Commun. Math. Phys. 247, 1–47 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-003-0996-0

Communications in
Mathematical Physics

A Two Dimensional Fermi Liquid. Part 1: Overview

Joel Feldman[1,⋆], **Horst Knörrer**[2], **Eugene Trubowitz**[2]

[1] Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada V6T 1Z2. E-mail: feldman@math.ubc.ca
[2] Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland.
E-mail: knoerrer@math.ethz.ch; trub@math.ethz.ch

Received: 21 September 2002 / Accepted: 12 August 2003
Published online: 6 April 2004 –

Abstract: In a series of ten papers (see the flow chart at the end of §I), of which this is the first, we prove that the temperature zero renormalized perturbation expansions of a class of interacting many–fermion models in two space dimensions have nonzero radius of convergence. The models have "asymmetric" Fermi surfaces and short range interactions. One consequence of the convergence of the perturbation expansions is the existence of a discontinuity in the particle number density at the Fermi surface. Here, we present a self contained formulation of our main results and give an overview of the methods used to prove them.

Contents

⋆ Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Forschungsinstitut für Mathematik, ETH Zürich.

I. Introduction

The concept of a Fermi liquid was introduced by L. D. Landau in [L1, L2, L3] and has become the generally accepted explanation for the unexpected success of the independent electron approximation. An elementary sketch of Landau's well known physical arguments can be found in [AM, pp. 345–351]. More thorough and technical discussions are presented in [AGD, pp. 154–203] and [N].

Roughly speaking, at temperature zero, the single particle excitations of a noninteracting Fermi gas become (almost stable) "quasi–particles" in a Fermi liquid. The quasi–particle spectrum has the "same structure" as the noninteracting single particle excitation spectrum and the quasi–particle density function $n(\mathbf{k})$ still has a jump at the "Fermi surface". The quasi–particle interaction at temperature zero is encoded in Landau's f–function $f(\mathbf{k}_F, \mathbf{k}'_F)$.

It is well known that there are a number of potential instabilities that can drive an interacting fermi gas away from the Fermi liquid state. See, for example, [MCD, §1.2,4.5]. One of the most celebrated is the BCS instability for the formation of Cooper pairs leading to superconductivity in 2 and 3 dimensions. This is a potential instability for any time reversal invariant system [KL, L].

Another important instability is the Luttinger instability. There are solvable models in one space dimension that exhibit qualitatively different behavior from that of a three dimensional Landau Fermi liquid. In particular, the quasi–particle density function $n(\mathbf{k})$ is continuous across the "Fermi surface" but has infinite slope there. These systems are called Luttinger liquids. For a rigorous treatment of Luttinger liquids in one dimension, see [BG] and the references therein.

Anderson [A1, A2] suggested that a two dimensional Fermi gas should exhibit behavior similar to a one dimensional Luttinger liquid. Theorem I.4, which is proven in this series of papers, rigorously shows that this is not the case for the class of models considered here. In particular, we show that the density function $n(\mathbf{k})$ has a jump discontinuity across the Fermi surface (Theorem I.5). The existence of the Landau f-function and its basic regularity properties follow directly from Theorem I.7.

The standard model for a gas of weakly interacting fermions in a d-dimensional crystal at low temperature is given in terms of

- a single particle dispersion relation (shifted by the chemical potential) $e(\mathbf{k})$ on $\mathbb{R}^d$,
- an ultraviolet cutoff $U(\mathbf{k})$ on $\mathbb{R}^d$,
- an interaction V.

Here $\mathbf{k}$ is the momentum variable dual to the position variable $\mathbf{x} \in \mathbb{R}^d$. The Fermi surface associated to the dispersion relation $e(\mathbf{k})$ is by definition

$$F = \left\{ \mathbf{k} \in \mathbb{R}^d \;\middle|\; e(\mathbf{k}) = 0 \right\}.$$

The ultraviolet cutoff is a smooth function with compact support that fulfills $0 \le U(\mathbf{k}) \le 1$ for all $\mathbf{k} \in \mathbb{R}^d$. We assume that it is identically one on a neighbourhood of the Fermi surface[1].

[1] In particular, we assume that F is compact.

We use renormalization group techniques to show that, for $d = 2$ and under the assumptions on the dispersion relation $e(\mathbf{k})$ specified in Hypotheses I.12below, such a system is a Fermi liquid whenever V is small enough (the precise statement is given in Theorem I.5 below). Renormalization is necessary since the Fermi surfaces for the noninteracting (that is $V = 0$) and interacting systems ($V \neq 0$) do not, in general, agree. We therefore select (V–dependent) counterterms $\delta e(\mathbf{k})$, from the space in Definition I.1, below, in such a way that the Fermi surface of the model with dispersion relation $e(\mathbf{k}) - \delta e(\mathbf{k})$ and interaction V is equal to F.

Definition I.1. *The space of counterterms, $\mathcal{E}$, consists of all functions $\delta e(\mathbf{k})$ on $\mathbb{R}^d$ that are supported in $\left\{ \mathbf{k} \in \mathbb{R}^d \;\middle|\; U(\mathbf{k}) = 1 \right\}$ and for which the L^1–norm of the Fourier transform is finite. That is*

$$\int d^d\mathbf{x} \left|\delta e^\wedge(\mathbf{x})\right| < \infty,$$

where $\delta e^\wedge(\mathbf{x}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{-\imath \mathbf{k}\cdot\mathbf{x}}\, \delta e(\mathbf{k})$.

The temperature Green's functions at temperature zero (also known as the Euclidean Green's functions) for this model can be described in field theoretic terms using the anticommuting fields $\psi_\sigma(x_0, \mathbf{x})$, $\bar\psi_\sigma(x_0, \mathbf{x})$, where $x_0 \in \mathbb{R}$ is the temperature (or Euclidean time) argument and $\sigma \in \{\uparrow, \downarrow\}$ is the spin argument. For $x = (x_0, \mathbf{x}, \sigma)$ we write $\psi(x) = \psi_\sigma(x_0, \mathbf{x})$ and $\bar\psi(x) = \bar\psi_\sigma(x_0, \mathbf{x})$.

For a model with dispersion relation $e(\mathbf{k}) - \delta e(\mathbf{k})$ and interaction $V = 0$, the Green's functions are the moments of the Grassmann Gaussian measure, $d\mu_{C(\delta e)}$, whose covariance is the Fourier transform of

$$C(k_0, \mathbf{k}; \delta e) = \frac{U(\mathbf{k})}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})}.$$

Precisely, for $x = (x_0, \mathbf{x}, \sigma),\ x' = (x_0', \mathbf{x}', \sigma') \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}$,

$$C(x, x'; \delta e) = \int \psi(x)\bar\psi(x')\, d\mu_{C(\delta e)}(\psi, \bar\psi) = \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath <k, x-x'>_-} C(k; \delta e),$$

where $< k, x - x' >_- = -k_0(x_0 - x_0') + \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')$ for $k = (k_0, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d$. To simplify notation we set

$$C(k) = C(k; 0) \qquad , \qquad C(x, x') = C(x, x'; 0).$$

The interaction between the fermions is determined by the effective potential

$$\mathcal{V}(\psi, \bar\psi) = \int_{(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^4} V(x_1, x_2, x_3, x_4)\, \bar\psi(x_1)\psi(x_2)\bar\psi(x_3)\psi(x_4)\, dx_1 dx_2 dx_3 dx_4.$$

We assume that V is translation invariant and spin independent. For some results, we also assume that V obeys

$$V(R_0x_1, R_0x_2, R_0x_3, R_0x_4) = \overline{V(-x_1, -x_2, -x_3, -x_4)} \tag{I.1}$$

and

$$V(-x_2, -x_1, -x_4, -x_3) = V(x_1, x_2, x_3, x_4), \tag{I.2}$$

where $R_0(x_0,\mathbf{x},\sigma) = (-x_0,\mathbf{x},\sigma)$ and $-(x_0,\mathbf{x},\sigma) = (-x_0,-\mathbf{x},\sigma)$. We call (I.1) "$k_0$–reversal reality" and (I.2)"bar/unbar exchange invariance". Precise definitions and a discussion of the properties of these symmetries are given in Appendix B of [FKTo2].

In the case of a two–body interaction $v(x_0,\mathbf{x})$, the interaction kernel is

$$V\big((x_{1,0},\mathbf{x}_1,\sigma_1),\cdots,(x_{4,0},\mathbf{x}_4,\sigma_4)\big) = -\tfrac{1}{2}\delta(x_1,x_2)\delta(x_3,x_4)\delta(x_{1,0}-x_{3,0})v(x_{1,0}-x_{3,0},\mathbf{x}_1-\mathbf{x}_3), \tag{I.3}$$

where $\delta\big((x_0,\mathbf{x},\sigma),(x_0',\mathbf{x}',\sigma')\big) = \delta(x_0-x_0')\delta(\mathbf{x}-\mathbf{x}')\delta_{\sigma,\sigma'}$. If the Fourier transform, $\check v(k_0,\mathbf{k})$, of the two–body interaction $v(x_0,\mathbf{x})$ obeys $\check v(-k_0,\mathbf{k}) = \overline{\check v(k_0,\mathbf{k})}$, then the interaction kernel V has all four symmetries mentioned above. In addition, $\mathcal{V}$ always conserves particle number.

We briefly discuss the norms imposed on interaction kernels. For a function $f(x_1,\cdots,x_n)$ on $(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^n$ we define its L_1–L_∞–norm as

$$|||f|||_{1,\infty} = \max_{1\le j_0\le n}\ \sup_{x_{j_0}\in\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}} \int \prod_{\substack{j=1,\cdots,n\\ j\ne j_0}} dx_j\ \ |f(x_1,\cdots,x_n)|.$$

A multiindex is an element $\delta = (\delta_0,\delta_1,\cdots,\delta_d)\in\mathbb{N}_0\times\mathbb{N}_0^d$. The length of a multiindex $\delta = (\delta_0,\delta_1,\cdots\delta_d)$ is $|\delta| = \delta_0+\delta_1+\cdots+\delta_d$ and its factorial is $\delta! = \delta_0!\delta_1!\cdots\delta_d!$. For two multiindices δ,δ' we say that $\delta\le\delta'$ if $\delta_i\le\delta_i'$ for $i=0,1,\cdots,d$. The spatial part of the multiindex $\delta = (\delta_0,\delta_1,\cdots\delta_d)$ is $\boldsymbol{\delta} = (\delta_1,\cdots\delta_d)\in\mathbb{N}_0^d$. It has length $|\boldsymbol{\delta}| = \delta_1+\cdots+\delta_d$. For a multiindex δ and $x = (x_0,\mathbf{x},\sigma),\ x' = (x_0',\mathbf{x}',\sigma')\in\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\ \downarrow\}$ set

$$(x-x')^\delta = (x_0-x_0')^{\delta_0}\,(\mathbf{x}_1-\mathbf{x}_1')^{\delta_1}\cdots(\mathbf{x}_d-\mathbf{x}_d')^{\delta_d}.$$

We fix $r_0, r\ge 6$ for the numbers of temporal and spatial momentum derivatives that we will control. The norm imposed on an interaction kernel will be

$$\max_{\substack{\delta_{i,j}\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \text{for } 1\le i<j\le 4\\ \Sigma|\boldsymbol{\delta}_{i,j}|\le r,\ \Sigma|\delta_{i,j;0}|\le r_0}} \Big|\Big|\Big| \prod_{1\le i<j\le 4}\tfrac{1}{\delta_{i,j}!}(x_i-x_j)^{\delta_{i,j}}\,V(x_1,x_2,x_3,x_4)\Big|\Big|\Big|_{1,\infty}.$$

When V is of the form (I.3),

$$\begin{aligned}\max_{\substack{\delta_{i,j}\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \text{for } 1\le i<j\le 4\\ \Sigma|\boldsymbol{\delta}_{i,j}|\le r,\ \Sigma|\delta_{i,j;0}|\le r_0}} &\Big|\Big|\Big| \prod_{1\le i<j\le 4}\tfrac{1}{\delta_{i,j}!}(x_i-x_j)^{\delta_{i,j}}\,V(x_1,x_2,x_3,x_4)\Big|\Big|\Big|_{1,\infty}\\ &= \max_{\substack{\boldsymbol{\delta}\in\mathbb{N}_0^d\\ |\boldsymbol{\delta}|\le r}} \tfrac{1}{\boldsymbol{\delta}!}\int \big|\mathbf{x}^{\boldsymbol{\delta}}v(\mathbf{x})\big|\,d\mathbf{x}.\end{aligned}$$

Formally, the generating function for the connected Green's functions is

$$\mathcal{G}(\phi,\bar\phi;\delta e) = \log\tfrac{1}{Z}\int e^{\phi J\psi}\,e^{\mathcal{V}(\psi,\bar\psi)}\,e^{-\mathcal{K}(\psi,\bar\psi;\delta e)}\,d\mu_C(\psi,\bar\psi), \tag{I.4}$$

where the source term is

$$\phi J\psi = \int dx\ \bar\phi(x)\psi(x)+\bar\psi(x)\phi(x). \tag{I.5}$$

The counterterm is implemented in

$$\mathcal{K}(\psi,\bar{\psi};\delta e)=\tfrac{1}{2}\sum_{\sigma\in\{\uparrow,\downarrow\}}\int d\tau d^d\mathbf{x}d^d\mathbf{y}\;\delta e^{\wedge}(\mathbf{x}-\mathbf{y})\;\bar{\psi}_\sigma(\tau,\mathbf{x})\psi_\sigma(\tau,\mathbf{y})$$

and $Z=\int e^{\mathcal{V}(\psi,\bar{\psi})}\,e^{-\mathcal{K}(\psi,\bar{\psi};\delta e)}\,d\mu_C(\psi,\bar{\psi})$ is the partition function. The fields $\phi,\bar{\phi}$ are called source fields and the fields $\psi,\bar{\psi}$ are called internal fields. The connected Green's functions themselves are determined by

$$\mathcal{G}(\phi,\bar{\phi};\delta e)=\sum_{n=1}^{\infty}\tfrac{1}{(n!)^2}\int\prod_{i=1}^{n}dx_i dy_i\;G_{2n}(x_1,y_1,\cdots,x_n,y_n;\delta e)\prod_{i=1}^{n}\bar{\phi}(x_i)\phi(y_i).$$

Observe that for $\delta e\in\mathcal{E}$, formally,

$$\mathcal{G}(\phi,\bar{\phi};\delta e)=\log\tfrac{1}{Z'}\int e^{\phi J\psi}\,e^{\mathcal{V}(\psi,\bar{\psi})}\,d\mu_{C(\delta e)}(\psi,\bar{\psi}),$$

where $Z'=\int e^{\mathcal{V}(\psi,\bar{\psi})}d\mu_{C(\delta e)}(\psi,\bar{\psi})$. See [FKTo2, Lemma C.2].

We show in this paper that, for $d=2$ and under the Hypotheses I.12 on $e(\mathbf{k})$, there exists, for every sufficiently small interaction, a counterterm $\delta e\in\mathcal{E}$ such that connected Green's functions $G_{2n}(\cdot;\delta e)$ exist and have Fermi surface F. This statement needs to be made precise, because the functional integrals in the definition of $\mathcal{G}$ are not, a priori, well defined due to the singularities of the propagator. This problem is dealt with by a multiscale analysis.

We introduce scales by slicing momentum space into shells around the Fermi surface. We choose a "scale parameter" $M>1$ and a function $\nu\in C_0^\infty([\frac{1}{M},2M])$ that takes values in $[0,1]$, is identically 1 on $[\frac{2}{M},M]$ and obeys

$$\sum_{j=0}^{\infty}\nu\big(M^{2j}x\big)=1$$

for $0<x<1$ (see also [FKTo2, §VIII]). The function ν may be constructed by choosing a function $\varphi\in C_0^\infty\big((-2,2)\big)$ that is identically one on $[-1,1]$ and setting $\nu(x)=\varphi(x/M)-\varphi(Mx)$ for $x>0$ and zero otherwise.

Definition I.2.

i) For $j\ge 1$, the j^{th} scale function on $\mathbb{R}\times\mathbb{R}^d$ is defined as

$$\nu^{(j)}(k)=\nu\left(M^{2j}(k_0^2+e(\mathbf{k})^2)\right).$$

By construction, $\nu^{(j)}$ is identically one on

$$\Big\{\,k=(k_0,\mathbf{k})\in\mathbb{R}\times\mathbb{R}^d\;\Big|\;\sqrt{\tfrac{2}{M}}\,\tfrac{1}{M^j}\le|ik_0-e(\mathbf{k})|\le\sqrt{M}\,\tfrac{1}{M^j}\,\Big\}.$$

The support of $\nu^{(j)}$ is called the j^{th} shell. By construction, it is contained in

$$\Big\{\,k\in\mathbb{R}\times\mathbb{R}^d\;\Big|\;\tfrac{1}{\sqrt{M}}\,\tfrac{1}{M^j}\le|ik_0-e(\mathbf{k})|\le\sqrt{2M}\,\tfrac{1}{M^j}\,\Big\}.$$

The momentum k is said to be of scale j if k lies in the j^{th} shell.

ii) For real $j \geq 1$, set

$$\nu^{(\geq j)}(k) = \varphi\big(M^{2j-1}(k_0^2 + e(\mathbf{k})^2)\big).$$

By construction, $\nu^{(\geq j)}$ is identically 1 on

$$\Big\{\, k \in \mathbb{R} \times \mathbb{R}^d \;\Big|\; |ik_0 - e(\mathbf{k})| \leq \sqrt{M}\tfrac{1}{M^j} \,\Big\}.$$

Observe that if j is an integer, then for $|ik_0 - e(\mathbf{k})| > 0$,

$$\nu^{(\geq j)}(k) = \sum_{i \geq j} \nu^{(i)}(k).$$

The support of $\nu^{(\geq j)}$ is called the j^{th} neighbourhood of the Fermi surface. By construction, it is contained in

$$\Big\{\, k \in \mathbb{R} \times \mathbb{R}^d \;\Big|\; |ik_0 - e(\mathbf{k})| \leq \sqrt{2M}\tfrac{1}{M^j} \,\Big\}.$$

The support of $\varphi\big(M^{2j-2}(k_0^2 + e(\mathbf{k})^2)\big)$ is called the j^{th} extended neighbourhood. It is contained in

$$\Big\{\, k \in \mathbb{R} \times \mathbb{R}^d \;\Big|\; |ik_0 - e(\mathbf{k})| \leq \sqrt{2}M\tfrac{1}{M^j} \,\Big\}.$$

iii) For real $\jmath \geq 1$, the covariance with infrared cutoff at scale $\jmath$ and counterterm δe is

$$C^{\mathrm{IR}(\jmath)}(k_0, \mathbf{k}; \delta e) = \frac{U(\mathbf{k}) - \nu^{(\geq \jmath)}(k)}{\imath k_0 - e(\mathbf{k}) + \delta e(\mathbf{k})[1 - \nu^{(\geq \jmath)}(k)]},$$

$$C^{\mathrm{IR}(\jmath)}(x, x'; \delta e) = \delta_{\sigma,\sigma'} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{\imath <k, x-x'>_-} C^{\mathrm{IR}(\jmath)}(k; \delta e).$$

Also write

$$C^{\mathrm{IR}(\jmath)}(k) = C^{\mathrm{IR}(\jmath)}(k; 0) \qquad , \qquad C^{\mathrm{IR}(\jmath)}(x, x') = C^{\mathrm{IR}(\jmath)}(x, x'; 0).$$

The Grassmann Gaussian measure $d\mu_{C^{\mathrm{IR}(\jmath)}(\delta e)}$ is characterized by

$$\int \psi(x)\bar\psi(x')\, d\mu_{C^{\mathrm{IR}(\jmath)}(\delta e)}(\psi, \bar\psi) = C^{\mathrm{IR}(\jmath)}(x, x'; \delta e),$$

$$\int \psi(x)\psi(x')\, d\mu_{C^{\mathrm{IR}(\jmath)}(\delta e)}(\psi, \bar\psi) = \int \bar\psi(x)\bar\psi(x')\, d\mu_{C^{\mathrm{IR}(\jmath)}(\delta e)}(\psi, \bar\psi) = 0.$$

Remark I.3.
i) As the scale parameter $M > 1$, the shells near the Fermi curve have $\jmath$ near $+\infty$, and the neighbourhoods shrink as $\jmath \to \infty$. Also,

$$\lim_{\jmath \to \infty} C^{\mathrm{IR}(\jmath)}(k) = C(k)$$

pointwise.

ii) Our ultraviolet cutoff $U(\mathbf{k})$ cuts off spatial directions only and does not restrict k_0. The entire ultraviolet regime, that is, large k_0, will be treated in a first step. To do so, we pick an integer $j_0 \geq 2$ and integrate out a covariance containing the factor $U(\mathbf{k}) - \nu^{(>j_0)}(k)$. See Theorem V.8.

