\,N(U'_\kappa;\alpha)\\
&\le \tfrac{2}{\alpha^2}\,\frac{N(W_\kappa;8\alpha)}{1-\frac{4}{\alpha^2}N(W_\kappa;8\alpha)}\,N\big(\tfrac{d}{d\kappa}W_\kappa;2\alpha\big).
\end{aligned}$$

The hypotheses of Prop. III.10 are fulfilled, at $\kappa = 0$, since, by hypothesis

$$N\big(U_0;4\alpha\big)_{\mathbf{0}} \;\le\; N\big(W_0;8\alpha\big)_{\mathbf{0}} < \tfrac{\alpha^2}{4}.$$

ii) Part (ii) follows from part (i) as Theorem IV.1 follows from Theorem II.28. □

Lemma IV.6. *Let $\mathfrak{c}$ be a contraction bound for C and $\mathfrak{c}'$ be a contraction bound for C'. If $\mathfrak{c} \le \frac{1}{\mu}\mathfrak{c}^2$, then*

$$N\Big(\sum_{i,j} \tfrac{\partial f}{\partial \xi_i}\, C'_{ij}\, \tfrac{\partial g}{\partial \xi_j};\alpha\Big) \le \tfrac{1}{\mu\alpha^2}\mathfrak{c}'\, N(f;2\alpha)\, N(g;2\alpha).$$

Proof. Write

$$f(\xi) = \sum_{m,n\ge 0} f_{m;n}(\xi) \qquad g(\xi) = \sum_{m',n'\ge 0} g_{m';n'}(\xi)$$

with $f_{m;n} \in A_m[n]$ and $g_{m';n'} \in A_{m'}[n']$. Then, by Lemma II.10 and Def. II.9,

$$\begin{aligned}\sum_{i,j} \tfrac{\partial f_{m;n}}{\partial \xi_i}\, C'_{ij}\, \tfrac{\partial g_{m';n'}}{\partial \xi_j} &= -n \underset{\xi\to\zeta}{\mathrm{Con}_{C'}} \big(f_{m;n}(\xi) g_{m';n'}(\zeta)\big)\Big|_{\zeta=\xi}\\ &= -nn' \underset{1\to n+1}{\mathcal{C}\mathrm{on}_{C'}} \big(f_{m;n}\otimes g_{m';n'}\big)\Big|_{\zeta=\xi}\end{aligned}$$

so that, by Lemma II.22.ii and Def. II.25,

$$\begin{aligned}\Big\| \sum_{i,j} \tfrac{\partial f_{m;n}}{\partial \xi_i}\, C'_{ij}\, \tfrac{\partial g_{m';n'}}{\partial \xi_j}\Big\| &\le nn' \Big\| \underset{1\to n+1}{\mathcal{C}\mathrm{on}_{C'}} \big(f_{m;n}\otimes g_{m';n'}\big)\Big\|\\ &\le nn'\mathfrak{c}'\, \|f_{m;n}\|\, \|g_{m';n'}\|.\end{aligned}$$

Hence, by Def. II.23,

$$\begin{aligned}N\Big(\sum_{i,j} \tfrac{\partial f}{\partial \xi_i}\, C'_{ij}\, \tfrac{\partial g}{\partial \xi_j};\alpha\Big) &\le \tfrac{1}{\mathrm{b}^2}\mathfrak{c} \sum_{m,m',n,n'\ge 0} (\alpha\mathrm{b})^{n+n'-2} \Big\| \sum_{i,j} \tfrac{\partial f_{m;n}}{\partial \xi_i}\, C'_{ij}\, \tfrac{\partial g_{m';n'}}{\partial \xi_j}\Big\|\\ &\le \tfrac{1}{\mu\alpha^2}\mathfrak{c}'\Big[\tfrac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n,\ge 0} n(\alpha\mathrm{b})^n\|f_{m;n}\|\Big]\Big[\tfrac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m',n',\ge 0} n'(\alpha\mathrm{b})^{n'}\|g_{m';n'}\|\Big]\\ &\le \tfrac{1}{\mu\alpha^2}\mathfrak{c}'\Big[\tfrac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m,n,\ge 0} (2\alpha\mathrm{b})^n\|f_{m;n}\|\Big]\Big[\tfrac{1}{\mathrm{b}^2}\mathfrak{c}\sum_{m',n',\ge 0} (2\alpha\mathrm{b})^{n'}\|g_{m';n'}\|\Big]\\ &= \tfrac{1}{\mu\alpha^2}\mathfrak{c}'\, N(f;2\alpha)\, N(g;2\alpha).\end{aligned}$$

□

Lemma IV.7. *Let, for κ in a neighbourhood of 0, C_κ be a covariance on V. Assume that C_0 has contraction bound $\mathfrak{c}$ and integral bound b, that $\frac{d}{d\kappa}C_\kappa\big|_{\kappa=0}$ has contraction bound $\mathfrak{c}'$ and that $\mathfrak{c}\le\frac{1}{\mu}\mathfrak{c}^2$. Let $W(\psi)\in\bigwedge_A V'$ be an even Grassmann function.*

i) Set

$$\tilde W_\kappa(\psi) \;=\; \Omega_{C_\kappa}(:W:_{\psi,C_\kappa}).$$

If $N\big(W;8\alpha\big)_{\mathbf{0}} < \frac{\alpha^2}{4}$ *, then*

$$N\Big(\tfrac{d}{d\kappa}\tilde W_\kappa\big|_{\kappa=0};\alpha\Big) \le \tfrac{1}{2\alpha^2}\,\frac{N(W;8\alpha)^2}{1-\frac{4}{\alpha^2}N(W;8\alpha)}\,\tfrac{1}{\mu}\mathfrak{c}'.$$

ii) Let D be a covariance on V with integral bound b. *Set*

$$:\tilde{W}_\kappa(\psi):_{\psi,D} \;=\; \Omega_{C_\kappa}(:W:_{\psi,C_\kappa+D}).$$

If $N\big(W;32\alpha\big)_{\mathbf{0}} < \alpha^2$ *, then*

$$N\big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0}\,;\alpha\big) \le \tfrac{1}{8\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}\,\tfrac{1}{\mu}\mathfrak{c}'.$$

Proof. Set $U(\psi,\xi)=W(\psi+\xi)$. By Prop. A.2.ii, $:U(\psi,\xi):_{\xi,C_\kappa} = :W:_{C_\kappa}(\psi+\xi)$ and by Remark II.24, $N\big(U;\alpha\big)\le N\big(W;2\alpha\big)$. By Lemma A.8.iv, setting $C'(\kappa)=\frac{d}{d\kappa}C_\kappa$,