Even for the cutoff, and hence bounded, covariance $C^{\mathrm{IR}(j)}$, it is not a priori clear that the generating functional

$$\begin{aligned}\mathcal{G}_j(\phi,\bar\phi;\mathcal{V},\delta e) &= \log\tfrac{1}{Z}\int e^{\phi J\psi}\,e^{\mathcal{V}(\psi,\bar\psi)}\,d\mu_{C^{\mathrm{IR}(j)}(\delta e)}(\psi,\bar\psi),\\ \text{where}\quad Z &= \int e^{\lambda\mathcal{V}(\psi,\bar\psi)}\,d\mu_{C^{\mathrm{IR}(j)}(\delta e)}(\psi,\bar\psi),\end{aligned}$$

or the corresponding connected Green's functions, $G_{2n;j}(x_1,y_1,\cdots,x_n,y_n)$, defined by

$$\mathcal{G}_j(\phi,\bar\phi;\mathcal{V},\delta e)=\sum_{n=1}^{\infty}\tfrac{1}{(n!)^2}\int\prod_{i=1}^{n}dx_i dy_i\;G_{2n;j}(x_1,y_1,\cdots,x_n,y_n)\prod_{i=1}^{n}\bar\phi(x_i)\phi(y_i),$$

make sense for a reasonable set of V's and δe's. On the other hand, it is easy to see, using graphs, that each term in the formal Taylor expansion of the Grassmann function[2] $\mathcal{G}_j(\phi,\bar\phi;\mathcal{V},\delta e(V))$ in powers of V is well–defined for a large class of V's and $\delta e(V)$'s. The Taylor expansion of $\int e^{\phi J\psi}\,e^{\mathcal{V}(\psi,\bar\psi)}\,d\mu_{C^{\mathrm{IR}(j)}}(\psi,\bar\psi)$ is $\sum_{n=1}^{\infty}\mathcal{G}_{j,n}(\mathcal{V},\ldots,\mathcal{V})$, where the n^{th} term is the multilinear form

$$\mathcal{G}_{j,n}(\mathcal{V}_1,\cdots,\mathcal{V}_n)=\tfrac{1}{n!}\int e^{\phi J\psi}\mathcal{V}_1(\psi,\bar\psi)\cdots\mathcal{V}_n(\psi,\bar\psi)\,d\mu_{C^{\mathrm{IR}(j)}}(\psi,\bar\psi)$$

restricted to the diagonal. Explicit evaluation of the Grassmann integral expresses $\mathcal{G}_{j,n}$ as the sum of all graphs with vertices $\mathcal{V}_1,\ \cdots,\ \mathcal{V}_n$ and lines $C^{\mathrm{IR}(j)}$. The (formal) Taylor coefficient $\frac{d}{dt_1}\cdots\frac{d}{dt_n}\mathcal{G}_j(\phi,\bar\phi;t_1\mathcal{V}_1+\cdots+t_n\mathcal{V}_n,0)\Big|_{t_1=\cdots=t_n=0}$ of $\mathcal{G}_j(\phi,\bar\phi;\mathcal{V},0)$ is similar, but with only connected graphs. Choosing δe to be an appropriate function of V produces renormalized connected graphs[3]. We prove here that, under suitable hypotheses, for each j, the renormalized formal Taylor series for $\mathcal{G}_j\big(\phi,\bar\phi;\mathcal{V},\delta e(V)\big)$ converges to an analytic[4] function of V with a radius of convergence that is independent of j. We further show that the limit as $j\to\infty$ exists.

Theorem I.4. *Assume that $d=2$ and that $e(\mathbf{k})$ fulfills the Hypotheses I.12 below. There is an open ball, centered on the origin, in the Banach space of translation invariant and spin independent interaction kernels V with norm*

$$\max_{\substack{\delta_{i,j}\in\mathbb{N}_0\times\mathbb{N}_0^d\\ \text{for }1\le i<j\le 4\\ \Sigma|\boldsymbol{\delta}_{i,j}|\le r,\ \Sigma|\delta_{i,j;0}|\le r_0}}\left|\!\left|\!\left|\prod_{1\le i<j\le 4}\tfrac{1}{\delta_{i,j}!}(x_i-x_j)^{\delta_{i,j}}V(x_1,x_2,x_3,x_4)\right|\!\right|\!\right|_{1,\infty},$$

and an analytic counterterm function $V\mapsto\delta e(V)\in\mathcal{E}$ on the ball, that vanishes for $V=0$, such that the following holds:
For any real $j\ge 1$, the formal Taylor series

$$\mathcal{G}_j(\phi,\bar\phi)=\log\tfrac{1}{Z}\int e^{\phi J\psi}\,e^{\lambda\mathcal{V}(\psi,\bar\psi)}\,d\mu_{C^{\mathrm{IR}(j)}(\delta e(V))}(\psi,\bar\psi)$$

---

[2] We shall, in Def. VI.7, introduce a norm on the Grassmann algebra generated by ϕ and $\bar\phi$. All of our generating functionals will in fact have finite norm.

[3] Under the hypotheses of Theorem I.4, below, when j is finite, both the propagator $C^{\mathrm{IR}(j)}$ and the vertex V are continuous in momentum space. The values of all connected graphs, whether renormalized or not, are well–defined. However renormalization is essential for the limit $j\to\infty$.

[4] For an elementary discussion of analytic maps between Banach spaces see, for example, Appendix A of [PT].

converges to an analytic function on the ball. As $\jmath \to \infty$, the Green's functions $G_{2n;\jmath}$ converge uniformly, in $x_1, \cdots, y_n$ and V, to a translation invariant, spin independent, particle number conserving function G_{2n} on $\big(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\big)^{2n}$ that is analytic in V. If, in addition, V is k_0–reversal real, as in (I.1)*, then $\delta e(\mathbf{k}; V)$ is real for all $\mathbf{k}$.*

The proof of Theorem I.4 follows the statement of Theorem VIII.5 in [FKTf2].

Theorem I.5. *Under the hypotheses of Theorem I.4 and the assumption that V obeys the symmetries* (I.1) *and* (I.2)*, the Fourier transform*

$$\begin{aligned}\check{G}_2(k_0,\mathbf{k}) &= \int dx_0 d^d\mathbf{x}\ e^{\imath(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\ G_2\big((0,0,\uparrow),(x_0,\mathbf{x},\uparrow)\big)\\ &= \int dx_0 d^d\mathbf{x}\ e^{\imath(-k_0x_0+\mathbf{k}\cdot\mathbf{x})}\ G_2\big((0,0,\downarrow),(x_0,\mathbf{x},\downarrow)\big)\end{aligned}$$

of the two–point function exists and is continuous, except on the Fermi surface (precisely, except when $k_0 = 0$ and $e(\mathbf{k}) = 0$). Define

$$n(\mathbf{k}) = \lim_{\tau\to 0+}\int \tfrac{dk_0}{2\pi}\ e^{\imath k_0\tau}\ \check{G}_2(k_0,\mathbf{k}).$$

Then $n(\mathbf{k})$ is continuous except on the Fermi surface F. If $\bar{\mathbf{k}}\in F$, then $\lim\limits_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} n(\mathbf{k})$ and $\lim\limits_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} n(\mathbf{k})$ exist and obey

$$\lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})<0}} n(\mathbf{k}) - \lim_{\substack{\mathbf{k}\to\bar{\mathbf{k}}\\ e(\mathbf{k})>0}} n(\mathbf{k}) > \tfrac{1}{2}.$$

The proof of Theorem I.5 follows Lemma XII.4 in [FKTf3].

Remark I.6. The quantity $n(\mathbf{k})$ is known as the momentum distribution function. The jump discontinuity in $n(\mathbf{k})$ at the Fermi surface exhibited in Theorem I.5 is generally viewed as the most basic characteristic of a Fermi liquid [MCD, §4.1]. The number $\frac{1}{2}$ in the bound of I.5 has no special significance. It may be replaced by any number strictly smaller than one, provided the interaction is made sufficiently weak.

Theorem I.7. *Let*

$$\begin{aligned}\check{G}_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1,k_2,k_3) = \int & G_4\big(x_1,x_2,x_3,(0,0,\sigma_4)\big)\\ &\times\prod_{\ell=1}^{3} e^{-(-1)^\ell\imath<k_\ell,x_\ell>_-}\ dx_{0,\ell}d^d\mathbf{x}_\ell\end{aligned}$$

be the Fourier transform of the four–point function and

$$\check{G}^A_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1,k_2,k_3) = \check{G}_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1,k_2,k_3)\prod_{\ell=1}^{4}\tfrac{1}{\check{G}_2(k_\ell)}\Big|_{k_4=k_1-k_2+k_3}$$

its amputation by the physical propagator. Under the hypotheses of Theorem I.5, $\check{G}_4^A$ is continuous on

$$\Big\{ (k_1, k_2, k_3) \ \Big|\ k_1 \neq k_2,\ k_2 \neq k_3, \\ U(\mathbf{k}_1) = U(\mathbf{k}_2) = U(\mathbf{k}_3) = U(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3) = 1 \Big\}.$$

Furthermore, $\check{G}_4^A$ has a decomposition

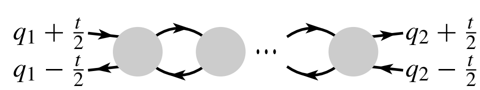
$$\begin{aligned} \check{G}^A_{4;\sigma_1,\sigma_2,\sigma_3,\sigma_4}(k_1, k_2, k_3) = N_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}&(k_1, k_2, k_3) \\ &+\tfrac{1}{2} L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\big(\tfrac{k_1+k_2}{2}, \tfrac{k_3+k_4}{2}, k_2 - k_1\big)\Big|_{k_4=k_1-k_2+k_3} \\ &-\tfrac{1}{2} L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\big(\tfrac{k_3+k_2}{2}, \tfrac{k_1+k_4}{2}, k_2 - k_3\big)\Big|_{k_4=k_1-k_2+k_3} \end{aligned}$$

with

- ∘ $N_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ *continuous*
- ∘ $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, t)$ *continuous except at* $t = 0$
- ∘ $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, (0, \mathbf{t}))$ *having an extension to* $\mathbf{t} = 0$ *which is continuous in* $(q_1, q_2, \mathbf{t})$
- ∘ $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, (t_0, \mathbf{0}))$ *having an extension to* $t_0 = 0$ *which is continuous in* (q_1, q_2, t_0).

The proof of Theorem I.7 is at the end of §XV in [FKTf3].

Remark I.8. $L_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(q_1, q_2, t)$ consists of particle–hole ladder contributions of the form



Remark I.9. Theorem I.4 defines $G_{2n;\underline{j}}(V)$ through its renormalized perturbation expansion. Alternatively, one could introduce an additional finite volume cutoff by replacing the space–time $\mathbb{R} \times \mathbb{R}^d$ with $(\mathbb{R}/L\mathbb{Z}) \times (\mathbb{R}^d/L\mathbb{Z}^d)$. The associated Green's functions, $G_{2n;\underline{j},L}(V)$, are trivially meromorphic functions of V that are analytic at $V = 0$. The methods of these papers could be used to show that the domain of analyticity of $G_{2n;\underline{j},L}(V)$ includes the ball of Theorem I.4 and that, as $L \to \infty$, $G_{2n;\underline{j},L}$ converges uniformly to $G_{2n;\underline{j}}$. We do not do so.

We also do not deal with the question of independence of the renormalization prescription. Suppose that $e(\mathbf{k})$, $\delta e(\mathbf{k}; V)$ and $e'(\mathbf{k})$, $\delta e'(\mathbf{k}; V)$ satisfy the appropriate regularity conditions and that $e(\mathbf{k}) - \delta e(\mathbf{k}; V) = e'(\mathbf{k}) - \delta e'(\mathbf{k}; V)$, for some specific V. Observe that $G_{2n;\underline{j}}$ depends on $e(\mathbf{k})$ and $\delta e(\mathbf{k}; V)$ only through the combinations $e(\mathbf{k}) - \delta e(\mathbf{k}; V)[1 - \nu^{(\geq \underline{j})}(k)]$ and $\nu^{(\geq \underline{j})}(k) = \varphi\big(M^{2j-1}(k_0^2 + e(\mathbf{k})^2)\big)$. Hence, it is likely that the limiting Green's functions constructed using $e(\mathbf{k})$, $\delta e(\mathbf{k}; V)$ coincide with those constructed using $e'(\mathbf{k})$, $\delta e'(\mathbf{k}; V)$, for the specific V, but we do not attempt to prove so here.

We now state the hypotheses on the dispersion relation $e(\mathbf{k})$ used in Theorems I.4, I.5 and I.7. First, we assume that the dispersion relation $e(\mathbf{k})$ is $r+6$ times continuously differentiable. Furthermore, we assume that the Fermi curve

$$F = \left\{ \mathbf{k} \in \mathbb{R}^2 \,\middle|\, e(\mathbf{k}) = 0 \right\}$$

is a strictly convex, smooth connected curve with curvature bounded away from zero. Since F is strictly convex, for each point $\mathbf{k} \in F$ there is a unique point $a(\mathbf{k}) \in F$ different from $\mathbf{k}$ such that the tangent lines to F at $\mathbf{k}$ and $a(\mathbf{k})$ are parallel. $a(\mathbf{k})$ is called the antipode of $\mathbf{k}$. Choose an orientation for F.

Definition I.10.

i) Let $\mathbf{k} \in F$, $\vec{\mathrm{t}}$ the oriented unit tangent vector to F at $\mathbf{k}$ and $\vec{\mathrm{n}}$ the inward pointing unit normal vector to F at $\mathbf{k}$. Then there is a function $\varphi_{\mathbf{k}}(s)$, defined on a neighbourhood of 0 in $\mathbb{R}$, such that $s \mapsto \mathbf{k} + s\vec{\mathrm{t}} + \varphi_{\mathbf{k}}(s)\vec{\mathrm{n}}$ is an oriented parametrization of F near $\mathbf{k}$.

ii) We say that F is strongly asymmetric if there is $n_0 \in \mathbb{N}$, with $n_0 \le r$, such that for each $\mathbf{k} \in F$ there exists an $n \le n_0$ with

$$\varphi_{\mathbf{k}}^{(n)}(0) \neq \varphi_{a(\mathbf{k})}^{(n)}(0).$$

Remark I.11.

i) By construction, $\varphi_{\mathbf{k}}(0) = \dot{\varphi}_{\mathbf{k}}(0) = 0$ and $\ddot{\varphi}_{\mathbf{k}}(0)$ is the curvature of F at $\mathbf{k}$.

ii) If F is symmetric about a point $\mathbf{p} \in \mathbb{R}^2$, that is $F = \{2\mathbf{p} - \mathbf{k} \,|\, \mathbf{k} \in F\}$, then $\varphi_{\mathbf{k}} = \varphi_{a(\mathbf{k})}$ for all $\mathbf{k} \in F$. Symmetry of the Fermi curve about a point promotes the formation of Cooper pairs and the phase transition to a superconducting state. Theorem I.4 shows that – at temperature zero – this is the only instability in a broad class of short range many fermion models, at least when $d = 2$. Sufficiently high temperature also blocks the Cooper instability and leads to Fermi liquid behaviour, even for a round Fermi surface. This was shown in [DR1, DR2] using the criterion proposed in [S]. See also [PS].

When $d = 1$, fermionic many–body models exhibit Luttinger liquid rather than Fermi liquid behaviour. See Chapter 11 of [BG] and the references therein. We would expect that results like Theorems I.4 and I.5 also hold for $d = 3$. There has been some progress in this direction [MR, DMR].

iii) In [FKTa] we show that independent fermions in a suitably chosen periodic electromagnetic background field have a dispersion relation whose associated Fermi curve, for suitably chosen chemical potential, is smooth, strictly convex, strongly asymmetric and has nonzero curvature everywhere.

Hypothesis I.12 (on the dispersion relation). *We assume that $e(\mathbf{k})$ is $r+6$ times continuously differentiable with $r \ge 6$, that the Fermi curve F is a strictly convex, smooth, strongly asymmetric, connected curve whose curvature is bounded away from zero and that $\nabla e(\mathbf{k})$ does not vanish on F.*

This paper is divided into three parts. This first part, which consists of Sects. I through III and Appendix A, contains an overview of the main ideas involved in the construction (§II) and the algebraic structure of the construction (§III). Part 2, [FKTf2], consists of Sect. IV through X and Appendix B and contains the proof of Theorem I.4 — using the results of the auxiliary papers [FKTl, FKTo1, FKTo2, FKTo3, FKTo4 and FKTr1, FKTr2]. The existence of a discontinuity in the particle number density at the Fermi surface, stated in Theorem I.5, is proven in the third part, [FKTf3]. Continuity properties of the amputated four point Green's functions, stated in Theorem I.7, are also proven in [FKTf3].

Our proof uses a multiscale analysis and discrete renormalization group flow, in the framework of fermionic functional integrals, and also uses renormalization of the Fermi surface. In the $j^{\rm th}$ step of the renormalization group flow we turn on interactions amongst particles whose momenta have distance from the Fermi surface about $\frac{1}{M^j}$. The output of the $j^{\rm th}$ step is an effective interaction for particles whose momenta have distance from the Fermi surface less than $\frac{1}{M^j}$. In [FKTr1] we build a tool for controlling Gaussian fermionic functional integrals of exponentials of effective interactions in a quite abstract setting[5], using Gram bounds for determinants. In [FKTo3, FKTo4] this tool is combined with a technique developed specifically to handle problems created by the fact that the singularity of the original propagator lies on an object with nontrivial geometry, the Fermi surface. The technique is "sectorization of the Fermi surface". It allows us to use the abstract results of [FKTr1] while retaining the ability to exploit conservation of momentum. In particular, conservation of momentum leads to improved power counting, due to "overlapping loops", for the two point function and the non–ladder contributions to the four point function. This mechanism is formulated abstractly in [FKTr2] and implemented concretely in [FKTo3]. Generalized particle–particle and particle–hole ladder diagrams require special treatment. Particle–particle ladders have improved power counting due to the assumed asymmetry of the Fermi surface[6]. This is proven in §XXII of [FKTo4]. The estimate on the sum of all contributions from particle–hole ladders exploits a sign cancellation between scales and is developed in [FKTl].

The treatment of the first renormalization group step, i.e. the ultraviolet regime, is a simple application of the expansion of [FKTr1] and appears in [FKTo2]. Control of the change of sectorization forced by the change of scales between renormalization group steps is achieved in [FKTo4].

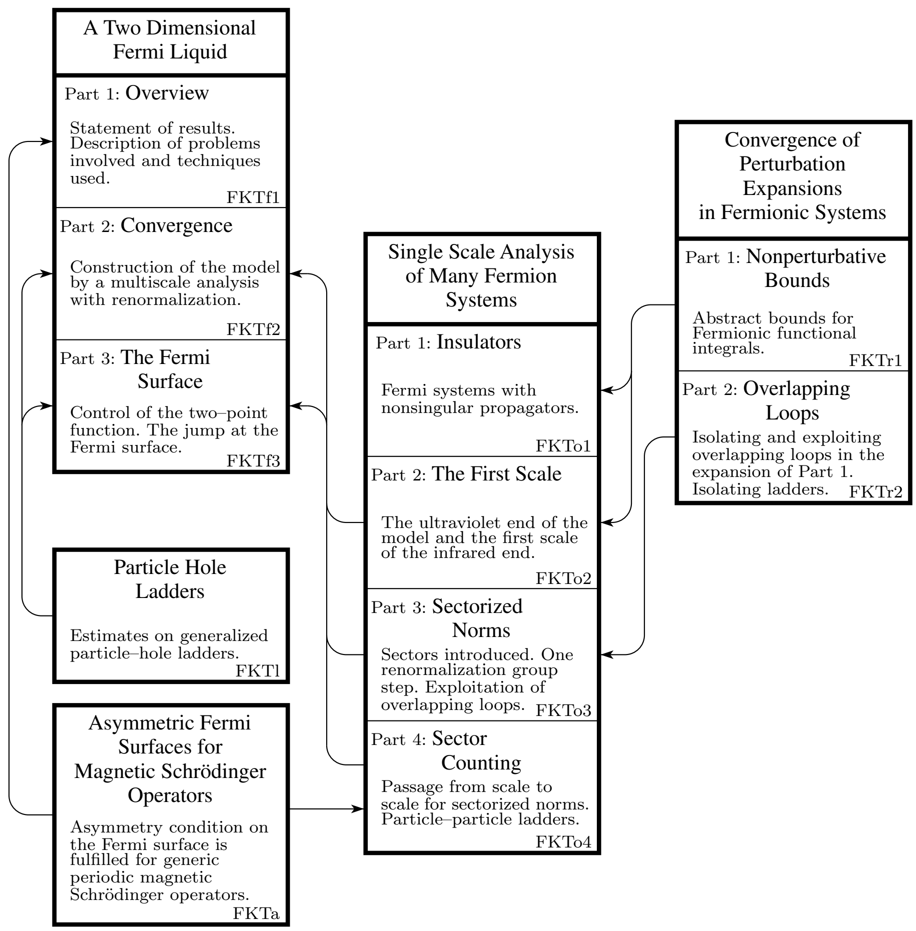
As mentioned above, §II and III of this paper give an overview of the construction. The paper [FKTo1], where nonsingular propagators (as is the case for insulators) are considered, can also serve as an introduction to some of the techniques. The set of eleven papers is self–contained. [FKTr1+r2], [FKTl] and [FKTo4] are each reasonably independent of the other papers in the series. The interface between each of these and the other papers is through a relatively small number of definitions and theorems. The flow chart below provides a thumbnail sketch of the contents of each of the papers and gives the logical connections between them.

Cumulative notation tables are provided at the end of each part. The construction described in these papers was outlined in [FKLT1, FKLT2, FKLT3].

[5] We expect that the results of [FKTr1] may be usable beyond the present problem and so may be of more general interest.

[6] The fact that Schrödinger operators with periodic magnetic potentials usually have such asymmetric Fermi surfaces is proven in the auxiliary paper [FKTa].

1. Flow Chart for the 2–d Fermi Liquid Construction



II. An Overview

In this section, we describe the main difficulties in the proof of Theorems I.4 and I.5 and outline our strategy to overcome them. For simplicity, we omit spins in this discussion. The notation in this section is close, but not always identical to that used in the rest of this paper. As the main theorems are for $d = 2$ space dimensions, we describe all constructions only for $d = 2$, even those that can be extended to other dimensions.

1. Renormalization of the Fermi Surface and the Dispersion Relation The Fermi surface of a fermionic many particle system is the locus in momentum space where the two point function $\check{G}_2(0, \mathbf{k})$ has a discontinuity – if such a discontinuity occurs at all.[7] For a system of non-interacting fermions, the Fermi surface coincides with the zero–set of the dispersion relation. In a metal or a crystal, the dispersion relation is a datum derived

[7] For example, in a superconductor there is no such discontinuity.

from first principles; it is determined by the associated periodic Schrödinger operator. On the other hand, the Fermi surface of the system of interacting fermions is accessible to measurement. See, for example, [AM].

As already mentioned in §I, the Fermi surface of a system of interacting fermions is, in general, different from that of the system of non-interacting fermions with the same dispersion relation. This shift in the Fermi surface is responsible for the divergence of many coefficients in naive perturbation expansions. It is controlled by renormalizing the dispersion relation. Theorems I.4 and I.5 state that, given a function $e(\mathbf{k})$ and an interaction V fulfilling all of the theorems' hypotheses, there is a function $\delta e(\mathbf{k})$, called the "counterterm", such that the system with dispersion relation $e(\mathbf{k})-\delta e(\mathbf{k})$ and interaction V has a Fermi surface and that Fermi surface is precisely $F = \{\mathbf{k}\big|\ e(\mathbf{k}) = 0\ \}$.

We pointed out in the previous paragraph that the data derived from first principles are the dispersion relation and the interaction V. Therefore it is desirable to prove that every reasonable function $e'(\mathbf{k})$ is of the form $e(\mathbf{k}) - \delta e(\mathbf{k})$ as above. This could be done by proving the invertibility of the map $e(\mathbf{k}) \mapsto e(\mathbf{k}) - \delta e(\mathbf{k})$ in an appropriate function space. To all orders in perturbation theory, this has been achieved in [FST4]. The bounds of this paper are not yet strong enough to prove the corresponding result non-perturbatively.

Even for C^∞ functions $e(\mathbf{k})$ and V, it is not known how smooth the counterterm $\delta e(\mathbf{k})$ is. In this paper, we show that δe is C^ϵ. In [FST3] it is shown that δe is $C^{2+\epsilon}$ to all orders in perturbation theory. Later in this overview (Subsect. 10) we shall point out where this lack of smoothness in the counterterm creates difficulties for the construction.

2. Multi Scale Analysis We cannot treat the functional integral (I.4) defining the formal Green's functions in one piece, because the propagator is singular. Similarly it is probably impossible to determine the counterterm $\delta e(\mathbf{k})$ in one step. Therefore we introduce scales adjusted to the size of the propagator in momentum space and to the infrared cut off propagators $C^{\mathrm{IR}(j)}(k_0, \mathbf{k}; \delta e)$ of Definition I.4, construct an appropriate counterterm for each scale j and take the limit $j \to \infty$. The limit is controlled by comparing, for each j, the model with covariance $C^{\mathrm{IR}(j+1)}(k_0, \mathbf{k}; \delta e)$ to that with covariance $C^{\mathrm{IR}(j)}(k_0, \mathbf{k}; \delta e)$. This comparison amounts to "integrating out scale j". We give an introduction to "integrating out a scale" in the next subsection. In §III, we describe, formally but in more detail, how the limit $j \to \infty$ is taken.

3. Integrating Out a Scale The discussion of the previous subsection shows that the essential estimates in our construction concern the effect of integrating out one scale as above. To simplify the discussion, we consider the case in which the covariance $C^{\mathrm{IR}(j)}(k; \delta e)$ is replaced by $C^{\mathrm{IR}(j)}(k; 0)$. That is, for the moment, we ignore the effect of the counterterm. Since $C^{\mathrm{IR}(j+1)}(k; 0) = C^{\mathrm{IR}(j)}(k; 0)+C^{(j)}(k)$ with $C^{(j)}(k) = \frac{\nu^{(j)}(k)}{\imath k_0-e(\mathbf{k})}$,

$$\begin{aligned}\mathcal{G}_{j+1}(\phi, \bar\phi) &= \log \tfrac{1}{Z'} \int\int e^{\phi J(\psi+\zeta)+\mathcal{V}(\psi+\zeta,\bar\psi+\bar\zeta)} d\mu_{C^{\mathrm{IR}(j)}}(\zeta, \bar\zeta) d\mu_{C^{(j)}}(\psi, \bar\psi)\\ &= \log \tfrac{1}{Z} \int e^{\phi J\psi+\mathcal{W}(\phi,\bar\phi,\psi,\bar\psi)} d\mu_{C^{(j)}}(\psi, \bar\psi),\end{aligned}$$

where

$$\mathcal{W}(\phi, \bar\phi, \psi, \bar\psi) = \log \tfrac{1}{Z_j} \int e^{\phi J\zeta+\mathcal{V}(\psi+\zeta,\bar\psi+\bar\zeta)} d\mu_{C^{\mathrm{IR}(j)}}(\zeta, \bar\zeta)$$

is the effective interaction at scale j. The partition functions Z, Z' and Z_j are chosen so that $\mathcal{G}_{j+1}(0,0) = \mathcal{W}(0,0,0,0) = 0$. To iterate this construction, we need an effective interaction

$$\mathcal{W}'(\phi,\bar\phi,\psi,\bar\psi) = \log \tfrac{1}{Z_{j+1}} \int e^{\phi J\zeta + \mathcal{V}(\psi+\zeta,\bar\psi+\bar\zeta)} d\mu_{C^{\mathrm{IR}(j+1)}}(\zeta,\bar\zeta)$$

at scale $j+1$. So the problem is to estimate

$$\mathcal{W}'(\phi,\bar\phi,\psi,\bar\psi) = \log \tfrac{1}{Z} \int e^{\phi J\zeta + \mathcal{W}(\phi,\bar\phi,\psi+\zeta,\bar\psi+\bar\zeta)} d\mu_{C^{(j)}}(\zeta,\bar\zeta)$$

in terms of estimates on $\mathcal{W}$. The main difficulties already occur when $\phi = \bar\phi = 0$, so we concentrate on this special case. Write

$$\mathcal{W}(0,0,\psi,\bar\psi) = \sum_{n\ge 0} \int dp_1\cdots dp_n\, dq_1\cdots dq_n\; w_{2n}(p_1,\cdots,p_n,q_1,\cdots,q_n)$$
$$\delta(p_1+\cdots+p_n-q_1-\cdots-q_n)\bar\psi(p_1)\cdots\bar\psi(p_n)\,\psi(q_1)\cdots\psi(q_n)$$

and

$$\mathcal{W}'(0,0,\psi,\bar\psi) = \sum_{n\ge 0} \int dp_1\cdots dp_n\, dq_1\cdots dq_n\; w'_{2n}(p_1,\cdots,p_n,q_1,\cdots,q_n)$$
$$\delta(p_1+\cdots+p_n-q_1-\cdots-q_n)\bar\psi(p_1)\cdots\bar\psi(p_n)\,\psi(q_1)\cdots\psi(q_n),$$

where $\psi(k)$ and $\bar\psi(k)$ are the Fourier transforms of $\psi(x)$ and $\bar\psi(x)$ respectively[8].

Then w'_{2n} can be written as a sum of values of connected directed graphs with vertices $w_2, w_4, \cdots$ and propagator $C^{(j)}(k)$. See [FW, Chap. 3]. Naive power counting just uses that

$$\|C^{(j)}(k)\|_\infty = \sup_k |C^{(j)}(k)| \text{ is of order } M^j \tag{II.1}$$

and

$$\text{volume of the } j^{\text{th}} \text{ shell is of order } \tfrac{1}{M^{2j}}. \tag{II.2}$$

This is because the j^{th} shell has width of order $\frac{1}{M^j}$ in the k_0 direction and in the $\mathbf{k}$–direction transversal to F and has circumference, in the direction along F, of order one. If one assumes that

$$\|w_{2n}\|_\infty \text{ is of order } M^{j(n-2)} \text{ for all } n \tag{II.3}$$

then every graph contributing to w'_{2n} is again of order

$$M^{j\Sigma_i(n_i-2)} M^{j\Sigma_i(2n_i-2n)/2} M^{-2j[\Sigma_i(2n_i-2n)/2-\Sigma_i 1+1]} = M^{j(n-2)}.$$

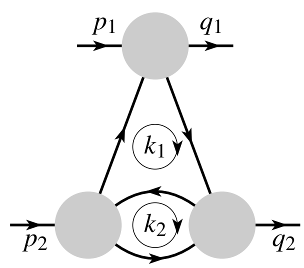
The three factors come from the suprema of the vertex functions w_{2n_i}, the suprema of the $\frac{\Sigma_i 2n_i-2n}{2}$ propagators and the volume of the domain of integration respectively. For $n\ge 3$ the Condition (II.3) grows nicely in j. Four legged vertices ($n=2$) are marginal — the estimate (II.3) alone would lead to divergences in powers of j when the sum

[8] Precise Fourier transform conventions are formulated in §VI.

over j is performed. We exploit "overlapping loops" and special estimates on ladders to derive better bounds on the four point functions. The counterterm is built up from contributions at each scale chosen so that (II.3) holds for n=1, too. Overlapping loops will be discussed in Subsect. 4, ladders in Subsect. 5 and 6, and the construction of the counterterm in Subsect. 10.

It is well known that the sum of the norms of all graphs diverges. In this paper we use cancellations between different graphs to derive convergent bounds. There are several schemes to implement such cancellations, all variants of the basic scheme of [C]. In all of these schemes the cancellation is only seen when one writes the model in position space variables. This is a major technical difficulty, since the geometry of the Fermi surface and special effects like "overlapping loops" and ladder estimates are naturally formulated in momentum space. We use the cancellation scheme developed in [FMRT, FKTcf, FKTr1, FKTr2]. It is sufficiently close to the graphical picture to allow us to implement overlapping loops in collections of diagrams within which cancellations take place.

4. Overlapping Loops We first describe the effect of overlapping loops in an example. Consider the diagram



contributing to $w'_4(p_1, p_2, q_1, q_2)$. If $w_4 = 1$, its value is

$$\Gamma(p_1, p_2, q_1, q_2) = \int dk_1\, dk_2\, C^{(j)}(k_1+q_1-p_1)\, C^{(j)}(k_1)\, C^{(j)}(k_1-k_2)\, C^{(j)}(q_2-k_2).$$

Naive power counting gives

$$\|\Gamma\|_\infty \le \|C^{(j)}\|_\infty^4 \cdot (\text{volume of the } j^{\text{th}} \text{ shell})^2 = O(1).$$

However, taking into account that $C^{(j)}$ is supported on the j^{th} shell,

$$\|\Gamma\|_\infty \le \|C^{(j)}\|_\infty^4 \cdot (\text{volume of } T),$$

where

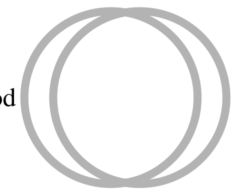
$$T = \{(k_1, k_2) \,\big|\, k_1, k_1 - k_2, q_2 - k_2 \in j^{\text{th}} \text{ neighbourhood}\}. \tag{II.4}$$

Let S be the set of all k_2 for which $q_2 - k_2$ lies in the j^{th} neighbourhood and S' be the set of all k_2 for which $|k_2| \ge \frac{1}{\sqrt{M^j}}$. Then $T \subset T_1 \cup T_2$, where

$$T_1 = \{(k_1, k_2) \,\big|\, k_1, k_1 - k_2 \in j^{\text{th}} \text{ neighbourhood},\ k_2 \in S \cap S'\},$$
$$T_2 = \{(k_1, k_2) \,\big|\, k_1 \in j^{\text{th}} \text{ neighbourhood},\ k_2 \in S \setminus S'\}.$$

For $k_2 \in S'$ the volume of the set

$$\{k_1 \,\big|\, k_1, k_1 - k_2 \in j^{\text{th}} \text{ neighbourhood}\} = \big(j^{\text{th}} \text{ neighbourhood}\big) \cap \big(k_2 + j^{\text{th}} \text{ neighbourhood}\big)$$

j^{th} neighbourhood  $k_2 + j^{\text{th}}$ neighbourhood

is of order $\frac{1}{M^{2j}} \cdot \frac{1}{\sqrt{M^j}}$, since this set has width of order $\frac{1}{M^j}$ in the k_0 direction and in the $\mathbf{k}$ direction transversal to F and width at most of order $\frac{1}{\sqrt{M^j}}$ in the direction along F. Here we use that the Fermi surface F has curvature bounded away from zero. Therefore the volume of T_1 is of order

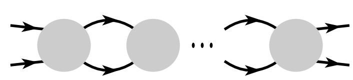
$$\tfrac{1}{M^{5j/2}} \cdot (\text{volume of } S) = \tfrac{1}{M^{5j/2}} \cdot O\big(\tfrac{1}{M^{2j}}\big) = O\big(\tfrac{1}{M^{9j/2}}\big).$$

Similarly the volume of T_2 is bounded by

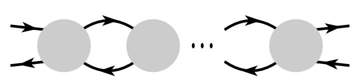
$$(\text{volume of } j^{\text{th}} \text{ neighbourhood}) \cdot (\text{volume of } S \setminus S') = O(\tfrac{1}{M^{2j}}) \cdot \tfrac{1}{M^{5j/2}} = O(\tfrac{1}{M^{9j/2}}).$$

Therefore, using (II.1), $\|\Gamma\|_\infty = O(\frac{1}{\sqrt{M^j}})$. The volume estimate on T derived above is not optimal. By [FST2, Theorem 1.1], the volume of T is $O\big(\frac{j}{M^j}\big)$.

Similar improvements are possible for all diagrams with "overlapping loops", i.e. with two different simple loops which have at least one line in common. If $w_2 = 0$, the only four legged diagrams without overlapping loops and tadpoles are particle–particle ladders



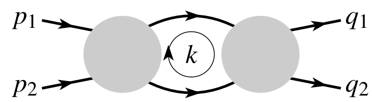
and particle–hole ladders



See [FST1, §2.4]. We shall Wick order the effective interactions with respect to the future covariance in order to exclude tadpoles. The condition that $w_2 = 0$ is achieved by moving the two legged part of the effective interaction into the covariance.

We have already mentioned that the effect of overlapping loops and special ladder estimates are used to get convergent bounds on the four point functions. The effect of overlapping loops has to be combined with the cancellation scheme between diagrams mentioned at the end of Subsect. 3 and discussed in Subsect. 9 below. Thus the cancellation scheme has to be sensitive enough to detect simultaneous overlapping loops in all mutually cancelling diagrams and to isolate ladder diagrams. Furthermore the geometric estimates on T above have to be exploitable in a position space setting. This is done using sectors. See Subsect. 8.

5. Particle–Particle Bubbles The strong asymmetry condition of Definition I.10 is used to get improved power counting on particle–particle ladders. We describe the effect in the example of the particle–particle bubble



again assuming that $w_4 = 1$. The value of this graph is

$$\int dk\, C^{(j)}(t-k)\, C^{(j)}(k),$$

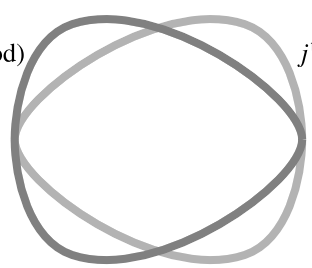
where $t = p_1 + p_2 = q_1 + q_2$ is the transfer momentum. Naive power counting again gives that this value is $O(1)$. On the other hand, taking the support condition of $C^{(j)}$ into account, the value of this graph is bounded by

$$\|C^{(j)}\|_\infty^2 \cdot (\text{volume of } \{k \mid k \text{ and } t-k \text{ lie in the } j^{\text{th}} \text{ neighbourhood}\}.$$

By the strong asymmetry condition of Definition I.10, F and the shifted reflected Fermi surface $t - F = \{t - k \mid k \in F\}$ have tangency of order at most n_0. From this one deduces that

$$\{k \mid k,\ t-k \in j^{\text{th}} \text{ neighbourhood}\} = \big(j^{\text{th}} \text{ neighbourhood}\big) \cap \big(t - (j^{\text{th}} \text{ neighbourhood})\big)$$

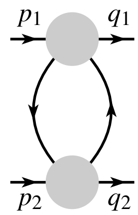
has width of order $\frac{1}{M^j}$ in the k_0 direction and in the $\mathbf{k}$ direction transversal to F and width at most of order $\frac{1}{M^{j/n_0}}$ in the direction along F. Therefore its volume is of order



$\frac{1}{M^{2j}} \frac{1}{M^{j/n_0}}$ and the value of the particle–particle bubble is of order $\frac{1}{M^{j/n_0}}$.

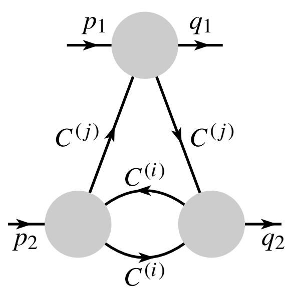
As in the case of overlapping loops, this estimate is based on a volume estimate in momentum space. Again sectors will used to implement this in position space variables.

6. Particle–Hole Ladders Our estimate on particle–hole ladders is not based on a geometric argument as in the case of particle–particle ladders or overlapping loops, but on cancellations between scales. The limit as $j \to \infty$ of the particle–hole bubble $B_j(p_1, p_2, q_1, q_2)$



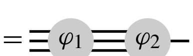
with $w_4 = 1$ and propagator $\sum_{i \le j} C^{(i)}$ has a discontinuity in the transfer momentum $t = p_1 - q_1$ at $t = 0$, but is continuous for $t \neq 0$ and smooth in a neighbourhood of the origin in $\{(t_0, \mathbf{t}) \,\big|\, t_0 = 0\}$. See the introduction to [FKTl], and in particular Lemma I.1 there. The proof of this lemma is based on integration by parts and thus cancellation between scales.

In the multi-scale analysis, we combine all the contributions of particle–hole ladders created at scales $\le j$, and give a uniform estimate on the result. A vertex of a particle–hole ladder at scale j may be a particle–hole ladder created at a previous scale $i < j$, as in the diagram:

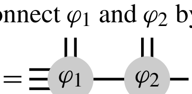


The iteration of these effects (and of the Wick ordering with respect to future covariances) leads to the concept of iterated particle–hole ladders of Definition VII.7. Uniform estimates on these iterated particle–hole ladders are stated in Theorem VII.8 and proven in [FKTl]. They have to be in position space, as they have to be combined with the other estimates derived from the "cancellation scheme" mentioned in Subsect. 3. This is technically difficult, because it amounts to taking Fourier transforms of quantities whose limit, as $j \to \infty$, is discontinuous.

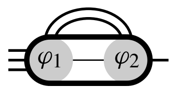
7. Power Counting in Position Space Proving power counting bounds on graphs is usually split into two steps. To illustrate the two steps, we consider the situation that we have two vertices φ_1 and φ_2 with n_1 and n_2 legs, respectively. For simplicity we ignore orientation of the lines. We assume we form a diagram Γ by connecting φ_1 and φ_2 by $1 \le r \le \min\{n_1, n_2\}$ lines.

$\Gamma =$ 

This can be done in two steps: First connect φ_1 and φ_2 by one line and call the result Γ'.

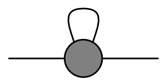
$\Gamma' =$ 

Secondly, pairwise contract $2r-2$ legs of Γ' to form $r-1$ lines.



The first estimate is to bound the norm of Γ' in terms of a constant times the product of the norms of φ_1 and φ_2. We call such a constant a "contraction bound" c. If one uses $\|\cdot\|_\infty$ norms in momentum space, then $\|C^{(j)}(k)\|_\infty$ is a contraction bound.

A tadpole bound is a number b with the following property. Let φ be any graph with at least two legs, and φ' the graph obtained from φ by connecting two legs to form a line.



Then the norm of φ' is bounded by b times the norm of φ. If one uses $\|\cdot\|_\infty$ norms in momentum space, then $\|C^{(j)}(k)\|_1$ is a tadpole bound.

Applying one contraction bound and $r-1$ tadpole bounds, one sees that the norm of Γ is bounded by $c\,b^{r-1}$ times the product of the norms of φ_1 and φ_2. By (II.1) and (II.2), for the $\|\cdot\|_\infty$ norms in momentum space, the contraction bound is of order M^j, while the tadpole bound is of order $\frac{1}{M^j}$. The power counting for the $\|\ \cdot\ \|_\infty$ norm in momentum space described in Subsect. 3 can be generalized to the abstract setting of contraction and tadpole bounds: If one assumes that

$$\text{the norm of } w_{2n} \text{ is of order } \tfrac{1}{cb^{n-1}} \text{ for all } n \tag{II.5}$$

then every graph contributing to w'_{2n} is again of order $\frac{1}{cb^{n-1}}$. For example, if such a graph Γ has two vertices, w_{2n_1} and w_{2n_2}, then there are $r=n_1+n_2-n$ connecting lines and the norm of Γ is bounded by

$$\begin{aligned} cb^{n_1+n_2-n-1}\big(\text{norm of } w_{2n_1}\big)\big(\text{norm of } w_{2n_2}\big) &= O\big(cb^{n_1+n_2-n-1}\tfrac{1}{cb^{n_1-1}}\tfrac{1}{cb^{n_2-1}}\big) \\ &= O\big(\tfrac{1}{cb^{n-1}}\big). \end{aligned}$$

A general graph may be bounded by building it up one vertex at a time.

As pointed out in Subsect. 3, we need to use position space variables. A natural position space norm for translation invariant vertices $\varphi(x_1,\cdots,x_n)$, which mimics the $\|\cdot\|_\infty$ norm in momentum space, is

$$|||\varphi|||_{1,\infty} = \max_{1\le p\le n}\ \sup_{x_p} \int \prod_{\substack{j=1,\cdots,n\\ j\ne p}} dx_j\ |\varphi(x_1,\cdots,x_n)|.$$

It is easily seen that the L^1 norm, $\|C^{(j)}(x)\|_1$, of the Fourier transform of $C^{(j)}(k)$ is a contraction bound for this norm and that the L^∞ norm of $C^{(j)}$ in position space, $\|C^{(j)}(x)\|_\infty$, is a tadpole bound. Clearly, $\|C^{(j)}(x)\|_\infty \le \|C^{(j)}(k)\|_1$, so that we again have a tadpole bound of order $\frac{1}{M^j}$. A naive computation, given in the next paragraph, gives a bound on $\|C^{(j)}(x)\|_1$ that is of order M^{2j}. A more refined argument, sketched in the next but one paragraph, gives a (realistic – see Example A.1) bound of order $M^{3j/2}$. In any event, $M^{3j/2} \gg \|C^{(j)}(k)\|_\infty$ and naive power counting in position space does not coincide with power counting in momentum space. Substituting $c=O\big(M^{3j/2}\big)$ and

$b = O\big(\frac{1}{M^j}\big)$ into (II.5) yields the requirement that $|||w_{2n}|||_{1,\infty}$ be order $M^{j(n-\frac{5}{2})}$. In particular the norm of the four point function would have to decrease like $\frac{1}{\sqrt{M^j}}$ as j increased. This is absurd, since the original interaction V is, at each scale, the dominant part of the four point function. Again, we use sectors to cope with this problem.