$$\begin{aligned}
\tfrac{d}{d\kappa}\tilde{W}_\kappa &= \frac{\frac{d}{d\kappa}\int e^{:W:_{C_\kappa}(\psi+\xi)}d\mu_{C_\kappa}(\xi)}{\int e^{:W:_{C_\kappa}(\psi+\xi)}d\mu_{C_\kappa}\xi)} \qquad \operatorname{mod} A_0\\
&= \frac{-\frac{1}{2}\int\big[\sum_{i,j}:\frac{\partial U}{\partial\xi_i}:_{\xi,C_t}C'_{ij}(\kappa)\;:\frac{\partial U}{\partial\xi_j}:_{\xi,C_\kappa}\big]e^{:U(\psi,\xi):_{\xi,C_\kappa}}\,d\mu_{C_\kappa}(\xi)}{\int e^{:U(\psi,\xi):_{\xi,C_\kappa}}d\mu_{C_\kappa}\xi)}\\
&= -\tfrac{1}{2}\mathcal{S}_{:U:_{\xi,C_\kappa},C_\kappa}\Big(\sum_{i,j}:\tfrac{\partial U}{\partial\xi_i}:_{\xi,C_\kappa}C'_{ij}(\kappa)\;:\tfrac{\partial U}{\partial\xi_j}:_{\xi,C_\kappa}\Big)\\
&= \mathcal{S}_{:U:_{\xi,C_\kappa},C_\kappa}(:V_\kappa:_{\xi,C_\kappa}),
\end{aligned}$$

where

$$:V_\kappa:_{\xi,C_\kappa} = -\tfrac{1}{2}\sum_{i,j}:\tfrac{\partial U}{\partial\xi_i}:_{\xi,C_\kappa}C'_{ij}(\kappa)\;:\tfrac{\partial U}{\partial\xi_j}:_{\xi,C_\kappa}.$$

As in Lemma IV.7, Prop. III.10 now implies that

$$\begin{aligned}
N\big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0}\,;\alpha\big) &= N'\big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0}\,;\alpha\big)\\
&\le N'\big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0} - V_0(\psi,0)\,;\alpha\big) + N'\big(V_0(\psi,0)\,;\alpha\big)\\
&= N'\big(\mathcal{S}_{:U:_{\xi,C_0},C_0}(:V_0:_{\xi,C_0}) - V_0(\psi,0)\big) + N'\big(V_0(\psi,0)\,;\alpha\big)\\
&\le \tfrac{2}{\alpha^2}\,\frac{N'(U;4\alpha)}{1-\frac{4}{\alpha^2}N'(U;4\alpha)}\,N'(V_0;\alpha) + N'\big(V_0\,;\alpha\big)\\
&\le \tfrac{4}{\alpha^2}\,\frac{N(U;4\alpha)}{1-\frac{4}{\alpha^2}N(U;4\alpha)}\,N(V_0;\alpha) + N(V_0;\alpha)\\
&\le \frac{1}{1-\frac{4}{\alpha^2}N(U;4\alpha)}\,N(V_0;\alpha)\\
&\le \frac{1}{1-\frac{4}{\alpha^2}N(W;8\alpha)}\,N\big(V_0;\alpha\big).
\end{aligned}$$

By Prop. A.2.i (three times)

$$\begin{aligned}
V_0(\psi,\xi) &= \int :V_0:_{\xi,C_0}(\psi,\xi+\xi')\,d\mu_{C_0}(\xi')\\
&= -\tfrac{1}{2}\sum_{i,j}\int \tfrac{\partial U}{\partial\xi_i}(\psi,\xi+\xi'+\zeta)\,C'_{ij}(0)\\
&\qquad\times\tfrac{\partial U}{\partial\xi_j}(\psi,\xi+\xi'+\zeta')\,d\mu_{-C_0}(\zeta)\,d\mu_{-C_0}(\zeta')\,d\mu_{C_0}(\xi')\\
&= -\tfrac{1}{2}\sum_{i,j}\int \tfrac{\partial W}{\partial\xi_i}(\psi+\xi+\xi'+\zeta)\,C'_{ij}(0)\\
&\qquad\times\tfrac{\partial W}{\partial\xi_j}(\psi+\xi+\xi'+\zeta')\,d\mu_{-C_0}(\zeta)\,d\mu_{-C_0}(\zeta')\,d\mu_{C_0}(\xi').
\end{aligned}$$

By Lemma II.31, with $s = 0$, followed by Lemma IV.6 and then Remark II.24,

$$\begin{aligned} N\big(V_0;\alpha\big) &\le \tfrac{1}{2}\sum_{i,j} N\Big(\tfrac{\partial W}{\partial \xi_i}(\psi+\xi+\xi'+\zeta)\, C'_{ij}(0)\, \tfrac{\partial W}{\partial \xi_j}(\psi+\xi+\xi'+\zeta')\,;\alpha\Big) \\ &\le \tfrac{1}{2\mu\alpha^2}\mathfrak{c}' N\big(W(\psi+\xi+\xi'+\zeta)\,;2\alpha\big) N\big(W(\psi+\xi+\xi'+\zeta')\,;2\alpha\big) \\ &\le \tfrac{1}{2\mu\alpha^2}\mathfrak{c}' N\big(W;8\alpha\big)^2. \end{aligned}$$

ii) Part (ii) follows from part (i) as Theorem IV.1 follows from Theorem II.28. □

Lemma IV.8. *Let C and, for κ in a neighbourhood of 0, D_κ be covariances on V. Assume that C has contraction bound $\mathfrak{c}$ and integral bound $\frac{1}{2}\mathrm{b}$, that D_0 has integral bound $\frac{1}{2}\mathrm{b}$ and that $\frac{d}{d\kappa}D_\kappa\big|_{\kappa=0}$ has integral bound $\frac{1}{2}\mathrm{b}'$. Let $W(\psi) \in \bigwedge_A V'$ be an even Grassmann function. Set*

$$:\tilde{W}_\kappa(\psi):_{\psi,D_\kappa} \;=\; \Omega_C(:W:_{\psi,C+D_\kappa}).$$

If $N\big(W;32\alpha\big)_{\mathbf{0}} < \alpha^2$, then

$$N\Big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0}\,;\alpha\Big) \le \frac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}\Big(\frac{\mathrm{b}'}{\mathrm{b}}\Big)^2.$$

Proof. Set $E_z = D_0 + z\frac{d}{d\kappa}D_\kappa\big|_{\kappa=0}$ and define V_z by

$$:V_z(\psi):_{\psi,E_z} \;=\; \Omega_C(:W:_{\psi,C+E_z}).$$

As D_κ and E_κ agree to order κ, $\frac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0} = \frac{d}{dz}V_z\big|_{z=0}$. Set $\varepsilon_d = \big(\frac{\mathrm{b}'}{\mathrm{b}}\big)^2$. Then, by Remark II.30,

$$\tfrac{1}{2}\mathrm{b} + \tfrac{1}{2}\sqrt{|z|}\,\mathrm{b}' \le \mathrm{b}$$

is an integral bound for E_z for all $|z| \le \frac{1}{\varepsilon_d}$. By Theorem IV.1,

$$N\big(V_z - W;\alpha\big) \le \frac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}$$

for all $|z| \le \frac{1}{\varepsilon_d}$. By the Cauchy integral formula, if $f(z)$ is analytic and bounded in absolute value by Q on $|z| \le r$, then

$$f'(z) = \frac{1}{2\pi\imath}\int_{|\zeta|=r}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$$

and

$$\big|f'(0)\big| \le \frac{1}{2\pi}\frac{Q}{r^2}2\pi r = Q\frac{1}{r}.$$

Hence

$$\begin{aligned} N\Big(\tfrac{d}{d\kappa}\tilde{W}_\kappa\big|_{\kappa=0}\,;\alpha\Big) &= N\Big(\tfrac{d}{dz}\big[V_z - W\big]_{z=0};\alpha\Big) \le \frac{\varepsilon_d}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)} \\ &= \frac{1}{2\alpha^2}\,\frac{N(W;32\alpha)^2}{1-\frac{1}{\alpha^2}N(W;32\alpha)}\Big(\frac{\mathrm{b}'}{\mathrm{b}}\Big)^2. \end{aligned}$$

□

Appendix A. Wick–Ordering

Let A be a superalgebra, V be a complex vector space and C an antisymmetric bilinear form (covariance) on V. Wick ordering with respect to a covariance C is the A–linear map on $\bigwedge_A V$,

$$f(\xi) \mapsto \; :f(\xi):_{\xi,C}$$

characterized by

$$:e^{\Sigma\, \xi_i \zeta_i}:_{\xi,C} \;=\; e^{1/2\, \Sigma\, \zeta_i C_{ij} \zeta_j}\, e^{\Sigma\, \xi_i \zeta_i}$$

for any set $\{\zeta_i\}$ of odd Grassmann variables. If the context admits, we delete the Wick ordering covariance C or the variable ξ (or both) from the symbol $:\cdot:_{\xi,C}$ for Wick ordering. Also recall that the Grassmann Gaussian integral with covariance C is characterized by

$$\int e^{\Sigma\, \xi_i \zeta_i}\, d\mu_C(\xi) = e^{-1/2\, \Sigma\, \zeta_i C_{ij} \zeta_j}.$$

Lemma A.1. *For $n, m \geq 0$,*

$$\int (:\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}:)\,(:\xi_{j_m}\xi_{j_{m-1}}\cdots\xi_{j_1}:)\, d\mu_C(\xi) \;=\; \begin{cases} \det\left(C_{i_k j_\ell}\right) & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

Proof. This follows easily from the definitions by induction. See, for example, Prop. I.31 of [FKTffi]. □

Choose a copy of V with generating system (ξ'_i) corresponding to the generating system (ξ_i) of V.

Proposition A.2. *(i) Let $f(\xi) = :\hat{f}(\xi):_\xi \; \in \bigwedge_A V$. Then*

$$f(\xi) = \int \hat{f}(\xi + i\xi')\, d\mu_C(\xi') = \int \hat{f}(\xi + \xi')\, d\mu_{-C}(\xi'),$$

$$\hat{f}(\xi) = \int f(\xi + \xi')\, d\mu_C(\xi').$$

In particular, $f(0) = \int \hat{f}(i\xi)\, d\mu_C(\xi) = \int \hat{f}(\xi)\, d\mu_{-C}(\xi)$.
(ii) Let $f(\xi) = :\hat{f}(\xi):_\xi \; \in \bigwedge_A V$. Then

$$f(\xi + \xi') = \; :\hat{f}(\xi + \xi'):_\xi \;=\; :\hat{f}(\xi + \xi'):_{\xi'}.$$

(iii) For $f_1(\xi), \cdots, f_\ell(\xi) \in \bigwedge_A V$,

$$:f_1(\xi):_\xi \cdots :f_\ell(\xi):_\xi \;=\; \vdots \int :f_1(\xi + \xi'):_{\xi'} \cdots :f_\ell(\xi + \xi'):_{\xi'}\, d\mu_C(\xi')\, \vdots_\xi.$$

(iv) For $f(\xi) \in \bigwedge_A V$,

$$e^{:f(\xi):_\xi} \;=\; \vdots \int e^{:f(\xi+\xi'):_{\xi'}}\, d\mu_C(\xi')\, \vdots_\xi.$$

Proof. (i) It suffices to prove the identities for $\hat{f}(\xi) = e^{\Sigma\,\xi_j\zeta_j}$. In this case

$$\int \hat{f}(\xi + i\xi')\, d\mu_C(\xi') = \int e^{\Sigma\,\xi_j\zeta_j}\, e^{i\Sigma\,\xi'_j\zeta_j}\, d\mu_C(\xi')$$
$$= e^{\Sigma\,\xi_j\zeta_j}\, e^{1/2\,\Sigma\,\zeta_i C_{ij}\zeta_j} = \,:\hat{f}(\xi):_C \,=\, f(\xi)$$

and

$$\int \hat{f}(\xi + \xi')\, d\mu_{-C}(\xi') = \int e^{\Sigma\,\xi_j\zeta_j}\, e^{\Sigma\,\xi'_j\zeta_j}\, d\mu_{-C}(\xi')$$
$$= e^{\Sigma\,\xi_j\zeta_j}\, e^{1/2\,\Sigma\,\zeta_i C_{ij}\zeta_j} = \,:\hat{f}(\xi):_C \,=\, f(\xi).$$

The statement about $\hat{f}(\xi)$ is proven in the same way, and the formula for $f(0)$ follows immediately.

(ii) Again we may assume that $\hat{f}(\xi) = e^{\Sigma\,\xi_i\zeta_i}$. Then

$$f(\xi) = e^{1/2\,\Sigma\,\zeta_i C_{ij}\zeta_j}\, e^{\Sigma\xi_i\zeta_i},$$
$$\hat{f}(\xi + \xi') = e^{\Sigma\,\xi_i\zeta_i}\, e^{\Sigma\,\xi'_i\zeta_i},$$

and

$$:\hat{f}(\xi + \xi'):_\xi \;= e^{\frac{1}{2}\Sigma\,\zeta_i C_{ij}\zeta_j + \Sigma\,\xi_i\zeta_i}\, e^{\Sigma\,\xi'_i\zeta_i}$$
$$= e^{1/2\,\Sigma\,\zeta_i C_{ij}\zeta_j + \Sigma\,(\xi_i + \xi'_i)\zeta_i}$$
$$= f(\xi + \xi').$$

Similarly one shows that $:\hat{f}(\xi + \xi'):_{\xi'} \,= f(\xi + \xi')$.

(iii) Let

$$:g(\xi): \,=\, :f_1(\xi):\cdots:f_\ell(\xi):.$$

By part (ii)

$$:g(\xi + \xi'):_{\xi'} \,=\, :f_1(\xi + \xi'):_{\xi'}\cdots:f_\ell(\xi + \xi'):_{\xi'}.$$

Therefore

$$g(\xi) = \int :g(\xi + \xi'):_{\xi'}\, d\mu_C(\xi')$$
$$= \int :f_1(\xi + \xi'):_{\xi'}\cdots:f_\ell(\xi + \xi'):_{\xi'}\, d\mu_C(\xi').$$

(iv) By part (iii)

$$e^{:f(\xi):} = \sum_{\ell=0}^{\infty} \tfrac{1}{\ell!}\left(:f(\xi):\right)^\ell$$
$$= \sum_{\ell=0}^{\infty} \tfrac{1}{\ell!}\, :\!\int \left(:f(\xi + \xi'):_{\xi'}\right)^\ell d\mu_C(\xi')\, :_\xi$$
$$= \,:\!\int e^{:f(\xi+\xi'):_{\xi'}} d\mu_C(\xi')\, :_\xi.$$

□

Corollary A.3. *For* $f_1(\xi), \cdots, f_\ell(\xi), g(\xi) \in \bigwedge_A V$,

$$
\begin{aligned}
&:f_1(\xi):_\xi \cdots :f_\ell(\xi):_\xi \ :g(\xi):_\xi \\
&\quad = \,\vdots \iint \vdots \Big(\prod_{i=1}^{\ell} :f_i(\xi+\xi'+\eta):_{\xi'} \Big) \vdots_\eta \ :g(\xi+\eta):_\eta \, d\mu_C(\xi')\, d\mu_C(\eta) \,\vdots_\xi .
\end{aligned}
$$