We first sketch the standard calculation that gives the naive bound on $\|C^{(j)}(x)\|_1$. For a multi-index $\delta = (\delta_0, \delta_1, \delta_2)$ of non-negative integers write $|\delta| = \delta_0 + \delta_1 + \delta_2$ and $x^\delta = x_0^{\delta_0} x_1^{\delta_1} x_2^{\delta_2}$. Then, integrating by parts $|\delta|$ times,

$$\big(\tfrac{x}{M^j}\big)^\delta |C^{(j)}(x)| \le \tfrac{1}{M^{j|\delta|}} \| \tfrac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)\|_1 = O(\tfrac{1}{M^j}),$$

since the support of $\frac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)$ has volume of order $\frac{1}{M^{2j}}$ and $\|\frac{\partial^{|\delta|}}{\partial k^\delta} C^{(j)}(k)\|_\infty$ is of order $M^{j(|\delta|+1)}$. Therefore

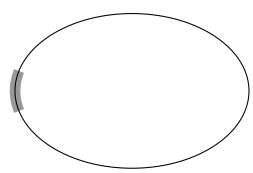
$$\Big(1 + \big(\tfrac{x_0}{M^j}\big)^2\Big)\Big(1 + \big(\tfrac{x_1}{M^j}\big)^2\Big)\Big(1 + \big(\tfrac{x_2}{M^j}\big)^2\Big)|C^{(j)}(x)| = O(\tfrac{1}{M^j}).$$

Dividing by $\prod_{\nu=0,1,2}\Big(1 + \big(\frac{x_\nu}{M^j}\big)^2\Big)$ and integrating over $\mathbb{R}^3$ gives the bound $\|C^{(j)}(x)\|_1 = O(M^{2j})$.

As it motivates the construction of sectors, we indicate how the more refined bound on $\|C^{(j)}(x)\|_1$ is derived. Assume first that F has a straight segment of length $\mathfrak{l}$ on the k_2 axis[9] and that $e(k) = k_1$ in a neighbourhood of this segment. Choose a cutoff function $\chi(k_2)$ that is identically one on most of the segment, zero outside the segment and for which $|\frac{\partial^n}{\partial k_2^n}\chi(k_2)|$ is of order $\frac{1}{\mathfrak{l}^n}$. Set $C_s^{(j)}(k) = \chi(k_2)C^{(j)}(k)$. An argument similar to that of the previous paragraph shows that

$$\Big(1 + \big(\tfrac{x_0}{M^j}\big)^2\Big)\Big(1 + \big(\tfrac{x_1}{M^j}\big)^2\Big)\Big(1 + (\mathfrak{l}x_2)^2\Big)|C_s^{(j)}(x)| = O(\tfrac{\mathfrak{l}}{M^j})$$

and therefore that $\|C_s^{(j)}(x)\|_1 = O(M^j)$. The same argument also works for a realistic Fermi surface, if one cuts out a "sector" of length $\mathfrak{l} \le \frac{1}{\sqrt{M^j}}$ as indicated in the figure below.



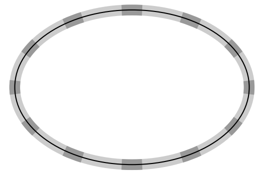
The precise computation is in [FKTo3, Prop. XIII.1 and Lemma XIII.2]. If the sector is too long, the curvature of the Fermi surface causes deterioration of the bounds on the derivatives parallel to the Fermi surface. One can divide up the j^{th} neighbourhood into $O(\frac{1}{\mathfrak{l}})$ sectors and use a partition of unity by functions like χ, to see that

$$\|C^{(j)}(x)\|_1 \le (\text{number of sectors}) \cdot O(M^j) = O(\tfrac{1}{\mathfrak{l}} M^j).$$

If one chooses $\mathfrak{l} = \frac{1}{\sqrt{M^j}}$, one gets the bound $\|C^{(j)}(x)\|_1 = O(M^{3j/2})$.

[9] This of course contradicts our hypotheses that F is strictly convex. We will remove this assumption immediately.

8. Sectors We cover the j^{th} neighbourhood by slightly overlapping sectors of length $\mathfrak{l} \le \frac{1}{\sqrt{M^j}}$ as indicated in the figure below.



The set Σ of sectors is called a sectorization of scale j and length $\mathfrak{l}$. Furthermore we select a partition of unity $\{\chi_s(k) \,\big|\, s \in \Sigma\}$ subordinate to the sectorization, such that each χ_s has properties analogous to those of the function χ in the last subsection[10]. Recall that we want to integrate out scale j and that

$$\mathcal{W}'(0,0,\psi,\bar\psi) = \tfrac{1}{Z}\int e^{\mathcal{W}(0,0,\psi+\zeta,\bar\psi+\bar\zeta)}d\mu_{C^{(j)}}(\zeta,\bar\zeta).$$

Decompose the kernel w_{2n} of the part of $\mathcal{W}$ that is homogeneous of degree $2n$ in the fields

$$w_{2n}(p_1,\cdots,p_n,q_1,\cdots,q_n) = \sum_{s_1,\cdots,s_{2n}\in\Sigma} \omega_{2n}\big((p_1,s_1),\cdots,(p_n,s_n),(q_1,s_{n+1}),\cdots,(q_n,s_{2n})\big)$$

for p_i, q_i in the j^{th} neighbourhood. Here, ω_{2n} is a function that vanishes unless p_i lies in the sector s_i and q_i lies in the sector s_{n+i}. ω_{2n} is called a Σ–sectorized representative of w_{2n}. A Σ–sectorized representative ω'_{2n} of w'_{2n} can then be written as a sum of values of connected directed graphs with vertices $\omega_2, \omega_4, \cdots$ and propagator $C^{(j)}(k)$, where the momentum integrals in the graphs also include sector sums. The main norm we use for the Fourier transforms $\hat\omega_{2n}\big((x_1,s_1),\cdots,(x_{2n},s_{2n})\big)$ of ω_{2n} is

$$|||\hat\omega_{2n}|||_{1,\Sigma} = \max_{1\le i_0\le 2n}\ \max_{s_{i_0}\in\Sigma} \sum_{\substack{s_i\in\Sigma \text{ for}\\ i\ne i_0}} |||\hat\omega_{2n}\big((x_1,s_1),\cdots,(x_{2n},s_{2n})\big)|||_{1,\infty}. \tag{II.6}$$

That is, we first fix one sector and then take the sum over all other sectors of the $|||\cdot|||_{1,\infty}$ norms of the functions $\hat\omega_{2n}\big((\cdot,s_1),\cdots,(\cdot,s_{2n})\big)$. With respect to this norm, we have a contraction bound of order M^j and a tadpole bound of order $\frac{\mathfrak{l}}{M^j}$.

We first indicate how the contraction bound is derived. Let $\varphi_1\big((x_1,s_1),\cdots,(x_n,s_n)\big)$ and $\varphi_2\big((x_1,s_1),\cdots,(x_m,s_m)\big)$ be vertices, and

$$\begin{aligned}\Gamma'\ &\big((x_1,s_1),\cdots,(x_{n-1},s_{n-1}),(x'_2,s'_2),\cdots,(x'_m,s'_m)\big)\\ &= \sum_{s_n,s'_1\in\Sigma}\int dx_n\,dx'_1\ \varphi_1\big((x_1,s_1),\cdots,(x_n,s_n)\big)\,C^{(j)}(x_n-x'_1)\,\varphi_2\big((x'_1,s'_1),\cdots,(x'_m,s'_m)\big)\end{aligned}$$

be the graph constructed by connecting φ_1 and φ_2 with one line. Write $C^{(j)}(k) = \sum_{s\in\Sigma} C^{(j)}_s(k)$, where $C^{(j)}_s(k) = \chi_s(k)\,C^{(j)}(k)$. Let $C^{(j)}_s(x)$ be the Fourier transform of $C^{(j)}_s(k)$. Then

[10] For precise definitions of sectors, sectorizations, and the partition of unity see Def. VI.2 through Def. VI.5.

$$\Gamma' = \sum_{s_n,s,s'_1\in\Sigma} \int dx_n\, dx'_1\; \varphi_1\big((x_1,s_1),\cdots,(x_n,s_n)\big)\, C_s^{(j)}(x_n-x'_1)\, \varphi_2\big((x'_1,s'_1),\cdots,(x'_m,s'_m)\big). \tag{II.7}$$

By conservation of momentum, the integral in (II.7) vanishes if $s_n \cap s \cap s'_1 = \emptyset$. For fixed sectors $s_1, \cdots, s_n,\ s'_1, \cdots, s'_m,\ s$, by double convolution

$$\begin{aligned} &\Big|\Big|\Big| \int dx_n\, dx'_1\; \varphi_1\big((x_1,s_1),\cdots,(x_n,s_n)\big)\, C_s^{(j)}(x_n-x'_1)\, \varphi_2\big((x'_1,s'_1),\cdots,(x'_m,s'_m)\big) \Big|\Big|\Big|_{1,\infty} \\ &\quad\le |||\varphi_1\big((\cdot,s_1),\cdots,(\cdot,s_n)\big)|||_{1,\infty}\, \|C_s^{(j)}(x)\|_1\, |||\varphi_2\big((\cdot,s'_1),\cdots,(\cdot,s'_m)\big)|||_{1,\infty} \end{aligned} \tag{II.8}$$

We consider the contribution to $|||\Gamma'|||_{1,\Sigma}$ having the sector s_1 fixed. By (II.7), (II.8) and conservation of momentum it is bounded by

$$\sum_{\substack{s_2,\cdots,s_{n-1}\in\Sigma\\ s'_2,\cdots,s'_m\in\Sigma}} \ \sum_{\substack{s_n,s,s'_1\in\Sigma\\ s_n\cap s\cap s'_1\neq\emptyset}} |||\varphi_1\big((\cdot,s_1),\cdots,(\cdot,s_n)\big)|||_{1,\infty}\, \|C_s^{(j)}(x)\|_1\, |||\varphi_2\big((\cdot,s'_1),\cdots,(\cdot,s'_m)\big)|||_{1,\infty}. \tag{II.9}$$

Observe that, for given s_n, there are at most three sectors s and at most three sectors s'_1 with $s_n \cap s \neq \emptyset,\ \ s_n \cap s'_1 \neq \emptyset$. Therefore (II.9) is bounded by

$$\begin{aligned} &9 \max_{s,s'_1\in\Sigma} \sum_{\substack{s_2,\cdots,s_{n-1},s_n\in\Sigma\\ s'_2,\cdots,s'_m\in\Sigma}} |||\varphi_1\big((\cdot,s_1),\cdots,(\cdot,s_n)\big)|||_{1,\infty}\, \|C_s^{(j)}(x)\|_1\, |||\varphi_2\big(\cdot,s'_1),\cdots,\cdot,s'_m)\big)|||_{1,\infty} \\ &\qquad\qquad \le 9\, |||\varphi_1|||_{1,\Sigma}\, |||\varphi_2|||_{1,\Sigma}\ \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_1. \end{aligned}$$

Fixing s_i with $2 \le i \le n-1$ or s'_i with $2 \le i \le m$ leads to the same bound, so that

$$|||\Gamma'|||_{1,\Sigma} \le 9\, |||\varphi_1|||_{1,\Sigma}\, |||\varphi_2|||_{1,\Sigma}\ \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_1. \tag{II.10}$$

As at the end of the previous subsection, one sees that $\max_{s\in\Sigma} \|C_s^{(j)}(x)\|_1 = O(M^j)$. This gives the contraction bound of order M^j.

To derive the tadpole bound, let $\varphi((x_1,s_1),\cdots,(x_n,s_n))$ be a vertex and

$$\begin{aligned} \Gamma\ &\big((x_1,s_1),\cdots,(x_{n-2},s_{n-2})\big) \\ &= \sum_{s_{n-1},s_n\in\Sigma} \int dx_{n-1}\, dx_n\; \varphi\big((x_1,s_1),\cdots,(x_{n-2},s_{n-2}),(x_{n-1},s_{n-1}),(x_n,s_n)\big)\, C^{(j)}(x_{n-1}-x_n) \end{aligned}$$

be obtained by joining the last two legs of φ to form a tadpole. As above, by conservation of momentum, for each choice of sectors $s_1, \cdots, s_{n-2}$,

$$\begin{aligned} &|||\Gamma\big((\cdot,s_1),\cdots,(\cdot,s_{n-2})\big)|||_{1,\infty} \\ &\quad\le \sum_{\substack{s_{n-1},s,s_n\in\Sigma\\ s_{n-1}\cap s\cap s_n\neq\emptyset}} \Big|\Big|\Big| \int dx_{n-1}\, dx_n\; \varphi\big(\cdot,s_1),\cdots,(\cdot,s_{n-2}),(x_{n-1},s_{n-1}),(x_n,s_n)\big)\, C_s^{(j)}(x_{n-1}-x_n) \Big|\Big|\Big|_{1,\infty} \\ &\quad\le 3 \sum_{s_{n-1},s_n\in\Sigma} |||\varphi\big((\cdot,s_1),\cdots,(\cdot,s_{n-2}),(\cdot,s_{n-1}),(\cdot,s_n)\big)|||_{1,\infty}\ \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_\infty \end{aligned}$$

as, for a given sector s_{n-1}, there are at most three sectors s for which $s \cap s_{n-1} \neq \emptyset$. Consequently

$$|||\Gamma|||_{1,\Sigma} \le 3\, |||\varphi|||_{1,\Sigma} \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_\infty .$$

As in the previous subsection, $\|C_s^{(j)}(x)\|_\infty = O\big(\frac{\mathfrak{l}}{M^j}\big)$ and the tadpole bound of order $\frac{\mathfrak{l}}{M^j}$ follows.

Substituting $c = O\big(M^j\big)$ and $b = O\big(\frac{\mathfrak{l}}{M^j}\big)$ into (II.5) yields the requirement that

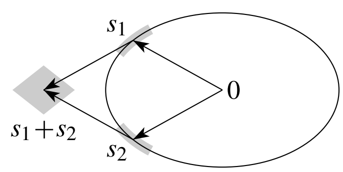
$$|||\hat\omega_{2n}|||_{1,\Sigma} \;\text{ is of order }\; \tfrac{M^{j(n-2)}}{\mathfrak{l}^{n-1}} \;\text{ for all } n. \tag{II.11}$$

In contrast to the norm $|||\cdot|||_{1,\infty}$ of Subsect. 7, this norm is compatible with change of scale. In §VI, we choose, for each scale i, a sectorization Σ_i of length $\mathfrak{l}_i$, where $\mathfrak{l}_i$ goes to zero as some power of $\frac{1}{M^i}$[11]. In going from scale j to scale $j+1$, we construct from a Σ_j–sectorized representative ω'_{2n} of w'_{2n} a Σ_{j+1}–sectorized representative ω''_{2n} of w'_{2n} that fulfills (II.11) with j replaced by $j+1$ and $\mathfrak{l}$ replaced by $\mathfrak{l}_{j+1}$. It is constructed using a partition of unity subordinate to Σ_{j+1}. See Example A.3.

To give an idea of the underlying mechanism, we show that the problem with the $|||\cdot|||_{1,\infty}$ norm of the four point function described in Subsect. 7 does not occur for the $|||\cdot|||_{1,\Sigma_j}$ norm. To do so, we need a $\Sigma = \Sigma_j$–sectorized representative for the original interaction kernel $V(p_1, p_2, q_1, q_2)$ whose $|||\cdot|||_{1,\Sigma}$ norm is of order $\frac{1}{\mathfrak{l}}$. A natural such representative is

$$v\big((p_1,s_1),(p_2,s_2),\ (q_1,s_3),(q_2,s_4)\big) = \chi_{s_1}(p_1)\,\chi_{s_2}(p_2)\,\chi_{s_3}(q_1)\,\chi_{s_4}(q_2) \\ \times V(p_1,p_2,q_1,q_2).$$

By conservation of momentum, the Fourier transform $\hat v\big((\cdot,s_1),(\cdot,s_2),\ (\cdot,s_3),(\cdot,s_4)\big)$ vanishes if $(s_1+s_2)\cap(s_3+s_4)=\emptyset$. Here, $s_1+s_2 = \{p_1+p_2 \,\big|\, p_1\in s_1,\ p_2\in s_2\}$.



One sees, by the same method that yielded $\|C_s(x)\|_1 = O(M^j)$, that the Fourier transform of each $\chi_s(k)$ fulfills $\|\hat\chi_s(x)\|_1 = O(1)$. Therefore, for each choice of sectors s_1, s_2, s_3, s_4,

$$|||\hat v\big((\cdot,s_1),(\cdot,s_2),\ (\cdot,s_3),(\cdot,s_4)\big)|||_{1,\infty} = O\big(|||\hat V|||_{1,\infty}\big).$$

Therefore, $|||\hat v|||_{1,\Sigma}$ is of the order

$$\max_{s_1\in\Sigma} \sharp\big\{\ (s_2,s_3,s_4)\in\Sigma^3 \ \big|\ (s_1+s_2)\cap(s_3+s_4)\neq\emptyset\ \big\}. \tag{II.12}$$

To estimate this number, fix a sector s_1. Observe that the map

$$F\times F \to \mathbb{R}^2,\ (k_1,k_2)\mapsto k_1+k_2 \tag{II.13}$$

[11] This is forced on us by the condition $\mathfrak{l}_i \le \frac{1}{\sqrt{M^i}}$ which is imposed to ensure that the curvature of the sector does not affect Fourier transform estimates.

is locally invertible at every $(k_1, k_2) \in F \times F$ for which the tangent vector to F at k_1 is not parallel to the tangent vector to F at k_2. From this one concludes that for a general choice of s_2 (such that $s_1 + s_2$ is not close to $2k$ or $k + \mathrm{antipode}(k)$ for all $k \in F$), there are only $O(1)$ choices of sectors s_3, s_4 such that $(s_1+s_2) \cap (s_3+s_4) \neq \emptyset$. Since there are $O\big(\frac{1}{\mathfrak{l}}\big)$ sectors, the contribution to (II.12) from all general sectors s_2 is $O\big(\frac{1}{\mathfrak{l}}\big)$. One can see that the few other sectors s_2 also only contribute $O\big(\frac{1}{\mathfrak{l}}\big)$. Thus, indeed $|||\hat{v}|||_{1,\Sigma} = O\big(\frac{1}{\mathfrak{l}}\big)$. The precise argument is given in [FKTo4, Prop. XIX.1].

The estimates on the contraction and tadpole bounds and about change of sectorization in this subsection all are specific to two space dimensions. In three space dimensions, similar arguments would give that:

a contraction bound for $C^{(j)}$ is of order M^j,
a tadpole bound for $C^{(j)}$ is of order $\frac{\mathfrak{l}^2}{M^j}$,
$|||\hat{v}|||_{1,\Sigma} = O\big(\frac{1}{\mathfrak{l}^3}\big)$, since the map (II.13) has one dimensional fibers in this case.

Thus, for the case of three space dimensions, the power counting suggested by sectors is not compatible with change of sectorization. This is the reason why, in this paper, we restrict ourselves to two space dimensions[12].

The advantage of sectors is that they allow exploitation of conservation of momentum, while working in position space. One example is the estimate on $|||\hat{v}|||_{1,\Sigma}$ derived two paragraphs ago. Another important example is that the improvements due to overlapping loops and particle–particle bubbles described in Subsect. 4 and 5 can be implemented using sectors. To this end we use a variant of the norm (II.6), with three sectors held fixed,

$$|||\hat{\omega}_{2n}|||_{3,\Sigma} = \max_{1\le i_1<i_2<i_3\le 2n} \ \max_{s_{i_1},s_{i_2},s_{i_3}\in\Sigma} \ \sum_{\substack{s_i\in\Sigma \text{ for} \\ i\neq i_1,i_2,i_3}} |||\hat{\omega}_{2n}((x_1,s_1),\cdots,(x_{2n},s_{2n}))|||_{1,\infty}. \tag{II.14}$$

Its power counting is better by one factor of $\mathfrak{l}$ than the power counting of the $|||\cdot|||_{1,\Sigma}$ norm. That is, we expect

$$|||\hat{\omega}_{2n}|||_{3,\Sigma} \text{ is of order } \Big(\frac{M^j}{\mathfrak{l}}\Big)^{n-2} \text{ for all } n, \tag{II.15}$$

and in particular that $|||\hat{\omega}_4|||_{3,\Sigma}$ is of order one[13]. This norm is particularly useful for the four point function. Since a given momentum can lie in at most two sectors,

$$\|\omega_4(p_1, p_2, q_1, q_2)\|_\infty \le 2^3 |||\hat{\omega}_4|||_{3,\Sigma}.$$

In Example A.2, we show how (II.11) and (II.15) can be used to get improved power counting for the $|||\cdot|||_{1,\Sigma}$ norm of the diagram discussed in Subsect. 4.

9. Cancellation Between Diagrams In Subsect. 7 and 8, we described how the power counting bound on a diagram Γ formed by connecting two vertices φ_1 and φ_2 by $r \ge 1$ lines[14] is obtained. First we connect φ_1 and φ_2 by one line and call the resulting graph

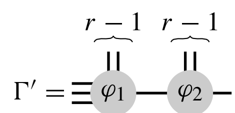
[12] Progress in the use of sectorization in three space dimensions has been made in [MR, DMR].
[13] The argument above that the sectorized representative v of the original interaction fulfills $|||\hat{v}|||_{1,\Sigma} = O\left(\frac{1}{\mathfrak{l}}\right)$ also shows that $|||\hat{v}|||_{3,\Sigma} = O(1)$.
[14] Again, for simplicity, we ignore orientation of the lines.