Proof. By iterated application of part (iii) of Prop. A.2,

$$
\begin{aligned}
&:f_1(\xi):_\xi \cdots :f_\ell(\xi):_\xi \ :g(\xi):_\xi \\
&\quad = \vdots \int \prod_{i=1}^{\ell} :f_i(\xi+\xi'):_{\xi'} \, d\mu_C(\xi') \,\vdots_\xi \ :g(\xi):_\xi \\
&\quad = \vdots \int \vdots \int \prod_{i=1}^{\ell} :f_i(\xi+\xi'+\eta):_{\xi'} \, d\mu_C(\xi') \,\vdots_\eta \ :g(\xi+\eta):_\eta \, d\mu_C(\eta) \,\vdots_\xi .
\end{aligned}
$$

□

Lemma A.4. *Let D be a second covariance on V, and let* $f(\xi) \in \bigwedge_A V$.

(i)
$$:f(\xi):_{\xi, C+D} = \vdots :f(\xi):_{\xi,C} \vdots_{\xi, D}$$

(ii)
$$:f:_{C+D}(\xi+\xi') = \vdots :f(\xi+\xi'):_{\xi,C} \vdots_{\xi', D},$$

(iii)
$$:f(\xi):_{\xi, C+D} = :f'(\xi):_{\xi,C},$$

where
$$f'(\xi) = \int f(\xi + i\xi')\, d\mu_D(\xi') = \int f(\xi+\xi')\, d\mu_{-D}(\xi').$$

Proof. Again we may assume that $f(\xi) = e^{\Sigma\, \xi_i \zeta_i}$. Set $\hat{f}(\xi) = :f(\xi):_{\xi,C} = e^{1/2\, \Sigma\, \zeta_i C_{ij} \zeta_j}$ $e^{\Sigma\, \xi_i \zeta_i}$.

(i)

$$
\begin{aligned}
:\hat{f}(\xi):_{\xi,D} &= e^{1/2\, \Sigma\, \zeta_i C_{ij} \zeta_j} \ :e^{\Sigma\, \xi_i \zeta_i}:_{\xi,D} \\
&= e^{1/2\, \Sigma\, \zeta_i (C_{ij}+D_{ij}) \zeta_j} \, e^{\Sigma\, \xi_i \zeta_i} \ = \ :f(\xi):_{\xi, C+D},
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\vdots :f(\xi+\xi'):_{\xi,C} \vdots_{\xi',D} &= e^{1/2\, \Sigma\, \zeta_i C_{ij} \zeta_j} \ :e^{\Sigma\, (\xi_i+\xi_i')\zeta_i}:_{\xi',D} \\
&= e^{1/2\, \Sigma\, \zeta_i (C_{ij}+D_{ij}) \zeta_j} \, e^{\Sigma\, (\xi_i+\xi_i')\zeta_i} \\
&= :f:_{C+D}(\xi+\xi').
\end{aligned}
$$

(iii) By part (i) and parts (i),(ii) of Prop. A.2,

$$
\begin{aligned}
:f(\xi):_{\xi,C+D} = :\hat{f}(\xi):_{\xi,D} &= \int \hat{f}(\xi + i\xi')\, d\mu_D(\xi') \\
&= \int :f(\xi+i\xi'):_{\xi,C}\, d\mu_D(\xi') = \vdots \int f(\xi+i\xi')\, d\mu_D(\xi') \,\vdots_{\xi,C}.
\end{aligned}
$$

□

Lemma A.5. *Let* $f(\xi), g(\xi), h(\xi) \in \bigwedge_A V$ *. Then*

$$\begin{aligned}&\int :f(\xi)g(\xi):_\xi h(\xi)\, d\mu_C(\xi)\\ &= \int :f(\xi):_\xi \int :g(\xi'):_{\xi'} :h(\xi+\xi'):_\xi \, d\mu_C(\xi')\, d\mu_C(\xi)\end{aligned}$$

Proof. We may assume that

$$f(\xi) = e^{\Sigma\, \xi_i \zeta_i}, \quad g(\xi) = e^{\Sigma\, \xi_i \zeta_i'}, \quad h(\xi) = e^{\Sigma\, \xi_i \eta_i},$$

with additional Grassmann variables $\zeta_i, \zeta_i', \eta_i$. Then

$$\begin{aligned}\int :f(\xi)g(\xi):_\xi h(\xi)\, d\mu_C(\xi) &= \int e^{1/2\, \Sigma\, (\zeta_i+\zeta_i')C_{ij}(\zeta_j+\zeta_j')}\, e^{\Sigma\, \xi_i(\zeta_i+\zeta_i'+\eta_i)}\, d\mu_C(\xi)\\ &= e^{1/2\, \Sigma\, [(\zeta_i+\zeta_i')C_{ij}(\zeta_j+\zeta_j')-(\zeta_i+\zeta_i'+\eta_i)C_{ij}(\zeta_j+\zeta_j'+\eta_j)]}\\ &= e^{-1/2\, \Sigma\, \eta_i C_{ij}\eta_j}\, e^{-\Sigma\, (\zeta_i+\zeta_i')C_{ij}\eta_j}.\end{aligned}$$

On the other hand

$$\begin{aligned}\int :g(\xi'):_{\xi'} :h(\xi+\xi'):_\xi \, d\mu_C(\xi') &= :\int e^{1/2\, \Sigma\, \zeta_i' C_{ij}\zeta_j'}\, e^{\Sigma\, \xi_i'(\zeta_i'+\eta_i)}\, e^{\Sigma\, \xi_i\eta_i}\, d\mu_C(\xi'):_\xi\\ &= :e^{1/2\, \Sigma\, [\zeta_i' C_{ij}\zeta_j' - (\zeta_i'+\eta_i)C_{ij}(\zeta_j'+\eta_j)]}\, e^{\Sigma\, \xi_i\eta_i}:_\xi\\ &= e^{1/2\, \Sigma\, [\zeta_i' C_{ij}\zeta_j' + \eta_i C_{ij}\eta_j - (\zeta_i'+\eta_i)C_{ij}(\zeta_j'+\eta_j)]}\, e^{\Sigma\, \xi_i\eta_i}\\ &= e^{-\Sigma\, \zeta_i' C_{ij}\eta_j}\, e^{\Sigma\, \xi_i\eta_i}.\end{aligned}$$

Therefore, also

$$\begin{aligned}&\int :f(\xi):_\xi \int :g(\xi'):_{\xi'} :h(\xi+\xi'):_\xi \, d\mu_C(\xi')\, d\mu_C(\xi)\\ &= \int e^{1/2\, \Sigma\, \zeta_i C_{ij}\zeta_j}\, e^{-\Sigma\, \zeta_i' C_{ij}\eta_j}\, e^{\Sigma\, \xi_i(\zeta_i+\eta_i)}\, d\mu_C(\xi)\\ &= e^{1/2\, [\Sigma\, \zeta_i C_{ij}\zeta_j - (\zeta_i+\eta_i)C_{ij}(\zeta_j+\eta_j)]}\, e^{-\Sigma\, \zeta_i' C_{ij}\eta_j}\\ &= e^{-1/2\, \Sigma\, \eta_i C_{ij}\eta_j}\, e^{-\Sigma\, (\zeta_i+\zeta_i')C_{ij}\eta_j}.\end{aligned}$$

□

Corollary A.6. *Let* $f(\xi), h(\xi) \in \bigwedge_A V$ *. Then*

$$\int f(\xi)h(\xi)\, d\mu_C(\xi) = \int f(\xi) :\int h(\xi+\xi')\, d\mu_C(\xi'):_\xi \, d\mu_C(\xi).$$

Proof. Set $g(\xi) = 1$ and replace $:f(\xi):_\xi$ by $f(\xi)$ in Lemma A.5. □

Lemma A.7. *Let* $f(\xi)$ *be a homogeneous Grassmann polynomial of degree two. Write*

$$f(\xi'+\xi'') = f(\xi') + f(\xi'') + f_{\text{mix}}(\xi', \xi'').$$

Furthermore let $g(\xi), h(\xi) \in \bigwedge_A V$*. Then*

$$\begin{aligned}\int :f(\xi):_\xi :g(\xi)h(\xi):_\xi \, d\mu_C(\xi) &= \iint f_{\text{mix}}(\xi', \xi'')\, :g(\xi'):_{\xi'} :h(\xi''):_{\xi''}\, d\mu_C(\xi')\, d\mu_C(\xi'')\\ &\quad + h(0)\int :f(\xi):_\xi :g(\xi):_\xi \, d\mu_C(\xi)\\ &\quad + g(0)\int :f(\xi):_\xi :h(\xi):_\xi \, d\mu_C(\xi).\end{aligned}$$

Proof. By Lemma A.5 and part (ii) of Prop. A.2,

$$\int :f(\xi):_\xi \, :g(\xi)h(\xi):_\xi \, d\mu_C(\xi)$$
$$= \iint \,\dot{\colon} :f(\xi'+\xi''):_{\xi'} \dot{\colon}_{\xi''} \, :g(\xi'):_{\xi'} \, :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'')$$
$$= \iint :\big[f(\xi') + f(\xi'') + f_{\rm mix}(\xi',\xi'')\big]:_{\xi',\xi''} \, :g(\xi'):_{\xi'} \, :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'')$$

Since $f_{\rm mix}$ has degree one in ξ' and in ξ''

$$\iint :f_{\rm mix}(\xi',\xi''):_{\xi',\xi''} \, :g(\xi'):_{\xi'} :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'')$$
$$= \iint \; f_{\rm mix}(\xi',\xi'') \, :g(\xi'):_{\xi'} \, :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'').$$

Clearly

$$\iint :f(\xi'):_{\xi',\xi''} :g(\xi'):_{\xi'} :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'') = h(0) \int :f(\xi):_\xi \, :g(\xi):_\xi \, d\mu_C(\xi),$$
$$\iint :f(\xi''):_{\xi',\xi''} :g(\xi'):_{\xi'} :h(\xi''):_{\xi''} \, d\mu_C(\xi')\, d\mu_C(\xi'') = g(0) \int :f(\xi):_\xi \, :h(\xi):_\xi \, d\mu_C(\xi).$$

□

Lemma A.8. *Let $C(\kappa)$ be a C^1 family of antisymmetric bilinear forms (covariances). Then*

i)

$$\tfrac{d}{d\kappa} \int f(\xi)\, d\mu_{C(\kappa)}(\xi) = \tfrac{1}{2}\tfrac{d^2}{dt^2} \iint f(\xi + t\xi')\, d\mu_{C(\kappa)}(\xi)\, d\mu_{C'(\kappa)}(\xi')\Big|_{t=0}$$
$$= -\tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \tfrac{\partial}{\partial \xi_i} \tfrac{\partial}{\partial \xi_j} f(\xi)\, d\mu_{C(\kappa)}(\xi).$$

ii)

$$\tfrac{d}{d\kappa} :f(\xi):_{C(\kappa)} = \tfrac{1}{2}\tfrac{d^2}{dt^2} \dot{\colon} \int f(\xi + t\xi')\, d\mu_{-C'(\kappa)}(\xi') \dot{\colon}_{\xi, C(\kappa)}\Big|_{t=0}$$
$$= \tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \dot{\colon} \tfrac{\partial}{\partial \xi_i} \tfrac{\partial}{\partial \xi_j} f(\xi) \dot{\colon}_{C(\kappa)}.$$

iii) If f is even,

$$\tfrac{d}{d\kappa} \int e^{f(\xi)}\, d\mu_{C(\kappa)}(\xi) = -\tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \Big[\tfrac{\partial f}{\partial \xi_i} \tfrac{\partial f}{\partial \xi_j} + \tfrac{\partial}{\partial \xi_i} \tfrac{\partial}{\partial \xi_j} f(\xi)\Big] e^{f(\xi)}\, d\mu_{C(\kappa)}(\xi).$$

iv) If f is even,

$$\tfrac{d}{d\kappa} \int e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi)$$
$$= -\tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \Big[\dot{\colon} \tfrac{\partial f}{\partial \xi_i} \dot{\colon}_{C(\kappa)} \; \dot{\colon} \tfrac{\partial f}{\partial \xi_j} \dot{\colon}_{C(\kappa)}\Big] e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi).$$

Proof. i)

$$
\begin{aligned}
\tfrac{d}{d\kappa}\int f(\xi)\, d\mu_{C(\kappa)}(\xi) &= \tfrac{1}{2}\tfrac{d^2}{dt^2}\int f(\xi)\, d\mu_{C(\kappa+t^2)}(\xi)\Big|_{t=0}\\
&= \tfrac{1}{2}\tfrac{d^2}{dt^2}\int f(\xi)\, d\mu_{C(\kappa)+t^2C'(\kappa)}(\xi)\Big|_{t=0}\\
&= \tfrac{1}{2}\tfrac{d^2}{dt^2}\iint f(\xi+\xi')\, d\mu_{C(\kappa)}(\xi)\, d\mu_{t^2C'(\kappa)}(\xi')\Big|_{t=0}\\
&= \tfrac{1}{2}\tfrac{d^2}{dt^2}\iint f(\xi+t\xi')\, d\mu_{C(\kappa)}(\xi)\, d\mu_{C'(\kappa)}(\xi')\Big|_{t=0}\\
&= -\tfrac{1}{2}\sum_{i,j}\iint \xi_i'\xi_j'\tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j}\, f(\xi)\, d\mu_{C(\kappa)}(\xi)\, d\mu_{C'(\kappa)}(\xi')\\
&= -\tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\int \tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j}\, f(\xi)\, d\mu_{C(\kappa)}(\xi).
\end{aligned}
$$

ii)