Γ' and apply a contraction bound. Then we form the remaining $r - 1$ lines, each time applying a tadpole bound. Iterating this procedure, adding vertex after vertex, leads to the power counting for arbitrary diagrams.

Since we are dealing with fermions, we may assume in our discussion that all vertex functions are antisymmetric. Let $n_1 \geq r$ and $n_2 \geq r$ be the number of legs of φ_1 and φ_2, respectively. Assume that $\max\{n_1, n_2\} > r$. Denote by $\mathcal{G}$ the set of all diagrams obtained from joining φ_1 and φ_2 by r lines. $\mathcal{G}$ has cardinality $\binom{n_1}{r}\binom{n_2}{r} r!$. If one bounds each individual diagram by power counting, and sums over all diagrams in $\mathcal{G}$, this large number of diagrams leads to divergences (due to the factor $r!$).

We first describe how one finds cancellations between diagrams of $\mathcal{G}$, then describe a blocking of diagrams that allows one to find similar cancellations for arbitrary numbers of vertices, and then show how, using this blocking, one may simultaneously exploit both these cancellations and overlapping loops.

The first step in constructing the diagrams of $\mathcal{G}$ is again to choose one leg of φ_1 and one leg of φ_2, form a line between these two legs, call the resulting graph Γ' and estimate the norm of Γ' in terms of the norms of φ_1 and φ_2 using a contraction bound. The second step is to choose $(r-1)$ additional legs of φ_1 and $(r-1)$ additional legs of φ_2.



The third step is to form all possible connections between the $(r-1)$ legs of φ_1 and the $(r-1)$ legs of φ_2 chosen in the second step. There are $n_1 n_2$ choices in the first step, $\binom{n_1-1}{r-1}\binom{n_2-1}{r-1}$ choices in the second step and $(r-1)!$ choices in the third step. Each diagram of $\mathcal{G}$ is obtained r times, since with the first step we distinguish one of the r lines. Observe that

$$\frac{1}{r}\, n_1 n_2 \binom{n_1-1}{r-1}\binom{n_2-1}{r-1}(r-1)! = \binom{n_1}{r}\binom{n_2}{r} r!$$

There are cancellations amongst the diagrams formed in the third step described above. For simplicity, assume that the $(r-1)$ legs of φ_1 chosen in the second step are labeled by $2, \cdots, r$ and are all incoming, and that the $(r-1)$ legs of φ_1 are labeled by $n_2 - r + 2, \cdots, n_2$ and are all outgoing. Then the sum of all diagrams formed in the third step is

$$\begin{aligned}&\int dx_2 \cdots dx_r\, dx'_{n_2-r+2} \cdots dx'_{n_2}\ \Gamma'(x_2, \cdots, x_r, \cdots x'_{n_2-r+2}, \cdots x'_{n_2})\\ &\times \int \bar{\psi}(x_2) \cdots \bar{\psi}(x_r)\, \psi(x'_{n_2-r+2}) \cdots \psi(x'_{n_2})\, d\mu_{C^{(j)}}.\end{aligned}$$

Its $\|\cdot\|_{1,\infty}$ norm is bounded by

$$\|\Gamma'\|_{1,\infty} \sup_{\substack{x_2, \cdots, x_r \\ x'_{n_2-r+2}, \cdots x'_{n_2}}} \left| \int \bar{\psi}(x_2) \cdots \bar{\psi}(x_r)\, \psi(x'_{n_2-r+2}) \cdots \psi(x'_{n_2})\, d\mu_{C^{(j)}} \right|.$$

The magnitude of the functional integral is

$$\left| \int \bar{\psi}(x_2) \cdots \bar{\psi}(x_r)\, \psi(x'_{n_2-r+2}) \cdots \psi(x'_{n_2})\, d\mu_{C^{(j)}} \right| = \left| \det \left[C^{(j)}(x_\mu - x'_{n_2-r+\nu}) \right]_{\mu,\nu=2,\cdots,r} \right|.$$

Observe that the μ–ν matrix entry

$$\begin{aligned} C^{(j)}(x_\mu - x'_{n_2-r+\nu}) &= \int dk\; e^{\imath<k,x_\mu - x'_{n_2-r+\nu}>_-} C^{(j)}(k) \\ &= \left\langle e^{\imath<k,x_\mu>_-}\sqrt{|C^{(j)}(k)|}\;,\; e^{\imath<k,x'_{n_2-r+\nu}>_-}\frac{C^{(j)}(k)}{\sqrt{|C^{(j)}(k)|}} \right\rangle_{L^2} \end{aligned}$$

(we are deliberately ignoring some unimportant factors of 2π) is the L^2 inner product of the vector $v_\mu(k) = e^{\imath<k,x_\mu>_-}\sqrt{|C^{(j)}(k)|}$ and the vector $v'_\nu(k) = e^{\imath<k,x'_{n_2-r+\nu}>_-}\frac{C^{(j)}(k)}{\sqrt{|C^{(j)}(k)|}}$. The vectors v_μ, v'_ν all have L^2 norm $\sqrt{\|C^{(j)}(k)\|_1}$. Therefore, by Gram's bound on determinants,

$$\Big| \int \psi(x_2)\cdot\cdots\cdot\bar\psi(x'_{n_2})\; d\mu_{C^{(j)}} \Big| \le \|C^{(j)}(k)\|_1^{r-1}.$$

Thus the bound on the sum of all diagrams formed at the third step is

$$|||\Gamma'|||_{1,\infty}\, \|C^{(j)}(k)\|_1^{r-1} \le c\, \|C^{(j)}(k)\|_1^{r-1}\, |||\varphi_1|||_{1,\infty}\, |||\varphi_2|||_{1,\infty},$$

where c is a contraction bound. Consequently, the $|||\cdot|||_{1,\infty}$ norms of the sum of all graphs in $\mathcal{G}$ is bounded by

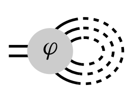
$$\tfrac{1}{r}\, n_1 n_2 \binom{n_1-1}{r-1}\binom{n_2-1}{r-1} \|C^{(j)}(x)\|_1\, \|C^{(j)}(k)\|_1^{r-1}\, |||\varphi_1|||_{1,\infty}\, |||\varphi_2|||_{1,\infty}$$

since, as seen in Subsect. 7, $\|C^{(j)}(x)\|_1$ is a contraction bound. This is smaller than the bound

$$(\sharp\mathcal{G})\, \|C^{(j)}(x)\|_1\, \|C^{(j)}(k)\|_1^{r-1}\, |||\varphi_1|||_{1,\infty}\, |||\varphi_2|||_{1,\infty}$$

derived by summing the graphwise bounds, by a factor of $\frac{1}{(r-1)!}$.

In general, we say that b is an integral bound, if the following holds: Let φ be any antisymmetric vertex, $2r \le n$, and $S(x_{2r+1}, \cdots, x_n)$ be the sum of the values of all diagrams obtained from φ by joining each of the legs labeled $1, \cdots, r$ to one and only one of the legs labeled $r+1, \cdots, 2r$.

$2r+1, \cdots \;, n\, \{$ 

Then the norm of S is bounded by b^{2r} times the norm of φ. The argument in the previous paragraph shows that $C^{(j)}$ has an integral bound with respect to the $|||\cdot|||_{1,\infty}$ norm that is of order $\sqrt{\|C^{(j)}(k)\|_1} = O(\frac{1}{\sqrt{M^j}})$. Similarly one sees that $C^{(j)}$ has an integral bound, with respect to the $|||\cdot|||_{1,\Sigma}$ norm, that is of order $O\Big(\sqrt{\frac{\mathfrak{l}}{M^j}}\Big)$. Thus, in both cases, the integral bound is of the order of the square root of the tadpole bound found before.

The discussion above shows how to get cancellations between diagrams that have only two vertices. The combinatorics for treating diagrams of arbitrary size was developed in [FMRT, FKTcf, FKTr1, FKTr2]. We sketch it here. As mentioned in Subsect. 4, we Wick order with respect to future covariances in order to avoid tadpoles. Thus, we write

$$\begin{aligned} \mathcal{W}(0,0,\psi,\bar\psi) &= {:}U(\psi,\bar\psi){:}_{C^{(\ge j)}}, \\ \mathcal{W}'(0,0,\psi,\bar\psi) &= {:}U'(\psi,\bar\psi){:}_{C^{(\ge j+1)}}, \end{aligned}$$

where $C^{(\geq j)} = \sum_{i\geq j} C^{(i)}$. Then the kernels of U' are sums of diagrams whose vertices are kernels of U and which have two kinds of lines. The first arises from integrating with respect to $d\mu_{C^{(j)}}$ and has propagator $C^{(j)}$. The second arises in Wick ordering $\mathcal{W}'$ and has propagator $C^{(\geq j+1)}$. The subgraph of each diagram obtained by deleting the $C^{(\geq j+1)}$ lines must be connected and tadpole–free. For clarity, we ignore the Wick lines. That is, we discuss a simplified situation in which $\mathcal{W}$ is Wick ordered with respect to $C^{(j)}$ and $\mathcal{W}'$ is not Wick ordered at all[15], so that we write

$$\begin{aligned}\mathcal{W}(0,0,\psi,\bar\psi) &= {:}\widetilde{\mathcal{W}}(\psi,\bar\psi){:}_{C^{(j)}}\\ r\widetilde{\mathcal{W}}(\psi,\bar\psi) &= \sum_{n\geq 0}\int dp_1\cdots dq_n\ \tilde w_{2n}(p_1,\cdots,q_n)\,\delta(p_1+\cdots+p_n-q_1-\cdots-q_n)\\ &\qquad\times\bar\psi(p_1)\cdots\bar\psi(p_n)\,\psi(q_1)\cdots\psi(q_n).\end{aligned}$$

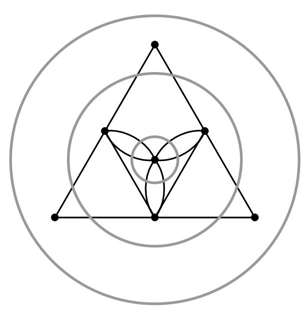
Then

$$w'_{2n} = \sum_\ell \frac{1}{\ell!}\left(\begin{matrix}\text{connected diagrams without tadpoles with}\\ 2n\text{ external legs and }\ell\text{ vertices from }\tilde w_2,\tilde w_4,\cdots\end{matrix}\right).$$

All diagrams are labeled. A rooted diagram is a diagram with one distinguished vertex, called the root. Clearly,

$$w'_{2n} = \sum_\ell \frac{1}{\ell!}\frac{1}{\ell}\left(\begin{matrix}\text{connected, rooted, tadpole–free diagrams with}\\ 2n\text{ external legs and }\ell\text{ vertices from }\tilde w_2,\tilde w_4,\cdots\end{matrix}\right).$$

The distance between two vertices in a diagram is the minimal number of lines needed to form a path connecting these two vertices. In a rooted diagram, the r^{th} ring is defined as the set of vertices of distance r from the root. Thus, the zero$^{\text{th}}$ ring is the root itself, and the first ring consists of all vertices that are directly connected to the root by a line. Observe that the full subgraph formed by the union of the first r rings of a diagram G is again a connected diagram G_r. Each leg emanating from a vertex of the r^{th} ring that is not part of a line in G_r (that is, each external leg of G_r) is either an external leg for the whole diagram or is connected to a vertex of the $(r+1)^{\text{st}}$ ring. Observe that, for each graph, there is an r_0 such that the r^{th} ring is empty for all $r \geq r_0$.



For simplicity, we only discuss the case that $n = 0$, i.e. that there are no external legs. Given a rooted diagram without external legs, we have the following combinatorial data:

- $\ell_r = \sharp(\text{vertices of the } r^{\text{th}} \text{ ring})$,

[15] General Wick ordering is treated in [FKTr2].

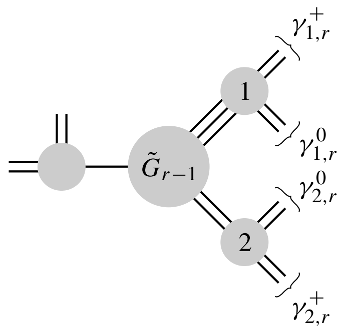
- For $i = 1, \cdots, \ell_r$, let $\begin{Bmatrix} \gamma_{i,r}^- \\ \gamma_{i,r}^0 \\ \gamma_{i,r}^+ \end{Bmatrix}$ be the number of legs of the i^{th} vertex in the r^{th} ring that are connected to a vertex in ring number $\begin{Bmatrix} r-1 \\ r \\ r+1 \end{Bmatrix}$.

By the definition of "ring", the i^{th} vertex in the r^{th} ring has $\gamma_{i,r}^- + \gamma_{i,r}^0 + \gamma_{i,r}^+$ legs and $\gamma_{i,r}^- \geq 1$ for all $1 \leq r \leq r_0$. The sequences $\vec{\ell} = (\ell_1, \ell_2, \cdots)$ and $\vec{\gamma} = (\gamma_{i,r}^-, \gamma_{i,r}^0, \gamma_{i,r}^+)_{\substack{r=1,2,\cdots \\ i=1,\cdots \ell_r}}$ are called the combinatorial data of the rooted graph.

We only exploit cancellations between connected rooted graphs with the same combinatorial data. So fix some combinatorial data $\vec{\ell}, \vec{\gamma}$ and let $\mathcal{G}$ be the set of all diagrams with these combinatorial data. Denote by $\tilde{G}_r$ the sum of the values of all subgraphs G_r of graphs $G \in \mathcal{G}$. View it as a single vertex. It has

$$\sum_{i=1}^{\ell_r} \gamma_{i,r}^+ = \sum_{i=1}^{\ell_{r+1}} \gamma_{i,r+1}^-$$

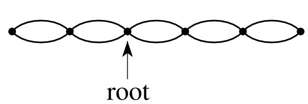
external legs. We describe how $\tilde{G}_r$ is formed from $\tilde{G}_{r-1}$ and how the norm of $\tilde{G}_r$ is bounded in terms of the norm of $\tilde{G}_{r-1}$: Connect each of the ℓ_r vertices of the r^{th} ring to $\tilde{G}_{r-1}$ by one line and apply contraction bounds. Then apply an integral bound for the $\sum_{i=1}^{\ell_r}(\gamma_{i,r}^- - 1)$ remaining connections between $\tilde{G}_{r-1}$ and the r^{th} ring.



Then, using a variant of the integral bound, form all connections between the $\gamma_{1,r}^0$ lines coming from the first vertex, the $\gamma_{2,r}^0$ lines coming from the second vertex, $\cdots$ and the $\gamma_{\ell_r,r}^0$ lines coming from the last vertex, always avoiding tadpoles. Repeat this procedure for all r for which $\ell_r \neq 0$. The bounds obtained in this way are summable over all combinatorial data.

Observe that ladders

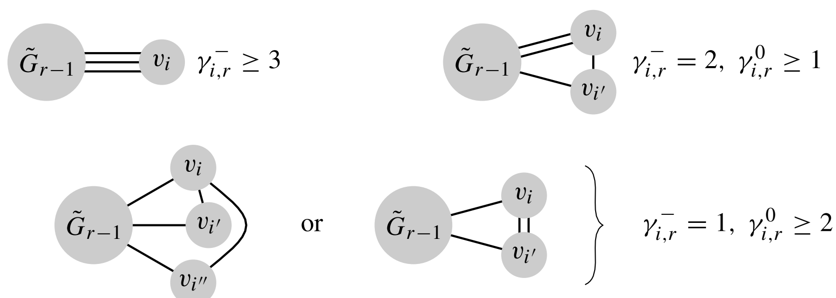
have very special combinatorial data: Each ℓ_r is either zero, one or two, $\gamma_{i,r}^- = 2$, $\gamma_{i,r}^0 = 0$ and $\gamma_{i,r}^+$ is either zero or two. In the example



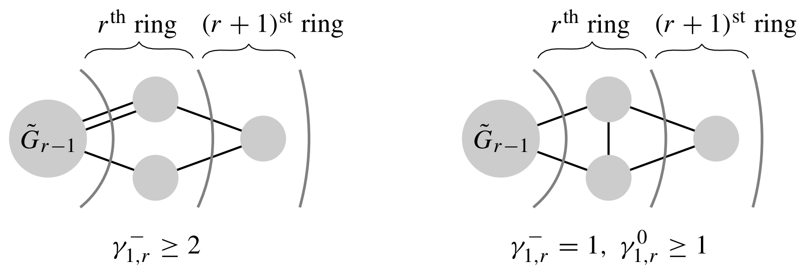
we have $\ell_1 = \ell_2 = 2, \ell_3 = 1, \gamma^+_{1,1} = \gamma^+_{2,1} = \gamma^+_{1,2} = 2, \gamma^+_{2,2} = \gamma^+_{1,3} = 0$. Not all diagrams with such combinatorial data are ladders, but there are so few of them that the non–ladder diagrams with these combinatorial data can be bounded individually without generating divergences.

In many cases, the combinatorial data of a diagram alone allow the detection of an overlapping loop. The two basic cases are:

(i) If, for some $r \ge 1$ and $1 \le i \le \ell_r$, we have $\gamma^-_{i,r} + \gamma^0_{i,r} \ge 3$ then there is an overlapping loop, as indicated in the following figures. Here v_i denotes the i^{th} vertex of the r^{th} ring.



(ii) If $\ell_r = 2$, $\ell_{r+1} = 1$ and $\gamma^-_{1,r} + \gamma^0_{1,r} \ge 2$, $\gamma^+_{1,r} \ge 1$, $\gamma^+_{2,r} \ge 1$ then there is an overlapping loop as seen in the following figures.



These overlapping loops can be used to generate improved estimates by the techniques mentioned at the end of Subsect. 8 without seriously affecting the cancellations described above.

If one also takes external legs into account and if $w_2 = 0$, then cases (i) and (ii) are enough to identify at least one overlapping loop in each four–legged diagram that does not have the combinatorial data of a ladder diagram. See §VII of [FKTr2].

10. The Counterterm The counterterm δe is constructed so that the proper self energy is bounded by $\frac{1}{2}|ik_0 - e(\mathbf{k})|$. That is, let $\check G_{2,j}(p)\,\delta(p-q)$ be the Fourier transform of the two point Green's function $G_{2,j}(x,y)$ constructed in Theorem I.4, and define $\Sigma_j(k)$ by

$$\check G_{2,j}(k) = \frac{U(k) - \nu^{(\ge j)}(k)}{\imath k_0 - e(\mathbf{k}) - \Sigma_j(k)}.$$

Then $|\Sigma_j(k)| \le \frac{1}{2}|ik_0 - e(\mathbf{k})|$.

To achieve this, we specify, for each scale j, a space $\mathcal{K}_j$ of "allowed future counterterm contributions for scales after j". It consists of functions $K(\mathbf{k})$ of the vector part of

$k = (k_0, \mathbf{k})$ only. These functions are required to be bounded by a small constant times $\frac{\mathfrak{l}_{j+1}}{M^{j+1}}$. The numerator reflects overlapping loop volume improvement. We construct a map δe_j from $\mathcal{K}_j$ to the space $\mathcal{E}$ of counterterms such that, if one writes

$$\check{G}_{2,j}\big(k; \delta e_j(K)\big) = \frac{U(k)-\nu^{(\geq j)}(k)}{\imath k_0 - e(\mathbf{k}) - \Sigma_j(k;K)}$$

then $\Sigma_j\big((0,\mathbf{k}); K\big) = K(\mathbf{k})$ in the j^{th} neighbourhood. Think of $\delta e_j(\mathbf{k}; K)$ as being of the form $\delta\tilde{e}_j(\mathbf{k}; K) + K(\mathbf{k})$ with $\delta\tilde{e}_j(\mathbf{k}; K)$ implementing cancellations that have already been identified and $K(\mathbf{k})$ reserved to implement as yet unidentified cancellations. Since $\Sigma_j(k; K)$ equals $\delta e_j(\mathbf{k}; K)$ plus higher order contributions, it is easy to solve $\Sigma_j\big((0,\mathbf{k}); K\big) = K(\mathbf{k})$ for $\delta\tilde{e}_j(\mathbf{k}; K)$. The algebra of this (recursive) construction is presented in detail in §III. We prove that $\delta e = \lim\limits_{j\to\infty} \delta e_j(0)$ exists and has the required properties.

At each scale, the properties of the counterterms are used to obtain the bound for the $w_2(p,q)$ of order $\frac{1}{M^j}$ necessary for power counting. The choice indicated above guarantees that such a bound holds for momenta $k = q - p$ for which the "temperature part" k_0 vanishes. To get this bound for arbitrary $k = (k_0, \mathbf{k})$ in the j^{th} neighbourhood, in particular when $|k_0| = O\big(\frac{1}{M^j}\big)$, we show that the k_0 derivative of this function is of order one. For this, we have to control derivatives (in momentum space) of all the data in the renormalization construction. Control of derivatives in momentum space is also needed to provide decay in position space. This is used, for example, in proving contraction bounds through L^1 norms in position space. We pointed out in Subsect. 1 that derivatives of $\delta e_j(0)$ might blow up as $j \to \infty$. Therefore we have to pay special attention to the behaviour of derivatives. This is the reason why we introduce, in Definition V.2, the "norm domain" that gives a convenient notation for bounds on derivatives of functions.