$$
\begin{aligned}
\tfrac{d}{d\kappa}\,{:}\,f(\xi)\,{:}_{C(\kappa)} &= \tfrac{d}{d\kappa}\int f(\xi+\xi'')\, d\mu_{-C(\kappa)}(\xi'')\\
&= \tfrac{1}{2}\tfrac{d^2}{dt^2}\iint f(\xi+\xi''+t\xi')\, d\mu_{-C(\kappa)}(\xi'')\, d\mu_{-C'(\kappa)}(\xi')\Big|_{t=0}\\
&= \int\left\{\tfrac{1}{2}\tfrac{d^2}{dt^2}\int f(\xi+t\xi'+\xi'')\, d\mu_{-C'(\kappa)}(\xi')\Big|_{t=0}\right\} d\mu_{-C(\kappa)}(\xi'')\\
&= {:}\,\tfrac{1}{2}\tfrac{d^2}{dt^2}\int f(\xi+t\xi')\, d\mu_{-C'(\kappa)}(\xi')\Big|_{t=0}\,{:}_{C(\kappa)}\\
&= \tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\,{:}\,\tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j}\, f(\xi)\,{:}_{C(\kappa)}.
\end{aligned}
$$

iii) By the first part

$$
\begin{aligned}
&\tfrac{d}{d\kappa}\int e^{f(\xi)}\, d\mu_{C(\kappa)}(\xi)\\
&\quad= -\tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\int \tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j}\, e^{f(\xi)}\, d\mu_{C(\kappa)}(\xi)\\
&\quad= -\tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\int \tfrac{\partial}{\partial\xi_i}\Big[e^{f(\xi)}\tfrac{\partial}{\partial\xi_j} f(\xi)\Big]\, d\mu_{C(\kappa)}(\xi)\\
&\quad= -\tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\int e^{f(\xi)}\Big[\tfrac{\partial f}{\partial\xi_i}(\xi)\,\tfrac{\partial f}{\partial\xi_j}(\xi) + \tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j} f(\xi)\Big]\, d\mu_{C(\kappa)}(\xi).
\end{aligned}
$$

iv) By parts (iii) and (ii),

$$
\begin{aligned}
&\tfrac{d}{d\kappa}\int e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi)\\
&\quad= -\tfrac{1}{2}\sum_{i,j} C_{ij}'(\kappa)\int \Big[\tfrac{\partial{:}f{:}}{\partial\xi_i}\tfrac{\partial{:}f{:}}{\partial\xi_j} + \tfrac{\partial}{\partial\xi_i}\tfrac{\partial}{\partial\xi_j}{:}f(\xi){:}\Big]e^{:f(\xi):}\, d\mu_{C(\kappa)}(\xi)\\
&\qquad+\int\Big[\tfrac{d}{d\kappa}{:}f(\xi){:}_{C(\kappa)}\Big]e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&= -\tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \Big[\tfrac{\partial :f:}{\partial \xi_i} \tfrac{\partial :f:}{\partial \xi_j} + \tfrac{\partial}{\partial \xi_i} \tfrac{\partial}{\partial \xi_j} :f(\xi): \Big] e^{:f(\xi):}\, d\mu_{C(\kappa)}(\xi) \\
&\quad + \tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \Big[\tfrac{\partial}{\partial \xi_i} \tfrac{\partial}{\partial \xi_j} \, {:f(\xi):}_{C(\kappa)} \Big] e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi) \\
&= -\tfrac{1}{2} \sum_{i,j} C'_{ij}(\kappa) \int \Big[{:\tfrac{\partial f}{\partial \xi_i}:}_{C(\kappa)} \, {:\tfrac{\partial f}{\partial \xi_j}:}_{C(\kappa)} \Big] e^{:f(\xi):_{C(\kappa)}}\, d\mu_{C(\kappa)}(\xi).
\end{aligned}
$$

□

Appendix B. Gram Bounds

Let C be an antisymmetric bilinear form (covariance) on the vector space V, and let $\{\xi_i\}$ be a system of generators for V.

Proposition B.1. *Suppose that V is the direct sum of two subspaces V_a and V_c such that each of the generators ξ_i lies either in V_a or V_c and*

$$C(\xi, \xi') = 0 \qquad \text{if both } \xi, \xi' \in V_a \text{ or both } \xi, \xi' \in V_c.$$

Assume furthermore that there is a Hilbert space $\mathcal{H}$, and that there is associated to each generator ξ_i a vector $w_i \in \mathcal{H}$. Set

$$S = \sup_i \|w_i\|.$$

i) If

$$C(\xi_i, \xi_j) = \langle w_i, w_j \rangle_{\mathcal{H}} \qquad \text{if } \xi_i \in V_a,\ \xi_j \in V_c$$

for all i, j, then

$$\Big| \int \xi_{i_1} \cdots \xi_{i_m}\, d\mu_C(\xi) \Big| \le S^m$$

for all $i_1, \cdots, i_m$.

ii) Assume that, for each generator ξ_i, there exists a real number τ_i such that

$$C(\xi_i, \xi_j) = \begin{cases} e^{-(\tau_i - \tau_j)} \langle w_i, w_j \rangle_{\mathcal{H}} & \text{if } \tau_i > \tau_j \\ 0 & \text{if } \tau_i \le \tau_j \end{cases}$$

for all i, j with $\xi_i \in V_a$, $\xi_j \in V_c$. Then again

$$\Big| \int \xi_{i_1} \cdots \xi_{i_m}\, d\mu_C(\xi) \Big| \le S^m$$

for all $i_1, \cdots, i_m$.

In both cases, $2S$ is an integral bound for the covariance C with respect to the norms of Example II.20.

Proof of part (i). If the integral does not vanish, we may reorder the factors in the integrand $\xi_{i_1}\cdots\xi_{i_m} = \pm\xi_{j_1}\xi_{\ell_1}\cdots\xi_{j_n}\xi_{\ell_n}$ so that $\xi_{j_p}\in V_a$ and $\xi_{\ell_p}\in V_c$ for all $1\le p\le n=\frac{m}{2}$. Then

$$\int \xi_{i_1}\cdots\xi_{i_m}\,d\mu_C(\xi) = \pm\det\big[C(\xi_{j_p},\xi_{\ell_q})\big]_{1\le p,q\le n}.$$

Part (i) now follows by Gram's inequality. Part (ii) shall be proven following Lemma B.4. □

Example B.2. In [FKTo3], part (i) of Prop. B.1 is applied to covariances of the form

$$C(\xi_i,\xi_j) = \delta_{\sigma_i,\sigma_j}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath<k,x_i-x_j>_-}\frac{\chi(k)}{\imath k_0 - e(\mathbf{k})},$$

where $\sigma_i,\sigma_j\in\{\uparrow,\downarrow\}$ are spins, $x_i=(\tau_i,\mathbf{x}_i),\ x_j=(\tau_j,\mathbf{x}_j)\in\mathbb{R}^{d+1}$ are points in space–(imaginary)time, $k=(k_0,\mathbf{k})\in\mathbb{R}\times\mathbb{R}^d$ and $<k,x_i-x_j>_- = -k_0(\tau_i-\tau_j)+\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)$. The function $e(\mathbf{k})$ is the dispersion relation, minus the chemical potential, and $\chi(k)$ is a nonnegative cutoff function. In this case, we may take $\mathcal{H}=L^2(\mathbb{R}^{d+1}\times\mathbb{C}^2)$,

$$w_i(k,\sigma) = \delta_{\sigma,\sigma_i}e^{-\imath<k,x_i>_-}\begin{cases}\sqrt{\frac{1}{(2\pi)^{d+1}}\frac{\chi(k)}{\imath k_0-e(\mathbf{k})}} & \text{if } \xi_i\in V_c\\ \overline{\sqrt{\frac{1}{(2\pi)^{d+1}}\frac{\chi(k)}{\imath k_0-e(\mathbf{k})}}} & \text{if } \xi_i\in V_a\end{cases}$$

for any single–valued square root, and

$$S=\sqrt{\int \tfrac{d^{d+1}k}{(2\pi)^{d+1}}\left|\tfrac{\chi(k)}{\imath k_0-e(\mathbf{k})}\right|}.$$