III. Formal Renormalization Group Maps

To simplify notation involving the fields, we define, for $\xi = (x_0, \mathbf{x}, \sigma, a) = (x, a) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$, the internal fields

$$\psi(\xi) = \begin{cases} \psi(x) = \psi_\sigma(x_0, \mathbf{x}) & \text{if } a{=}0 \\ \bar\psi(x) = \bar\psi_\sigma(x_0, \mathbf{x}) & \text{if } a{=}1 \end{cases}.$$

Similarly, we define for an external variable $\eta = (y_0, \mathbf{y}, \tau, b) = (y, b) \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$, the source fields

$$\phi(\eta) = \begin{cases} \phi(y) = \phi_\sigma(y_0, \mathbf{y}) & \text{if } b{=}0 \\ \bar\phi(y) = \bar\phi_\sigma(y_0, \mathbf{y}) & \text{if } b{=}1 \end{cases}.$$

$\mathcal{B} = \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\} \times \{0, 1\}$ is called the "base space" parameterizing the fields. An antisymmetric function $S(\xi, \xi')$ on $\mathcal{B} \times \mathcal{B}$ is called a covariance and determines a Grassmann Gaussian measure by

$$\int \psi(\xi)\psi(\xi')\, d\mu_S(\psi) = S(\xi, \xi').$$

A function $S(k)$ on momentum space, $\mathbb{R}\times\mathbb{R}^d$, defines a function $S(x,x')$ on position space $\big(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\big)^2$ by

$$S(x,x')=\delta_{\sigma,\sigma'}\int\frac{d^{d+1}k}{(2\pi)^{d+1}}e^{\imath<k,x-x'>_-}S(k).\tag{III.1}$$

Any function $S(x,x')$ on position space, $\big(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\}\big)^2$ defines a unique antisymmetric function $S(\xi,\xi')$ on $\mathcal{B}\times\mathcal{B}$ by

$$S\big((x,a),(x',a')\big)=\begin{cases}S(x,x') & \text{if } a{=}0, a'{=}1\\ -S(x',x) & \text{if } a{=}1, a'{=}0\\ 0 & \text{if } a{=}a'\end{cases}.\tag{III.2}$$

We denote the associated Grassmann Gaussian measure again by $d\mu_S$.

With the notation introduced above the source term of (I.5) is

$$\phi J\psi=\int d\xi\, d\xi'\,\phi(\xi)J(\xi,\xi')\psi(\xi')=\psi J\phi,$$

where the operator J has kernel

$$J\big((x_0,\mathbf{x},\sigma,a),(x_0',\mathbf{x}',\sigma',a')\big)=\delta(x_0-x_0')\delta(\mathbf{x}-\mathbf{x}')\delta_{\sigma,\sigma'}\begin{cases}1 & \text{if } a{=}1, a'{=}0\\ -1 & \text{if } a{=}0, a'{=}1.\\ 0 & \text{otherwise}\end{cases}\tag{III.3}$$

Definition III.1 (Renormalization Group Maps). *Let S be a covariance and $\mathcal{W}(\phi,\psi)$ a Grassmann function for which $Z=\int e^{\mathcal{W}(0,\zeta)}\,d\mu_S(\zeta)\neq 0$. We set*

$$\Omega_S(\mathcal{W})(\phi,\psi)=\log\tfrac{1}{Z}\int e^{\mathcal{W}(\phi,\psi+\zeta)}\,d\mu_S(\zeta),$$
$$\tilde\Omega_S(\mathcal{W})(\phi,\psi)=\log\tfrac{1}{Z}\int e^{\phi J\zeta}\,e^{\mathcal{W}(\phi,\psi+\zeta)}\,d\mu_S(\zeta).$$

Ω_S and $\tilde\Omega_S$ map Grassmann functions in the variables ϕ, ψ, to Grassmann functions in the same variables. They obey the semigroup property

$$\Omega_{S_1+S_2}=\Omega_{S_1}\circ\Omega_{S_2}\qquad,\qquad\tilde\Omega_{S_1+S_2}=\tilde\Omega_{S_1}\circ\tilde\Omega_{S_2}.\tag{III.4}$$

By Lemma VII.3 of [FKTo2], they are related by

$$\tilde\Omega_S(\mathcal{W})(\phi,\psi)=\tfrac12\phi JSJ\phi+\Omega_S(\mathcal{W})(\phi,\psi+SJ\phi)\tag{III.5}$$

where, for any covariance S, $\phi S\phi=\int d\xi_1 d\xi_2\,\phi(\xi_1)\,S(\xi_1,\xi_2)\,\phi(\xi_2)$. Clearly

$$\mathcal{G}_{j}(\phi;\delta e)=\tilde\Omega_{C^{\mathrm{IR}(j)}(\delta e)}(\widetilde{\mathcal{V}})(\phi,0)\qquad\text{with}\quad\widetilde{\mathcal{V}}(\phi,\psi)=\mathcal{V}(\psi).$$

Observe that

$$C^{\mathrm{IR}(j)}(k;0)=C^{\mathrm{IR}(1)}(k;0)+C^{(1)}(k)+\cdots+C^{(j-2)}(k)+C^{(j-1)}(k),$$

where

$$C^{(i)}(k_0,\mathbf{k}) = \frac{\nu^{(i)}(k)}{\imath k_0 - e(\mathbf{k})}.$$

Therefore, by induction and the semigroup property,

$$\mathcal{G}_{j}(\phi;0) = \tilde{\Omega}_{C^{(j-1)}} \circ \tilde{\Omega}_{C^{(j-2)}} \circ \cdots \circ \tilde{\Omega}_{C^{(1)}} \circ \tilde{\Omega}_{C^{\mathrm{IR}(1)}(0)}(\widetilde{\mathcal{V}})(\phi,0). \qquad \text{(III.6)}$$

Since $C^{(i)}(k)$ is supported on the i^{th} shell only, (III.6) would provide a convenient framework for a multiscale analysis of the unrenormalized generating functional $\mathcal{G}_{j}(\phi;0)$, by recursively controlling the effective interactions

$$\begin{aligned}\mathcal{G}_i(\phi,\psi;0) &= \tilde{\Omega}_{C^{(i-1)}} \circ \cdots \circ \tilde{\Omega}_{C^{(1)}} \circ \tilde{\Omega}_{C^{\mathrm{IR}(1)}(0)}(\widetilde{\mathcal{V}})(\phi,\psi)\\ &= \tilde{\Omega}_{C^{(i-1)}}\big(\mathcal{G}_{i-1}(0)\big)(\phi,\psi). \qquad \text{(III.7)}\end{aligned}$$

We now describe an analog of (III.6) for the case $\delta e \neq 0$. It incorporates a number of technical modifications needed to maintain control over the bounds. These modifications include the introduction of a scale–dependent Wick ordering and a scale–dependent contribution to the counterterm, periodic shifting of a portion of the interaction into the covariance and the periodic isolation of purely ϕ dependent terms. To this end, the effective interaction $\mathcal{G}_i(\phi,\psi;0)$ of (III.7) is replaced by a triple $(\mathcal{W},\mathcal{G},u)$ with

- ∘ $\mathcal{G}(\phi)$ being the purely ϕ dependent part of the effective interaction,
- ∘ $\mathcal{W}(\phi,\psi)$ being the rest of the interaction, and
- ∘ u being the kernel of a quadratic Grassmann function that has been moved from the effective interaction into the covariance.

To help clarify the algebraic structure of this more complicated setting, we outline the construction in the category of formal power series, without specifying the bounds that will ultimately be proven. To avoid formal power series in infinitely many variables, we introduce a coupling constant λ into the interaction

$$\mathcal{V}(\psi) = \lambda \int_{(\mathbb{R}\times\mathbb{R}^d\times\{\uparrow,\downarrow\})^4} V(x_1,x_2,x_3,x_4)\,\bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4)\,dx_1dx_2dx_3dx_4$$

and deal with Grassmann algebra valued formal power series in λ. Correspondingly, the final counterterm, $\delta e(\mathbf{k})$ is made λ–dependent.

Definition III.2. *The space of formal (final) counterterms, $\mathcal{E}^{form}$, consists of the space of all formal power series $\delta e(\mathbf{k},\lambda) = \sum_{n=1}^{\infty} \delta e_n(\mathbf{k})\lambda^n$ in λ each of whose coefficients is supported in $\big\{\, \mathbf{k}\in\mathbb{R}^d \;\big|\; U(\mathbf{k}) = 1 \,\big\}$.*

As indicated above the final counterterm δe is built up from contributions at each scale.

Definition III.3. *The space $\mathfrak{K}_j^{form}$ of formal (future) counterterms for scale j is the space of all formal power series in λ whose coefficients are antisymmetric, translation invariant functions of $\mathbf{x}$, $\mathbf{x}'$. The coefficient of λ^0 vanishes and the Fourier transform of each coefficient is supported on* $\operatorname{supp}\nu^{(\ge j+1)}\big((0,\mathbf{k})\big)$.

Definition III.4. *A formal interaction triple at scale j is a triple $(\mathcal{W},\mathcal{G},u)$ that obeys the following conditions:*

- ∘ *$\mathcal{W}(\phi, \psi; K)$ is a formal power series in λ, whose coefficients are functions of $K \in \mathfrak{K}_j^{form}$ that take values in the Grassmann algebra generated by the fields $\phi(\xi)$ and $\psi(\xi)$. Furthermore $\mathcal{W}(\phi, 0; K) = 0$ and $\mathcal{W}(\phi, \psi; K)\big|_{\lambda=0} = 0$. The coefficients of $\mathcal{W}$ are translation invariant, spin independent and particle–number conserving.*
- ∘ *$\mathcal{G}(\phi; K)$ is a formal power series in λ whose coefficients are functions of $K \in \mathfrak{K}_j^{form}$ that take values in the Grassmann algebra generated by the fields $\phi(\xi)$. The constant term $\mathcal{G}(0; K) = 0$. The coefficients of $\mathcal{G}$ are translation invariant, spin independent and particle–number conserving.*
- ∘ *$u(\xi_1, \xi_2; K)$ is a formal power series in λ whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of $\xi_1, \xi_2 \in \mathcal{B}$ and $K \in \mathfrak{K}_j^{form}$. The Fourier transform[16] $\check{u}(k; K)$ of u obeys*

$$\check{u}\big((0, \mathbf{k}); K\big) = -\check{K}(\mathbf{k}) \qquad\qquad \check{u}\big(k; K\big)\big|_{\lambda=0} = -\check{K}(\mathbf{k}).$$

The condition that $\mathcal{W}(\phi, 0; K) = 0$ ensures that $\mathcal{G}(\phi; K)$ contains the full pure ϕ part of the effective interaction.

As mentioned above, u is the kernel of a quadratic Grassmann function that has been moved from the effective interaction into the covariance. Precisely,

Definition III.5. *(i) Let $u(\xi_1, \xi_2)$ be a formal power series in λ whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of $\xi_1, \xi_2 \in \mathcal{B}$. Then*

$$C_u^{(j)}(k) = \frac{\nu^{(j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k)}.$$

(ii) Let $u(\xi_1, \xi_2; K)$ be a formal power series in λ whose coefficients are antisymmetric, spin independent, particle number conserving, translation invariant functions of $\xi_1, \xi_2 \in \mathcal{B}$. Then, for $K \in \mathfrak{K}_j^{form}$,

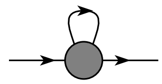
$$\begin{aligned} C_j(u; K)(k) &= \tfrac{\nu^{(\ge j)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K) - \check{K}(\mathbf{k})\nu^{(\ge j+2)}(k)}, \\ D_j(u; K)(k) &= \tfrac{\nu^{(\ge j+1)}(k)}{\imath k_0 - e(\mathbf{k}) - \check{u}(k;K) - \check{K}(\mathbf{k})\nu^{(\ge j+2)}(k)}. \end{aligned}$$

For the formal interaction triple $(\mathcal{W}, \mathcal{G}, u)$ at scale j, integrating out scale j involves the evaluation of an integral with respect to the Gaussian measure with covariance $C_{u(K)}^{(j)}$. The effective interaction $\mathcal{W}$ will be Wick ordered with respect to the covariance $C_j(u(K); K)$. One important property of the Wick ordering covariances is that $D_j(u; K) = C_j(u; K) - C_{u(K)}^{(j)}$, so that

$$\int :f(\psi + \zeta):_{C_j(u,K)}\, d\mu_{C_{u(K)}^{(j)}}(\zeta) = :f(\psi):_{D_j(u,K)}$$

for all Grassmann functions $f(\psi)$, by Proposition A.2.i and Lemma A.4.ii of [FKTr1]. This property prevents the formation of Wick self–contractions

[16] A systematic set of Fourier transform conventions will be given in Def. VI.1. In the present context $\check{u}(k; K)$ is the Fourier transform of $u\,((0, \mathbf{0}, \uparrow, 1), (x_0, \mathbf{x}, \uparrow, 0); K)$ as in Theorem I.5.



and ensures that the effective interaction resulting from integrating out scale j is naturally Wick ordered with respect to the "output Wick ordering covariance" $D_j(u, K)$. Also the Wick ordering covariances have been chosen so that, for k of scale at least $j+3$, $\check{u}(k; K) + \check{K}(\mathbf{k})\nu^{(\geq j+2)}(k) = \check{u}(k; K) + \check{K}(\mathbf{k})$ vanishes for $k_0 = 0$. This property ensures that the denominator still vanishes only on the Fermi surface.

Definition III.6. *Integrating out the fields of scale j is implemented by the map Ω_j, which maps a formal interaction triple $(\mathcal{W}, \mathcal{G}, u)$ of scale j to the triple $(\mathcal{W}', \mathcal{G}', u)$ determined by*

$$:\mathcal{W}'(\phi, \psi; K):_{\psi, D_j(u;K)} = \log \tfrac{1}{Z(\phi)} \int e^{\phi J \zeta}\, e^{:\mathcal{W}(\phi, \psi+\zeta; K):_{\psi, C_j(u;K)}}\, d\mu_{C^{(j)}_{u(K)}}(\zeta)$$
$$\mathcal{G}'(\phi) = \mathcal{G}(\phi) + \log \tfrac{Z(\phi)}{Z(0)},$$

where

$$\log Z(\phi) = \int \Big[\log \int e^{\phi J \zeta}\, e^{:\mathcal{W}(\phi, \psi+\zeta; K):_{\psi, C_j(u;K)}}\, d\mu_{C^{(j)}_{u(K)}}(\zeta) \Big] d\mu_{D_j(u;K)}(\psi).$$

In fact $(\mathcal{W}', \mathcal{G}', u)$ is again a formal interaction triple of scale j. That $\mathcal{W}'(\phi, 0; K) = 0$ follows by inserting the definitions into

$$\mathcal{W}'(\phi, 0; K) = \int :\mathcal{W}'(\phi, \psi; K):_{\psi, D_j(u,K)}\, d\mu_{D_j(u,K)}(\psi).$$

To verify $\mathcal{W}'(\phi, \psi; K)\big|_{\lambda=0} = 0$, observe that for $\lambda = 0$,

$$\int e^{\phi J \zeta}\, e^{:\mathcal{W}(\phi, \psi+\zeta; K):_{\psi, C_j(u;K)}}\, d\mu_{C^{(j)}_{u(K)}}(\zeta) = \int e^{\phi J \zeta}\, d\mu_{C^{(j)}_{u(K)}}(\zeta) = e^{\frac{1}{2}\phi J C^{(j)}_{u(K)} J \phi}$$

is independent of ψ. To verify the various symmetries, apply Remark B.5 of [FKTo2].

Remark III.7. Define, for all $1 \leq i \leq j \leq \infty$,

$$C_u^{[i,j)}(k) = \begin{cases} \frac{\nu^{(\geq i)}(k) - \nu^{(\geq j)}(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k)[1 - \nu^{(\geq j)}(k)]} & \text{if } j < \infty \\ \frac{\nu^{(\geq i)}(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k)} & \text{if } j = \infty \end{cases}.$$

We also write $C_u^{(\geq i)} = C_u^{[i,\infty)}$.

Let $(\mathcal{W}', \mathcal{G}', u) = \Omega_j(\mathcal{W}, \mathcal{G}, u)$. Then, for any infrared cutoff $j + 2 \leq \not{j} \leq \infty$, formally, ignoring the problems engendered by the infrared singularity,

$$\mathcal{G}(\phi) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}(\phi, \psi; K):_{\psi, C_j(u;K)}}\, d\mu_{C_u^{[j,\not{j})}}(\psi)}{\int e^{:\mathcal{W}(0, \psi; K):_{\psi, C_j(u;K)}}\, d\mu_{C_u^{[j,\not{j})}}(\psi)}$$
$$= \mathcal{G}'(\phi) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}'(\phi, \psi; K):_{\psi, D_j(u;K)}}\, d\mu_{C_u^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}'(0, \psi; K):_{\psi, D_j(u;K)}}\, d\mu_{C_u^{[j+1,\not{j})}}(\psi)}.$$

Proof. Since $C_u^{[j,\not j)} = C_u^{(j)} + C_u^{[j+1,\not j)}$,

$$\int f(\phi,\psi)\, d\mu_{C_u^{[j,\not j)}}(\psi) = \iint f(\phi,,\psi+\zeta)\, d\mu_{C_u^{(j)}}(\zeta)\, d\mu_{C_u^{[j+1,\not j)}}(\psi)$$

by Proposition I.21 of [FKTffi]. Hence, by Proposition A.2.ii of [FKTr1],

$$\begin{aligned}
&\log \int e^{\phi J\psi} e^{:\mathcal{W}(\phi,\psi;K):_{\psi,C_j(u;K)}}\, d\mu_{C_u^{[j,\not j)}}(\psi)\\
&= \log \int\int e^{\phi J(\psi+\zeta)}\, e^{:\mathcal{W}(\phi,\psi+\zeta;K):_{\psi,C_j(u;K)}}\, d\mu_{C_u^{(j)}}(\zeta)\, d\mu_{C_u^{[j+1,\not j)}}(\psi)\\
&= \log \int e^{\phi J\psi}\, e^{:\mathcal{W}'(\phi,\psi;K):_{\psi,D_j(u;K)}+\log Z(\phi)}\, d\mu_{C_u^{[j+1,\not j)}}(\psi)\\
&= \log \int e^{\phi J\psi}\, e^{:\mathcal{W}'(\phi,\psi;K):_{\psi,D_j(u;K)}}\, d\mu_{C_u^{[j+1,\not j)}}(\psi) + \log Z(\phi)\\
&= \log \int e^{\phi J\psi}\, e^{:\mathcal{W}'(\phi,\psi;K):_{\psi,D_j(u;K)}}\, d\mu_{C_u^{[j+1,\not j)}}(\psi) + \log Z(0) + \mathcal{G}'(\phi) - \mathcal{G}(\phi).
\end{aligned}$$

Subtracting the same equation with $\phi = 0$ gives the desired result. □

When we derive bounds on the map Ω_j, we get improvements on the two– and four–legged contributions to $\mathcal{W}'(0,\psi)$ by exploiting overlapping loops. See Subsect. 4 of §II and the introduction to [FKTr2]. However to ensure the presence of and to detect sufficiently many overlapping loops, we need that the two–legged part of $\mathcal{W}(0,\psi)$ vanishes. See the end of Subsect. 9 of §II and Remark VI.7 of [FKTr2]. Therefore, we wish that the formal interaction triple $(\mathcal{W},\mathcal{G},u)$ input to Ω_j be an element of

Definition III.8 (Formal Input Data). *The space $\mathcal{D}_{\rm in}^{(j,form)}$ of formal input data consists of the set of all formal interaction triples $(\mathcal{W},\mathcal{G},u)$ at scale j, in the sense of Definition III.4, obeying*

(i) If the effective interaction $\mathcal{W}(K) = \sum_{m,n\ge 0} \mathcal{W}_{m,n}$ with

$$\mathcal{W}_{m,n} = \int \prod_{i=1}^{m} d\eta_i \prod_{\ell=1}^{n} d\xi_\ell\ W_{m,n}(\eta_1,\cdots,\eta_m,\xi_1,\cdots,\xi_n)\phi(\eta_1)\cdots\phi(\eta_m)\psi(\xi_1)\cdots\psi(\xi_n),$$

then $W_{0,2} = 0$.

(ii) The coefficient of λ^0 in $\mathcal{G}(\phi;K)$ is $\frac{1}{2}\phi J C_{-K}^{(<j)} J\phi$. Here $C_u^{(<j)} = \frac{U(\mathbf{k})-\nu^{(\ge j)}(k)}{ik_0 - e(\mathbf{k}) - \check{u}(k)}$.

When Ω_j is applied to an element of $\mathcal{D}_{\rm in}^{(j,form)}$, the output no longer satisfies condition (i) of Definition III.8. Rather, the output lies in

Definition III.9 (Formal Output Data). *The space $\mathcal{D}_{\rm out}^{(j,form)}$ of formal output data consists of the set of all formal interaction triples $(\mathcal{W},\mathcal{G},u)$ at scale j, in the sense of Definition III.4, for which the coefficient of λ^0 in $\mathcal{G}(\phi;K)$ is $\frac{1}{2}\phi J C_{-K}^{(\le j)} J\phi$.*

Lemma III.10. *Let $(\mathcal{W},\mathcal{G},u) \in \mathcal{D}_{\rm in}^{(j,form)}$. Then $\Omega_j(\mathcal{W},\mathcal{G},u) \in \mathcal{D}_{\rm out}^{(j,form)}$.*

Proof. Set $(\mathcal{W}', \mathcal{G}', u) = \Omega_j(\mathcal{W}, \mathcal{G}, u)$. Then

$$\begin{aligned}\mathcal{G}'(\phi)\big|_{\lambda=0} &= \mathcal{G}(\phi)\big|_{\lambda=0} + \log \tfrac{Z(\phi)}{Z(0)}\big|_{\lambda=0}\\ &= \tfrac{1}{2}\phi J C_{-K}^{(<j)} J\phi + \log \tfrac{Z(\phi)}{Z(0)}\big|_{\lambda=0}.\end{aligned}$$

Since

$$\log Z(\phi)\big|_{\lambda=0} = \int \Big[\log \int e^{\zeta J\phi}\, d\mu_{C_{-K}^{(j)}}(\zeta)\Big] d\mu_{D_j(u;K)}(\psi) = \tfrac{1}{2} J\phi C_{-K}^{(j)} J\phi$$

the result follows. □

Elements of the space $\mathcal{D}_{\rm out}^{(j,form)}$ are not of the form desired for the application of Ω_{j+1}, the map implementing the integration out of scale $j+1$. In particular, the two–point part of the effective interaction is nonzero and $\mathfrak{K}_j^{form}$ is not the appropriate space of counterterms. Below, just before Proposition III.12, we construct maps

$$\mathcal{O}_j : \mathcal{D}_{\rm out}^{(j,form)} \to \mathcal{D}_{\rm in}^{(j+1,form)},$$

and, for each $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}_{\rm out}^{(j,form)}$,

$$\mathrm{ren}_{j,j+1}(\,\cdot\,, \mathcal{W}, u) : \mathfrak{K}_{j+1}^{form} \to \mathfrak{K}_j^{form}$$

with the following property: If $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}_{\rm out}^{(j,form)}$ and $(\mathcal{W}', \mathcal{G}', u') = \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u)$ and if $K' \in \mathfrak{K}_{j+1}^{form}$ and $K = \mathrm{ren}_{j,j+1}(K', \mathcal{W}, u)$, then, for any infrared cutoff $j+1 \le \not{j} \le \infty$, formally, ignoring the problems engendered by the infrared singularity,

$$\begin{aligned}&\mathcal{G}(\phi; K) + \log \frac{\int e^{\phi J\psi}\, e^{:\mathcal{W}(\phi,\psi;K):_{\psi,D_j(u;K)}}\, d\mu_{C_{u(K)}^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}(0,\psi;K):_{\psi,D_j(u;K)}}\, d\mu_{C_{u(K)}^{[j+1,\not{j})}}(\psi)}\\ &\quad= \mathcal{G}'(\phi; K') + \log \frac{\int e^{\phi J\psi}\, e^{:\mathcal{W}'(\phi,\psi;K'):_{\psi,C_{j+1}(u';K')}}\, d\mu_{C_{u'(K')}^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}'(0,\psi;K'):_{\psi,C_{j+1}(u';K')}}\, d\mu_{C_{u'(K')}^{[j+1,\not{j})}}(\psi)}. \qquad \text{(III.8)}\end{aligned}$$

The map $\mathcal{O}_j$ moves the two–point part of $\mathcal{W}$ into the covariance, through u', and updates the Wick ordering covariance (for a precise statement, see (III.11) below). The map $\mathrm{ren}_{j,j+1}$ introduces the contribution of the current scale into the counterterm.

Using the maps Ω_j of Definition III.6 and the maps $\mathcal{O}_j$ and $\mathrm{ren}_{j,j+1}$ we can describe the renormalization group flow. We start by choosing an arbitrary but fixed $j_0 \ge 2$ and integrate out all scales from 1 to j_0 to arrive at the initial effective interaction triple $(\mathcal{W}_{j_0}^{\rm out}, \mathcal{G}_{j_0}^{\rm out}, u_{j_0}) \in \mathcal{D}_{\rm out}^{(j_0,form)}$ with

$$\begin{aligned}\mathcal{W}_{j_0}^{\rm out} &= \tilde{\Omega}_{C_{-K}^{(\le j_0)}}(\widetilde{\mathcal{V}})(\phi,\psi) - \tilde{\Omega}_{C_{-K}^{(\le j_0)}}(\widetilde{\mathcal{V}})(\phi,0),\\ \mathcal{G}_{j_0}^{\rm out} &= \tilde{\Omega}_{C_{-K}^{(\le j_0)}}(\widetilde{\mathcal{V}})(\phi,0),\\ u_{j_0} &= -K.\end{aligned}$$

The renormalization group flow is the concatenation of the maps $\mathcal{O}_{j_0}$, Ω_{j_0+1}, $\mathcal{O}_{j_0+1}$, $\Omega_{j_0+2}, \cdots, \Omega_j, \mathcal{O}_j, \cdots$ applied to the initial datum. Set

$$\begin{aligned}(\mathcal{W}_j^{\rm in}, \mathcal{G}_j^{\rm in}, u_j) &= \mathcal{O}_{j-1} \circ \Omega_{j-1} \circ \mathcal{O}_{j-2} \circ \cdots \circ \Omega_{j_0+1} \circ \mathcal{O}_{j_0}(\mathcal{W}_{j_0}^{\rm out}, \mathcal{G}_{j_0}^{\rm out}, u_{j_0}) \\ &\in \mathcal{D}_{\rm in}^{(j,form)}, \\ (\mathcal{W}_j^{\rm out}, \mathcal{G}_j^{\rm out}, u_j) &= \Omega_j \circ \mathcal{O}_{j-1} \circ \Omega_{j-1} \circ \mathcal{O}_{j-2} \circ \cdots \circ \Omega_{j_0+1} \circ \mathcal{O}_{j_0}(\mathcal{W}_{j_0}^{\rm out}, \mathcal{G}_{j_0}^{\rm out}, u_{j_0}) \\ &\in \mathcal{D}_{\rm out}^{(j,form)}. \end{aligned} \tag{III.9}$$

We recursively define maps $\mathrm{ren}_{i,j} : \mathfrak{K}_j^{form} \to \mathfrak{K}_i^{form}$, $j_0 \le i \le j$ by

$$\begin{aligned}\mathrm{ren}_{j,j}(K) &= K, \\ \mathrm{ren}_{i,j}(K) &= \mathrm{ren}_{i,j-1}\big(\mathrm{ren}_{j-1,j}(K)\big) \quad \text{for } j > i.\end{aligned}$$

We define for $K \in \mathfrak{K}_j^{form}$,

$$\delta e_j(K) = \mathrm{ren}_{j_0,j}(K)$$

and show that, for the generating function of the connected Green's functions at scale $\not{j}$ of Theorem I.4,

$$\begin{aligned}\mathcal{G}_{\not{j}}(\phi, \bar{\phi}; \delta e_j(K)) &= \mathcal{G}_j^{\rm in}(\phi; K) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}_j^{\rm in}(\phi,\psi;K):_{\psi, C_j(u_j;K)}}\, d\mu_{C_{u_j(K)}^{[j,\not{j})}}(\psi)}{\int e^{:\mathcal{W}_j^{\rm in}(0,\psi;K):_{\psi, C_j(u_j;K)}}\, d\mu_{C_{u_j(K)}^{[j,\not{j})}}(\psi)} \\ &= \mathcal{G}_j^{\rm out}(\phi; K) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}_j^{\rm out}(\phi,\psi;K):_{\psi, D_j(u_j;K)}}\, d\mu_{C_{u_j(K)}^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}_j^{\rm out}(0,\psi;K):_{\psi, D_j(u_j;K)}}\, d\mu_{C_{u_j(K)}^{[j+1,\not{j})}}(\psi)}\end{aligned} \tag{III.10}$$

for $K \in \mathfrak{K}_j^{form}$ and $j_0 \le j \le \not{j} - 2$. When $\not{j} = \infty$, (III.10) holds with $\mathcal{G}_\infty$ being the formal generating function of the connected Green's functions (I.4).

Equation (III.10) is proven by induction on j, combining Remark III.7 and (III.8). To start the induction, observe that $\mathcal{G}_{j_0}^{\rm out}(\phi; K) + \mathcal{W}_{j_0}^{\rm out}(\phi, \psi; K) = \tilde{\Omega}_{C_{-K}^{(\le j_0)}}(\widetilde{\mathcal{V}})$, so that, by the semigroup property (III.4),

$$\begin{aligned}\mathcal{G}_{\not{j}}(\phi, \bar{\phi}; K) &= \mathcal{G}_{j_0}^{\rm out}(\phi; K) + \log \frac{\int e^{\phi J \psi}\, e^{\mathcal{W}_{j_0}^{\rm out}(\phi,\psi;K)}\, d\mu_{C_{-K}^{(j_0,\not{j})}}(\psi)}{\int e^{\mathcal{W}_{j_0}^{\rm out}(0,\psi;K)}\, d\mu_{C_{-K}^{(j_0,\not{j})}}(\psi)} \\ &= \mathcal{G}_{j_0}^{\rm out}(\phi; K) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}_{j_0}^{\rm out}(\phi,\psi;K):_{\psi, D_{j_0}(u_{j_0};K)}}\, d\mu_{C_{u_{j_0}(K)}^{(j_0,\not{j})}}(\psi)}{\int e^{:\mathcal{W}_{j_0}^{\rm out}(0,\psi;K):_{\psi, D_{j_0}(u_{j_0};K)}}\, d\mu_{C_{u_{j_0}(K)}^{(j_0,\not{j})}}(\psi)}\end{aligned}$$

with $D_{j_0} = 0$[17].

[17] Wick ordering with respect to the covariance zero is the same as not Wick ordering at all. $D_{j_0} = 0$ is a convention that we introduce purely for notational consistency.

In Theorem VIII.5, we shall prove bounds that show that the limits $\delta e = \lim_{j\to\infty} \delta e_j(0)$ and $\mathcal{G}(\phi,\bar\phi;\delta e) = \lim_{j\to\infty} \mathcal{G}_j^{\rm out}(\phi;0) = \lim_{j\to\infty} \mathcal{G}_j^{\rm in}(\phi;0)$ exist. To prove Theorem I.4 we show that $\lim_{j\to\infty} \big(\mathcal{G}_j(\phi,\bar\phi) - \mathcal{G}_j^{\rm out}(\phi;0)\big) = 0$.

We now describe the passage from output data to input data, that is the maps

$$\mathcal{O}_j : \mathcal{D}_{\rm out}^{(j,form)} \to \mathcal{D}_{\rm in}^{(j+1,form)} \qquad {\rm ren}_{j,j+1}(\,\cdot\,,\mathcal{W},u) : \mathfrak{K}_{j+1}^{form} \to \mathfrak{K}_j^{form}.$$

Let $(\mathcal{W},\mathcal{G},u) \in \mathcal{D}_{\rm out}^{(j,form)}$ and write

$$\mathcal{W}(0,\psi;K) = \sum_{m\ge 2} \int d\xi_1\cdots d\xi_m\ W_{0,m}(\xi_1,\cdots,\xi_m;K)\ \psi(\xi_1)\cdots\psi(\xi_m)$$

with each $W_{0,m}(\xi_1,\cdots,\xi_m;K)$ antisymmetric under permutation of the ξ_i's.

To perform the reWick ordering, observe that, if E is any covariance and

$$:\mathcal{W}(\phi,\psi;K):_{\psi,D_j(u;K)} = :\tilde{\mathcal{W}}(\phi,\psi;K):_{\psi,D_j(u;K)+E}$$

then, by Lemma A.2.i of [FKTr1],

$$\tilde{\mathcal{W}}(\phi,\psi;K) = \int \mathcal{W}(\phi,\psi+\psi';K)\, d\mu_E(\psi').$$

We wish to choose the covariance E such that, after we reWick order $:\mathcal{W}(\phi,\psi;K):_{\psi,D_j(u;K)}$ to $:\tilde{\mathcal{W}}(\phi,\psi;K):_{\psi,D_j(u;K)+E}$ and move the quadratic part of $\tilde{\mathcal{W}}(\phi,\psi;K)$ into the covariance, replacing $u(K)$ by $u'(K')$, then the Wick ordering covariance $D_j(u;K)+E$ is exactly $C_{j+1}(u';K')$. When we replace $u(K)$ by $u'(K')$, we choose $K\in\mathfrak{K}_j^{form}$ as a function of $K'\in\mathfrak{K}_{j+1}^{form}$ in such a way that $u'(K')$ fulfills the third condition of Definition III.4. This function will be denoted $K(K') = {\rm ren}_{j,j+1}(K',\mathcal{W},u) = K' + \delta K(K')$.

The unknowns in the scheme outlined in the last paragraph are E and δK. They are determined implicitly by the requirements of the last paragraph. We choose to express E and δK in terms of one function $q(\xi_1,\xi_2;K')$, with $\frac{1}{2}q$ being the kernel of $\tilde{\mathcal{W}}_{0,2}$. Once q is determined, we set

$$\begin{aligned} \delta\check K(\mathbf{k};K';q) &= \check q\big((0,\mathbf{k});K'\big)\,\nu^{(\ge j+1)}((0,\mathbf{k})),\\ K(K';q) &= K' + \delta K(K';q),\\ \check u'(k;K';q) &= \check u(k;K(K';q)) + \check q(k;K')\nu^{(\ge j+1)}(k),\\ E(K';q) &= C_{j+1}\big(u'(\,\cdot\,;q);K'\big) - D_j(u;K(K';q)). \end{aligned} \tag{III.11}$$

Set

$$\tilde{\mathcal{W}}(\phi,\psi;K';q) = \int \mathcal{W}(\phi,\psi+\psi';K(K';q))\, d\mu_{E(K';q)}(\psi')$$

and expand

$$\tilde{\mathcal{W}}(0,\psi;K';q) = \sum_{m\ge 0} \int d\xi_1\cdots d\xi_m\ \tilde W_{0,m}(\xi_1,\cdots,\xi_m;K';q)\ \psi(\xi_1)\cdots\psi(\xi_m).$$

The requirement that $\frac{1}{2}q$ be the kernel of $\tilde{\mathcal{W}}_{0,2}$ is now an implicit equation.

Lemma III.11. *There is a unique formal power series $q_0(\xi_1, \xi_2; K')$ in λ that solves the equation*

$$\tfrac{1}{2}q(K') = \tilde{W}_{0,2}\big(K'; q(K')\big). \tag{III.12}$$

The coefficient of λ^0 in q_0 vanishes.

Proof. Equation (III.12) is the form $q = F(\lambda, q)$, with F being C^∞ in λ and q and with $F(\lambda, 0)$ being of order at least λ. An easy formal power series argument yields the result. □

We define, for each $K' \in \mathfrak{K}_{j+1}^{form}$,

$$\begin{aligned}
\tilde{\mathcal{W}}(\phi, \psi; K') &= \tilde{\mathcal{W}}(\phi, \psi; K'; q_0(K')),\\
\mathcal{W}'(\phi, \psi; K') &= \tilde{\mathcal{W}}\big(\phi, \psi; K'\big) - \tilde{\mathcal{W}}\big(\phi, 0; K'\big) - \tfrac{1}{2}\int d\xi_1 d\xi_2\, q_0(\xi_1, \xi_2; K')\psi(\xi_1)\psi(\xi_2),\\
\mathcal{G}'(\phi; K') &= \mathcal{G}(\phi; K(K')) + \tilde{\mathcal{W}}(\phi, 0; K') - \tilde{\mathcal{W}}(0, 0; K'),\\
u'(K') &= u'(K'; q_0(K')),
\end{aligned}$$

and

$$\mathcal{O}_j(\mathcal{W}, \mathcal{G}, u) = (\mathcal{W}', \mathcal{G}', u').$$

We also define

$$\mathrm{ren}_{j,j+1}(K', \mathcal{W}, u) = K(K') = K(K'; q_0(K')) \in \mathfrak{K}_j^{form}.$$

Proposition III.12. *Let $j \geq j_0$, $(\mathcal{W}, \mathcal{G}, u) \in \mathcal{D}_{\mathrm{out}}^{(j, form)}$ and $(\mathcal{W}', \mathcal{G}', u') = \mathcal{O}_j(\mathcal{W}, \mathcal{G}, u)$. Then*

a)

$$(\mathcal{W}', \mathcal{G}', u') \in \mathcal{D}_{\mathrm{in}}^{(j+1, form)}.$$

b) If $K' \in \mathfrak{K}_{j+1}^{form}$ and $K = \mathrm{ren}_{j,j+1}(K', \mathcal{W}, u)$ then, formally, ignoring the problems engendered by the infrared singularity,

$$\begin{aligned}
&\mathcal{G}(\phi; K) + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}(\phi,\psi;K):_{\psi, D_j(u;K)}}\, d\mu_{C_{u(K)}^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}(0,\psi;K):_{\psi, D_j(u;K)}}\, d\mu_{C_{u(K)}^{[j+1,\not{j})}}(\psi)}\\
&= \mathcal{G}'(\phi; K') + \log \frac{\int e^{\phi J \psi}\, e^{:\mathcal{W}'(\phi,\psi;K'):_{\psi, C_{j+1}(u';K')}}\, d\mu_{C_{u'(K')}^{[j+1,\not{j})}}(\psi)}{\int e^{:\mathcal{W}'(0,\psi;K'):_{\psi, C_{j+1}(u';K')}}\, d\mu_{C_{u'(K')}^{[j+1,\not{j})}}(\psi)}
\end{aligned}$$

if $j + 1 < \not{j} \leq \infty$.

Proof. Let $K' \in \mathfrak{K}_{j+1}^{form}$ and set $K = \mathrm{ren}_{j,j+1}(K', \mathcal{W}, u)$.

a) We first verify that $(\mathcal{W}', \mathcal{G}', u')$ is a formal interaction triple. The only condition of Definition III.4 that is not trivially satisfied is

$$\begin{aligned}\check{u}'\big((0,\mathbf{k});K'\big) &= \check{u}\big((0,\mathbf{k});K\big(K';q_0(K')\big)\big) + \check{q}_0((0,\mathbf{k});K')\nu^{(\geq j+1)}(0,\mathbf{k})\\ &= -\check{K}(\mathbf{k};K';q_0(K')) + \delta\check{K}(\mathbf{k};K';q_0(K'))\\ &= -\check{K}'(\mathbf{k}).\end{aligned}$$

We now verify the conditions of Definition III.8. That $W'_{0,2}$ vanishes amounts to $\tilde{W}_{0,2} = \frac{1}{2}q_0$ which is Lemma III.11. Condition (ii) of Definition III.8 is trivially fulfilled.

b) Observe that

$$C^{[j+1,\not{j})}_{u'(K')} = C^{[j+1,\not{j})}_{u(K(K'))+q_0(K')\nu^{(\geq j+1)}}. \tag{III.13}$$

Set $\mathcal{U}(K') = \frac{1}{2}\int d\xi_1 d\xi_2\, q_0(\xi_1,\xi_2;K'){:}\psi(\xi_1)\psi(\xi_2){:}_{C_{j+1}(u';K')}$. By Lemma C.2 of [FKTo2] and (III.13),

$$\begin{aligned}&\mathcal{G}(\phi;K) + \log\int e^{\phi J\psi} e^{:\mathcal{W}(\phi,\psi;K):_{\psi,D_j(u;K)}}\, d\mu_{C^{[j+1,\not{j})}_{u(K)}}(\psi)\\ &= \mathcal{G}'(\phi;K') + \log\int e^{\phi J\psi} e^{:\tilde{\mathcal{W}}(\phi,\psi;K):_{\psi,C_{j+1}(u';K')} - \mathcal{U} + \mathcal{U} + (\mathcal{G}(\phi;K)-\mathcal{G}'(\phi;K'))}\, d\mu_{C^{[j+1,\not{j})}_{u(K)}}(\psi)\\ &= \mathcal{G}'(\phi;K') + \log\int e^{\phi J\psi}\, e^{:\mathcal{W}'(\phi,\psi;K'):_{\psi,C_{j+1}(u';K')} + \mathcal{U}}\, d\mu_{C^{[j+1,\not{j})}_{u(K)}}(\psi) + \text{const}\\ &= \mathcal{G}'(\phi;K') + \log\int e^{\phi J\psi} e^{:\mathcal{W}'(\phi,\psi;K'):_{\psi,C_{j+1}(u';K')}}\, d\mu_{C^{[j+1,\not{j})}_{u(K)+q_0(K')\nu^{(\geq j+1)}}}(\psi) + \text{const}\\ &= \mathcal{G}'(\phi;K') + \log\int e^{\phi J\psi} e^{:\mathcal{W}'(\phi,\psi;K'):_{\psi,C_{j+1}(u';K')}}\, d\mu_{C^{[j+1,\not{j})}_{u'(K')}}(\psi) + \text{const}\,.\end{aligned}$$

Subtracting the same equation with $\phi = 0$ gives part b). □

Appendix A. Model Computations

This appendix provides a number of model computations that illustrate important features of the present construction. Various other model computations are given in the introductory sections of this paper and other papers in this series. Here is a table of model computations and their locations.

Topic	Location
Overlapping loop volume improvement	§II, Subsect. 4
Particle–particle bubble volume improvement	§II, Subsect. 5
Particle–hole bubble sign cancellation	[FKTl, Lemma I.1]
Sectorization and conservation of momentum	Example A.1
Power counting with sectorization	§II, Subsect. 8
Overlapping loops and sectors	Example A.2
Sectorization and change of scale	Example A.3
Cancellations at high orders of perturbation theory	§II, Subsect. 9
	[FKTr2, §X]
	[FKTo1, §V]

Example A.1 (Sectorization and Conservation of Momentum). Sectors enable us to apply L^1 norms to functions for which the L^1 norm well approximates the L^∞ norm of the Fourier transform. We provide a simple illustration of the use of sectorization as a tool for bounding $\| \ \|_{1,\infty}$ norms of Green's functions built from $C^{(j)}$.