Part (ii) of Prop. B.1 will be used in [FKTo2]. It is designed to deal with covariances of the form

$$C(\xi_i,\xi_j) = \delta_{\sigma_i,\sigma_j}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath<k,x_i-x_j>_-}\frac{U(\mathbf{k})}{\imath k_0-1}$$

in which the nonnegative cutoff function $U(\mathbf{k})$ is independent of k_0. In this case $\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\left|\frac{U(\mathbf{k})}{\imath k_0-1}\right|$ diverges. As

$$\int\frac{dk_0}{2\pi}e^{-\imath k_0(\tau_i-\tau_j)}\frac{U(\mathbf{k})}{\imath k_0-1} = \begin{cases}-U(\mathbf{k})e^{-(\tau_i-\tau_j)} & \text{if } \tau_i>\tau_j\\ 0 & \text{if } \tau_i\le\tau_j\end{cases}$$

(actually, the case $\tau_i=\tau_j$ is defined by the limit $\tau_j\to\tau_i+$), we may take $\mathcal{H}=L^2(\mathbb{R}^d\times\mathbb{C}^2)$,

$$w_i(\mathbf{k},\sigma)=\delta_{\sigma,\sigma_i}e^{-\imath\mathbf{k}\cdot\mathbf{x}_i}\sqrt{\tfrac{1}{(2\pi)^d}U(\mathbf{k})}\begin{cases}-1 & \text{if } \xi_i\in V_a\\ 1 & \text{if } \xi_i\in V_c\end{cases}$$

and then

$$S=\sqrt{\int\tfrac{d^d\mathbf{k}}{(2\pi)^d}\big|U(\mathbf{k})\big|}.$$

To prepare for the proof of part (ii) of Prop. B.1, let $\bigwedge \mathcal{H} = \bigoplus_{n\ge 0} \bigwedge^n \mathcal{H}$ be the Grassmann algebra over $\mathcal{H}$. The element $1 \in \mathbb{C} = \bigwedge^0 \mathcal{H}$ is also denoted by Ω_0 (the ground state). On each of the summands $\bigwedge^n \mathcal{H}$ there is an inner product such that

$$\langle v_1 \cdots v_n,\ v_1' \cdots v_n' \rangle = \det\Big(\big\langle v_i,\, v_j' \big\rangle \Big)_{i,j=1,\cdots,n}$$

for all $v_1, \cdots, v_n,\ v_1', \cdots, v_n' \in \mathcal{H}$. On $\mathcal{H}$ there is an inner product determined by the requirement that $\bigoplus_{n\ge 0} \bigwedge^n \mathcal{H}$ be an orthogonal direct sum. For $v \in \mathcal{H}$, let $a^\dagger(v)$ be the operator on $\bigwedge \mathcal{H}$ that maps $f \in \bigwedge^n \mathcal{H}$ to $vf \in \bigwedge^{n+1} \mathcal{H}$, and $a(v)$ its adjoint. We have the standard

Lemma B.3. *For all* $v, w \in \mathcal{H}$*,*

i)

$$\begin{aligned} \{a(v), a(w)\} &= \{a^\dagger(v), a^\dagger(w)\} = 0, \\ \{a(v), a^\dagger(w)\} &= \langle v,\ w \rangle\,. \end{aligned}$$

ii) The operator norms $\|a(v)\|$, $\|a^\dagger(v)\|$ *are bounded by* $\|v\|_{\mathcal{H}}$.

Proof. Let $\big(e_j\big)_{j\in\mathcal{J}}$ be an orthonormal basis for $\mathcal{H}$, indexed by a totally ordered set $\mathcal{J}$. For each finite subset J of $\mathcal{J}$ set

$$e_J = e_{j_1} \cdots e_{j_n} \qquad \text{when } J = \{j_1, \cdots, j_n\} \text{ with } j_1 \prec \cdots \prec j_n.$$

The elements e_J, $|J| = n$ are an orthonormal basis for $\bigwedge^n \mathcal{H}$. Then

$$\begin{aligned} a^\dagger(e_j)e_J &= \begin{cases} \epsilon_{J,j}\ e_{J\cup\{j\}} & \text{if } j \notin J \\ 0 & \text{if } j \in J \end{cases}, \\ a(e_j)e_J &= \begin{cases} \epsilon_{J\setminus\{j\},j}\ e_{J\setminus\{j\}} & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}, \end{aligned}$$

where, for $j \notin J$, $\epsilon_{J,j}$ is the sign of the permutation that brings the sequence j, J to standard order.

In the case $v = e_j,\ w = e_{j'}$, part (i) of the lemma follows directly from this description. The general case follows from this special case, since all terms are bilinear in v and w.

To prove part (ii) of the lemma observe that, for all $u \in \mathcal{H}$ and $f \in \bigwedge \mathcal{H}$,

$$\|a(u)f\|^2_{\wedge\mathcal{H}} + \|a^\dagger(u)f\|^2_{\wedge\mathcal{H}} = \big\langle \{a(u), a^\dagger(u)\}f,\ f \big\rangle = \|u\|^2_{\mathcal{H}}\ \|f\|^2_{\wedge\mathcal{H}}$$

so that

$$\|a(u)f\|_{\wedge\mathcal{H}} \le \|u\|_{\mathcal{H}}\ \|f\|_{\wedge\mathcal{H}} \qquad \|a^\dagger(u)f\|_{\wedge\mathcal{H}} \le \|u\|_{\mathcal{H}}\ \|f\|_{\wedge\mathcal{H}}.$$

□

Let N be the number operator on $\bigwedge \mathcal{H}$. By definition, its restriction to $\bigwedge^n \mathcal{H}$ is multiplication by n. For each index i, labeling a generator ξ_i, set

$$a_i = \begin{cases} e^{\tau_i N} a(w_i) e^{-\tau_i N} = e^{-\tau_i} a(w_i) & \text{if } \xi_i \in V_a \\ e^{\tau_i N} a^\dagger(w_i) e^{-\tau_i N} = e^{\tau_i} a^\dagger(w_i) & \text{if } \xi_i \in V_c \end{cases}.$$

A sequence $i_1, \cdots, i_m$ is called time ordered if $\tau_{i_1} \geq \cdots \geq \tau_{i_m}$ and for $1 \leq k < \ell \leq m$,

$$\tau_{i_k} = \tau_{i_\ell}, \;\; \xi_{i_k} \in V_a \quad \text{implies} \;\; \xi_{i_\ell} \in V_a.$$

Lemma B.4. *Let $i_1, \cdots, i_m$ be a time ordered sequence. Then, under the assumptions of part (ii) of Prop. B.1,*

$$\int \xi_{i_1} \cdots \xi_{i_m} \, d\mu_C(\xi) = \left\langle \Omega_0, \, a_{i_1} \cdots a_{i_m} \Omega_0 \right\rangle.$$

Proof. The proof is by induction on m. The cases $m = 0$ and $m = 1$ are trivial. We perform the induction step $m - 2 \to m$.

Assume first that $\xi_{i_1} \in V_a$. Then

$$\{a_{i_1}, \, a_{i_k}\} = \begin{cases} e^{-\tau_{i_1} + \tau_{i_k}} \{a(w_{i_1}), \, a^\dagger(w_{i_k})\} = e^{-(\tau_{i_1} - \tau_{i_k})} \left\langle w_{i_1}, \, w_{i_k} \right\rangle & \text{if } \xi_{i_k} \in V_c \\ 0 & \text{if } \xi_{i_k} \in V_a \end{cases}$$

and $a_{i_1} \Omega_0 = 0$. Therefore

$$\begin{aligned} \left\langle \Omega_0, \, a_{i_1} \cdots a_{i_m} \Omega_0 \right\rangle &= \sum_{\substack{k=2 \\ \xi_{i_k} \in V_c}}^{m} (-1)^k \, e^{-(\tau_{i_1} - \tau_{i_k})} \left\langle w_{i_1}, \, w_{i_k} \right\rangle \\ &\quad \times \left\langle \Omega_0, \, a_{i_2} \cdots a_{i_{k-1}} a_{i_{k+1}} \cdots a_{i_m} \, \Omega_0 \right\rangle \\ &= \sum_{k=2}^{m} (-1)^k C(\xi_{i_1}, \xi_{i_k}) \int \xi_{i_2} \cdots \xi_{i_{k-1}} \xi_{i_{k+1}} \cdots \xi_{i_m} \, d\mu_C(\xi) \\ &= \int \xi_{i_1} \cdots \xi_{i_m} \, d\mu_C(\xi). \end{aligned}$$

For the second equality we used the assumption on C, the fact that the sequence $i_1, \cdots, i_m$ is time ordered, and the induction hypothesis. The third equality is the integration by parts formula.

Now assume that $\xi_{i_1} \in V_c$. Then

$$\left\langle \Omega_0, \, a_{i_1} \cdots a_{i_m} \Omega_0 \right\rangle = e^{\tau_{i_1}} \left\langle a(w_{i_1}) \Omega_0, \, a_{i_2} \cdots a_{i_m} \Omega_0 \right\rangle = 0$$

and, by the integration by parts formula

$$\int \xi_{i_1} \cdots \xi_{i_m} \, d\mu_C(\xi) = \sum_{k=2}^{m} (-1)^k C(\xi_{i_1}, \xi_{i_k}) \int \xi_{i_2} \cdots \xi_{i_{k-1}} \xi_{i_{k+1}} \cdots \xi_{i_m} \, d\mu_C(\xi) = 0,$$

since $C(\xi_{i_1}, \xi_{i_k}) = -C(\xi_{i_k}, \xi_{i_1}) = 0$ for $k = 2, \cdots, m$. □

Proof of part (ii) of Prop. B.1. We may assume that the sequence $i_1, \cdots, i_m$ is time ordered. If $\#\{\nu \,|\, \xi_{i_\nu} \in V_c\} \neq \#\{\nu \,|\, \xi_{i_\nu} \in V_a\}$ then $\int \xi_{i_1} \cdots \xi_{i_m} \, d\mu_C(\xi) = 0$. Otherwise, by Lemma B.4 and Lemma B.3,

$$\begin{aligned}
\Big| \int \xi_{i_1} \cdots \xi_{i_m} \, d\mu_C(\xi) \Big| &= \big| \langle \Omega_0,\ a_{i_1} \cdots a_{i_m} \Omega_0 \rangle \big| \\
&= \Big| \Big\langle \Omega_0,\ a^{(\dagger)}(w_{i_1}) e^{-(\tau_{i_1} - \tau_{i_2})N} a^{(\dagger)}(w_{i_2}) \cdots e^{-(\tau_{i_{m-1}} - \tau_{i_m})N} a^{(\dagger)}(w_{i_m}) \Omega_0 \Big\rangle \Big| \\
&\le \big\| a^{(\dagger)}(w_{i_1}) \big\| \, \big\| e^{-(\tau_{i_1} - \tau_{i_2})N} \big\| \, \big\| a^{(\dagger)}(w_{i_2}) \big\| \cdots \big\| e^{-(\tau_{i_{m-1}} - \tau_{i_m})N} \big\| \, \big\| a^{(\dagger)}(w_{i_m}) \big\| \\
&\le \prod_{k=1}^{m} \big\| a^{(\dagger)}(w_{i_k}) \big\| \ \le\ S^m .
\end{aligned}$$

Here, we used that $a_{i_1} \cdots a_{i_m} \Omega_0 \in \bigwedge^0 \mathcal{H}$ and that the restriction of the number operator N to $\bigwedge^0 \mathcal{H}$ is identically zero. □

Notation

Not’n	Description	Reference
$\mathcal{Z}(f)$	degree zero component of f	Def. II.1.iii
$\bigwedge V$	Grassmann algebra over V	Example II.2
$\bigwedge_A V$	Grassmann algebra over V with coefficients in A	Example II.2
$A_m[n_1, \cdots, n_r]$	partially antisymmetric elements of $A_m \otimes V^{\otimes(n_1+\cdots+n_r)}$	Def. II.21
$\int e^{\Sigma \xi_i \zeta_i} \, d\mu_C(\xi)$	$= e^{-1/2 \, \Sigma \, \zeta_i C_{ij} \zeta_j}$ Grassmann Gaussian integral	before Def. II.3
$\Omega_C(W)(\psi)$	$= \log \frac{1}{Z} \int e^{W(\psi+\xi)} d\mu_C(\xi)$ renormalization group map	Definitions II.3, II.27
$\mathcal{S}(f)$	$= \frac{1}{Z(U,C)} \int f(\xi)\, e^{U(\xi)} \, d\mu_C(\xi)$ Schwinger functional	before Remark III.1
R	R–operator	before Theorem III.2
$R_C(K_1, \cdots, K_\ell)$	$\ell^{\rm th}$ Taylor coefficient of R	(III.2)
$\mathcal{R}_{K,C}(f)$	$\vdots \iint :e^{:K(\xi,\xi',\eta):_{\xi'}} - 1:_{\eta} f(\eta) d\mu_C(\xi')\, d\mu_C(\eta) \vdots_{\xi}$	Def. III.4
$:e^{\Sigma \xi_i \zeta_i}:_{\xi,C}$	$= e^{1/2 \, \Sigma \, \zeta_i C_{ij} \zeta_j} \, e^{\Sigma \xi_i \zeta_i}$ Wick ordering	after Remark II.4
$\mathcal{C}\mathrm{on}_{C \atop i \to j}$, $\mathrm{Con}_{C \atop \xi \to \xi'}$	contractions	Definitions II.5, II.9
$\mathfrak{N}_d$	norm domain	Def. II.14
$\mathfrak{c}$	contraction bound	Def. II.25.i
b	integral bound	Def. II.25.ii
$N(f; \alpha)$	$\frac{1}{\mathrm{b}^2} \, \mathfrak{c} \sum_{m,n_1,\cdots,n_r \ge 0} \alpha^{\lvert n \rvert} \, \mathrm{b}^{\lvert n \rvert} \, \lVert f_{m;n_1,\cdots,n_r} \rVert$	Def. II.23

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