Recall that $C^{(j)}(k) = \frac{\nu^{(j)}(k)}{ik_0 - e(\mathbf{k})}$ and that, on the support of $\nu^{(j)}$, $\big|ik_0 - e(\mathbf{k})\big| \approx \frac{1}{M^j}$. To make this example as explicit as possible, we suppress k_0, choose $d = 3$, choose $e(\mathbf{k}) = |\mathbf{k}|^2 - 1$, replace $ik_0 - e(\mathbf{k})$ by $\frac{|\mathbf{k}|}{M^j}$ and replace $\nu^{(j)}(k)$ by $\varphi\big(M^j(|\mathbf{k}| - 1)\big)$, where $\varphi \in C^\infty([-1, 1])$ is real and even. So we define

$$c^{(j)}(\mathbf{k}) = \frac{M^j}{|\mathbf{k}|}\varphi\big(M^j(|\mathbf{k}| - 1)\big) \qquad c^{(j)}(\mathbf{x}, \mathbf{x}') = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}c^{(j)}(\mathbf{k}).$$

We have rigged things so that we can compute $c^{(j)}(\mathbf{x}, \mathbf{x}')$ relatively explicitly using the Fourier transform $\int_{S^{d-1}} d\sigma(\mathbf{k}')\ e^{ir\mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}')} = 2^{\frac{d}{2}-1}\Gamma\big(\frac{d}{2}\big)\omega_d\ (r|\mathbf{x} - \mathbf{x}'|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(r|\mathbf{x} - \mathbf{x}'|)$ of the unit sphere in $\mathbb{R}^d$. Here Γ, ω_d and J_ν are the Gamma function, the surface area of S^{d-1} and the Bessel function of order ν, respectively. The answer is

$$c^{(j)}(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\ \sin\big(|\mathbf{x} - \mathbf{x}'|\big)\hat{\varphi}\big(\tfrac{|\mathbf{x}-\mathbf{x}'|}{M^j}\big).$$

In particular

$$\big\|c^{(j)}(\mathbf{x}, \mathbf{x}')\big\|_{1,\infty} = \int d^3\mathbf{x}\ \frac{1}{2\pi^2|\mathbf{x}|}\ \Big|\sin\big(|\mathbf{x}|\big)\hat{\varphi}\big(\tfrac{|\mathbf{x}|}{M^j}\big)\Big| = \frac{2}{\pi}\int_0^\infty dr\ r\Big|\sin(r)\hat{\varphi}\big(\tfrac{r}{M^j}\big)\Big|.$$

For large j, $|\sin(r)|$ is much more rapidly varying than $\hat{\varphi}\big(\frac{r}{M^j}\big)$ and replacing $|\sin(r)|$ by its average value, $\frac{2}{\pi}$, introduces only a small error,

$$\begin{aligned}\big\|c^{(j)}(\mathbf{x}, \mathbf{x}')\big\|_{1,\infty} &= \Big(\frac{4}{\pi^2} + O\big(\tfrac{1}{M^j}\big)\Big)\int_0^\infty dr\ r\Big|\hat{\varphi}\big(\tfrac{r}{M^j}\big)\Big| \\ &= M^{2j}\Big(\frac{4}{\pi^2} + O\big(\tfrac{1}{M^j}\big)\Big)\int_0^\infty dr\ r\big|\hat{\varphi}(r)\big|. \qquad \text{(A.1)}\end{aligned}$$

Observe that $\big\|c^{(j)}(\mathbf{x}, \mathbf{x}')\big\|_{1,\infty}$ is a factor of about M^j larger than $\sup_{\mathbf{k}}\big|c^{(j)}(\mathbf{k})\big| \sim M^j$.

Now introduce a sectorization Σ of the Fermi surface $\big\{\ \mathbf{k} \in \mathbb{R}^3\ \big|\ |\mathbf{k}| = 1\ \big\}$ as in Subsect. 8 of §II (for details, see Def. VI.2) using approximately square sectors of side $\frac{1}{M^{j/2}}$. We may construct, also as in Subsect. 8 of §II (for details, see (XIII.2) of [FKTo3] and Lemma XII.3 of [FKTo3]), a partition of unity, χ_s, $s \in \Sigma$, of the support of $\varphi\big(M^j(|\mathbf{k}| - 1)\big)$ such that

$$c_s^{(j)}(\mathbf{k}) = \frac{M^j}{|\mathbf{k}|}\varphi\big(M^j(|\mathbf{k}| - 1)\big)\chi_s(\mathbf{k}) \qquad c_s^{(j)}(\mathbf{x}, \mathbf{x}') = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}c_s^{(j)}(\mathbf{k})$$

obeys $\big\|c_s^{(j)}(\mathbf{x}, \mathbf{x}')\big\|_{1,\infty} \le \mathit{const}\ M^j$. The proof of this bound was sketched in Subsect. 7 of §II. For details, see Prop. XIII.1and Lemma XIII.2of [FKTo3]. Observe that, with sectorization,

$$\big\|c_s^{(j)}(\mathbf{x}, \mathbf{x}')\big\|_{1,\infty} \le \mathit{const}\ \sup_{\mathbf{k}}\big|c_s^{(j)}(\mathbf{k})\big|.$$

Had we chosen sectors of side $\frac{1}{M^{\aleph j}}$ with $\aleph < \frac{1}{2}$, this would no longer be the case. Also observe that, since $|\Sigma|$ is of order $\left(M^{j/2}\right)^2$,

$$\left\|c^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} \le \sum_{s\in\Sigma} \left\|c_s^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} \le \mathit{const} \left(M^{j/2}\right)^2 M^j = \mathit{const}\, M^{2j}$$

recovers (A.1), up to an unimportant constant, by a technique that extends to nonround Fermi surfaces, for which explicit computations of $c^{(j)}(\mathbf{x},\mathbf{x}')$ are not available.

Finally, consider

$$G_2(\mathbf{x},\mathbf{x}') = \int d\mathbf{y}\; c^{(j)}(\mathbf{x},\mathbf{y})c^{(j)}(\mathbf{y},\mathbf{x}').$$

This would be a (first order) contribution to a model with covariance $c^{(j)}$ and an ultralocal quadratic interaction. If we attempt to bound $\left\|G_2(\mathbf{x},\mathbf{x}')\right\|_{1,\infty}$ just using $\left\|c^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty}$, we get

$$\begin{aligned}\left\|G_2(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} &= \int d\mathbf{x}\left|G_2(\mathbf{x},\mathbf{0})\right| = \int d\mathbf{x}\left|\int d\mathbf{y}\; c^{(j)}(\mathbf{x},\mathbf{y})c^{(j)}(\mathbf{y},\mathbf{0})\right| \\ &\le \int d\mathbf{x}\int d\mathbf{y}\left|c^{(j)}(\mathbf{x},\mathbf{y})c^{(j)}(\mathbf{y},\mathbf{0})\right| = \left\|c^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty}^2 \sim \text{const}\, M^{4j},\end{aligned}$$

which is a terrible answer: $G_2(\mathbf{x},\mathbf{x}')$ is the Fourier transform of $c^{(j)}(\mathbf{k})^2 = \frac{M^{2j}}{|\mathbf{k}|^2}\varphi(M^j(|\mathbf{k}|-1))^2$ and $\frac{1}{|\mathbf{k}|^2}\varphi(M^j(|\mathbf{k}|-1))^2$ is very much like $\frac{1}{|\mathbf{k}|}\varphi(M^j(|\mathbf{k}|-1))$, so the real behaviour of $\left\|G_2(\mathbf{x},\mathbf{x}')\right\|_{1,\infty}$ is M^{3j}. We may recover this real behaviour using sectors,

$$\begin{aligned}\left\|G_2(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} &= \int d\mathbf{x}\left|\int d\mathbf{y}\; c^{(j)}(\mathbf{x},\mathbf{y})c^{(j)}(\mathbf{y},\mathbf{0})\right| \\ &= \int d\mathbf{x}\left|\sum_{s,s'\in\Sigma}\int d\mathbf{y}\; c_s^{(j)}(\mathbf{x},\mathbf{y})c_{s'}^{(j)}(\mathbf{y},\mathbf{0})\right| \\ &\le \sum_{s,s'\in\Sigma}\int d\mathbf{x}\left|\int d\mathbf{y}\; c_s^{(j)}(\mathbf{x},\mathbf{y})c_{s'}^{(j)}(\mathbf{y},\mathbf{0})\right|.\end{aligned}$$

But $\int d\mathbf{y}\; c_s^{(j)}(\mathbf{x},\mathbf{y})c_{s'}^{(j)}(\mathbf{y},\mathbf{0})$ is the Fourier transform of $c_s^{(j)}(\mathbf{k})c_{s'}^{(j)}(\mathbf{k}) = c^{(j)}(\mathbf{k})^2\chi_s(\mathbf{k})\chi_{s'}(\mathbf{k})$, which vanishes identically unless the supports of χ_s and $\chi_{s'}$ overlap. For each fixed $s\in\Sigma$, there are at most 9 sectors $s'\in\Sigma$ that overlap with s. Hence

$$\begin{aligned}\left\|G_2(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} &\le 9|\Sigma|\max_{s,s'\in\Sigma}\int d\mathbf{x}\left|\sum_{s,s'\in\Sigma}\int d\mathbf{y}\; c_s^{(j)}(\mathbf{x},\mathbf{y})c_{s'}^{(j)}(\mathbf{y},\mathbf{0})\right| \\ &\le 9|\Sigma|\max_{s,s'\in\Sigma}\left\|c_s^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty}\left\|c_{s'}^{(j)}(\mathbf{x},\mathbf{x}')\right\|_{1,\infty} \le \text{const}\,|\Sigma| M^{2j} \\ &\le \text{const}\, M^{3j}\end{aligned}$$

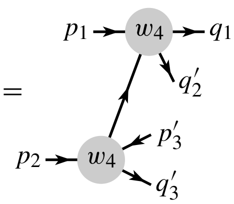
as desired.

Example A.2 (Overlapping Loops and Sectors). Let Γ be the diagram of Subsect. 4 in §II with vertices w_4 fulfilling the bounds

$$|||\hat{\omega}_4|||_{1,\Sigma} = O\big(\tfrac{1}{\mathfrak{l}}\big), \qquad\qquad |||\hat{\omega}_4|||_{3,\Sigma} = O\big(1\big)$$

of (II.11) and (II.15). The naive power counting bound (II.11) gives that $|||\hat{\Gamma}|||_{1,\Sigma} = O\big(\frac{1}{\mathfrak{l}}\big)$. We show that exploiting overlapping loop volume improvement leads to the bounds $|||\hat{\Gamma}|||_{1,\Sigma} = O(1)$ and $|||\hat{\Gamma}|||_{3,\Sigma} = O(\mathfrak{l})$.

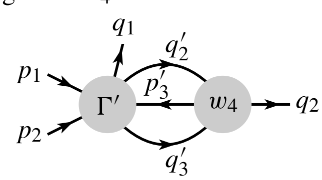
Let $\Gamma'(p_1, p_2, p_3', q_1, q_2', q_3')$ be the six legged subgraph consisting of the left two vertices of Γ.

$\Gamma' =$ 

As in (II.10),

$$|||\hat{\Gamma}'|||_{1,\Sigma} \le 9\, |||\hat{\omega}_4|||_{1,\Sigma}^2 \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_1 \le O\big(\tfrac{M^j}{\mathfrak{l}^2}\big),$$
$$|||\hat{\Gamma}'|||_{3,\Sigma} \le 9\, |||\hat{\omega}_4|||_{3,\Sigma} |||\hat{\omega}_4|||_{1,\Sigma} \max_{s\in\Sigma} \|C_s^{(j)}(x)\|_1 \le O\big(\tfrac{M^j}{\mathfrak{l}}\big).$$

Clearly, Γ is obtained by joining Γ' to w_4 with three lines.

$\Gamma =$ 

As in Subsect. 8 of §II, by conservation of momentum,

$$\begin{aligned}&\hat{\Gamma}\big((x_1,s_1),(x_2,s_2),(x_3,s_3),(x_4,s_4)\big)\\ &\quad= \sum_{\substack{\sigma_i,\sigma_i',\sigma_i''\in\Sigma\\ \sigma_i\cap\sigma_i'\cap\sigma_i''\neq\emptyset\\ \text{for } i=1,2,3}} \int dy_1dy_2dy_3\, dz_1dz_2dz_3\; \hat{\Gamma}'\big((x_1,s_1),(x_2,s_2),(y_1,\sigma_1),(x_3,s_3),(y_2,\sigma_2),(y_3,\sigma_3)\big)\\ &\qquad\qquad \times C^{(j)}_{\sigma_1'}(y_1-z_1)\, C^{(j)}_{\sigma_2'}(z_2-y_2)\, C^{(j)}_{\sigma_3'}(z_3-y_3)\; \hat{\omega}_4\big((z_2,\sigma_2''),(z_3,\sigma_3''),(z_1,\sigma_1''),(x_4,s_4)\big),\end{aligned}$$

so that

$$\begin{aligned}&\big\|\hat{\Gamma}\big((\cdot,s_1),(\cdot,s_2),(\cdot,s_3),(\cdot,s_4)\big)\big\|_{1,\infty}\\ &\quad\le \sum_{\substack{\sigma_i,\sigma_i',\sigma_i''\in\Sigma\\ \sigma_i\cap\sigma_i'\cap\sigma_i''\neq\emptyset\\ \text{for } i=1,2,3}} \|\hat{\Gamma}'\big((\cdot,s_1),(\cdot,s_2),(\cdot,\sigma_1),(\cdot,s_3),(\cdot,\sigma_2),(\cdot,\sigma_3)\big)\|_{1,\infty}\\ &\qquad\qquad \times \|C^{(j)}_{\sigma_1'}(x)\|_1 \|C^{(j)}_{\sigma_2'}(x)\|_\infty \|C^{(j)}_{\sigma_3'}(x)\|_\infty\; \|\hat{\omega}_4\big((\cdot,\sigma_2''),(\cdot,\sigma_3''),(\cdot,\sigma_1''),(\cdot,s_4)\big)\|_{1,\infty}.\end{aligned}$$

For fixed sectors s_1, s_2, s_3,

$$\begin{aligned}
&\sum_{s_4} \big\| \hat{\Gamma}\big((\cdot,s_1),(\cdot,s_2),(\cdot,s_3),(\cdot,s_4)\big) \big\|_{1,\infty} \\
&\quad \le 3^6 \sum_{\sigma_1,\sigma_2,\sigma_3\in\Sigma} \| \hat{\Gamma}'((\cdot,s_1),(\cdot,s_2),(\cdot,\sigma_1),(\cdot,s_3),(\cdot,\sigma_2),(\cdot,\sigma_3)) \|_{1,\infty} \Big(\max_{\sigma_1'\in\Sigma} \| C^{(j)}_{\sigma_1'}(x) \|_1 \Big) \\
&\qquad \times \Big(\max_{\sigma'\in\Sigma} \| C^{(j)}_{\sigma'}(x) \|_\infty \Big)^2 \max_{\sigma_1'',\sigma_2'',\sigma_3''\in\Sigma} \sum_{s_4} \| \hat{\omega}_4((\cdot,\sigma_2''),(\cdot,\sigma_3''),(\cdot,\sigma_1''),(\cdot,s_4)) \|_{1,\infty} \\
&\quad \le O\Big(M^j \big(\tfrac{\mathfrak{l}}{M^j}\big)^2 \Big) |||\hat{\omega}_4|||_{3,\Sigma} \sum_{\sigma_1,\sigma_2,\sigma_3\in\Sigma} \| \hat{\Gamma}'((\cdot,s_1),(\cdot,s_2),(\cdot,\sigma_1),(\cdot,s_3),(\cdot,\sigma_2),(\cdot,\sigma_3)) \|_{1,\infty},
\end{aligned}$$

since, for a given sector σ_i, there are three sectors σ_i' and three sectors σ_i'' with $\sigma_i \cap \sigma_i' \ne \emptyset$, $\sigma_i \cap \sigma_i'' \ne \emptyset$. We see that the contribution to $|||\hat{\Gamma}|||_{3,\Sigma}$ with fixed s_1, s_2, s_3 is bounded by

$$O\big(M^j \big(\tfrac{\mathfrak{l}}{M^j}\big)^2 \big) |||\hat{\omega}_4|||_{3,\Sigma} |||\Gamma'|||_{3,\Sigma} = O\Big(M^j \ \tfrac{\mathfrak{l}^2}{M^{2j}} \ 1 \ \tfrac{M^j}{\mathfrak{l}} \Big) = O(\mathfrak{l})$$

and, taking the sector sum over s_2, s_3, the contribution to $|||\hat{\Gamma}|||_{1,\Sigma}$ with fixed s_1 is bounded by

$$O\big(M^j \big(\tfrac{\mathfrak{l}}{M^j}\big)^2 \big) |||\hat{\omega}_4|||_{3,\Sigma} |||\Gamma'|||_{1,\Sigma} = O\Big(M^j \ \tfrac{\mathfrak{l}^2}{M^{2j}} \ 1 \ \tfrac{M^j}{\mathfrak{l}^2} \Big) = O(1).$$

The contributions with other sectors fixed are estimated in the same way.

Example A.3 (Sectorization and Change of Scale). Each time the renormalization group flows to a new scale, the associated sector decomposition changes. Therefore, in the notation of §II, we have to choose new sectorized representatives for $w_2,\ w_4,\ \cdots$ and compare the norms of these new sectorized representatives with respect to the new sectorization to the norms of the old sectorized representatives with respect to the old sectorization. To isolate this problem, suppose that $j' > j$, that Σ and Σ', respectively, are sectorizations of scale j and j', respectively, of length $\mathfrak{l}$ and $\mathfrak{l}'$, respectively and that $\mathfrak{l}' < \mathfrak{l}$. Furthermore, let $\omega_{2n}((p_1,s_1),\ \cdots,\ (p_n,s_n),\ (q_1,s_{n+1}),\ \cdots,\ (q_n,s_{2n}))$ be a Σ–sectorized representative of w_{2n}. Using the partition of unity $\{\chi_{s'}\}_{s'\in\Sigma'}$ subordinate so Σ', one constructs the Σ'–sectorized representative

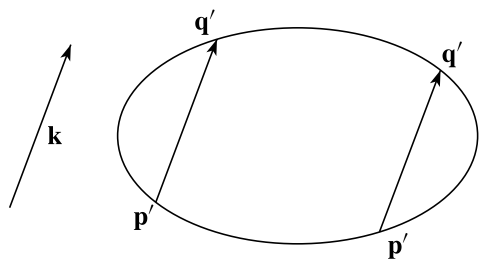
$$\omega'_{2n}((p_1,s_1'),\ \cdots,\ (q_n,s_{2n}')) = \sum_{\substack{s_i'\in\Sigma' \\ s_i'\cap s_i\ne\emptyset \\ 1\le i\le 2n}} \chi_{s_1'}(p_1)\cdots\chi_{s_{2n}'}(q_n)\ \omega_{2n}((p_1,s_1),\ \cdots,\ (q_n,s_{2n}))$$

of w_{2n}. The norm $\big|\big|\big|\hat{\omega}'_{2n}\big|\big|\big|_{1,\Sigma'}$ of (II.6) is defined in terms of a supremum over $s' \in \Sigma'$ of a sum over

$$\begin{aligned}
&\mathrm{Mom}_i(s') \\
&\quad = \Big\{ (s_1',\cdots,s_{2n}') \in \Sigma'^{2n} \ \Big|\ s_i' = s' \ \text{ and there exist } p_\ell \in s_\ell',\ q_\ell \in s_{n+\ell}',\ 1 \le \ell \le n \\
&\qquad\qquad \text{such that } p_1 + \cdots + p_n = q_1 + \cdots + q_n \Big\}.
\end{aligned}$$

It is natural to partition the sum over $\mathrm{Mom}_i(s')$ into a sum over $(s_1,\cdots,s_{2n}) \in \Sigma^{2n}$ followed by a sum over elements of $\mathrm{Mom}_i(s')$ that obey $s_\ell \cap s_\ell' \ne \emptyset$ for all $1 \le \ell \le 2n$. Now, we will try to motivate that, "morally", for any fixed $s_1,\ \cdots\ s_{2n} \in \Sigma$, there are at most $\left[\text{const}\,\frac{\mathfrak{l}}{\mathfrak{l}'}\right]^{2n-3}$ elements of $\mathrm{Mom}_i(s')$ obeying $s_\ell \cap s'_\ell \neq \emptyset$ for all $1 \le \ell \le 2n$. We may assume that $i = 1$. Then s'_1 must be s'. Denote by I_ℓ the interval on the Fermi curve F that has length $\mathfrak{l} + 2\mathfrak{l}'$ and is centered on $s_\ell \cap F$. If $s' \in \Sigma'$ intersects s_ℓ, then $s' \cap F$ is contained in I_ℓ. Every sector in Σ' contains an interval of F of length $\frac{3}{4}\mathfrak{l}'$ that does not intersect any other sector in Σ'. (The specific number $\frac{3}{4}$ comes from Def. VI.2. It is not important.) At most $\left[\frac{4}{3}\frac{\mathfrak{l}+2\mathfrak{l}'}{\mathfrak{l}'}\right]$ of these "hard core" intervals can be contained in I_ℓ. Thus there are at most $\left[\frac{4}{3}\frac{\mathfrak{l}}{\mathfrak{l}'} + 3\right]^{2n-3}$ choices for $s'_i,\ i \neq 1, n, 2n$.

Fix $s'_i,\ i \neq n, 2n$. Once s'_n is chosen, s'_{2n} is essentially uniquely determined by conservation of momentum. But the desired bound demands more. It says, roughly speaking, that both s'_n and s'_{2n} are essentially uniquely determined. As p_ℓ and q_ℓ run over s'_ℓ and $s'_{n+\ell}$, respectively, for $1 \le \ell \le n-1$, the sum $\mathbf{p}_1 + \cdots + \mathbf{p}_{n-1} - \mathbf{q}_1 - \cdots - \mathbf{q}_{n-1}$ runs over a small set centered on some point $\mathbf{k}$. In order for $(s'_1, \cdots, s'_{2n})$ to be in $\mathrm{Mom}_1(s')$, there must exist $\mathbf{p}' \in s'_n \cap F$ and $\mathbf{q}' \in s'_{2n} \cap F$ with $\mathbf{q}' - \mathbf{p}'$ very close to $\mathbf{k}$. But $\mathbf{q}' - \mathbf{p}'$ is a secant joining two points of the Fermi curve F. We have assumed that F is strictly convex.



Consequently, for any given $\mathbf{k} \neq 0$ in $\mathbb{R}^2$ there exist at most two pairs $(\mathbf{p}', \mathbf{q}') \in F^2$ with $\mathbf{q}' - \mathbf{p}' = \mathbf{k}$. So, if $\mathbf{k}$ is not near the origin, s'_n and s'_{2n} are almost uniquely determined. If $\mathbf{k}$ is close to zero, then $\mathbf{p}_1 + \cdots + \mathbf{p}_{n-1} - \mathbf{q}_1 - \cdots - \mathbf{q}_{n-1}$ must be close to zero and the number of allowed $s'_i,\ i \neq n, 2n$ is reduced. Careful application of these types of arguments yields (Prop. XIX.4 of [FKTo4])

$$\begin{aligned}
\left|\!\left|\!\left| \hat{\omega}'_{2n} \right|\!\right|\!\right|_{1,\Sigma'} &\le const^n \left[\tfrac{\mathfrak{l}}{\mathfrak{l}'}\right]^{2n-3} \Big(|||\hat{\omega}_{2n}|||_{1,\Sigma} + \tfrac{1}{\mathfrak{l}}\, |||\hat{\omega}_{2n}|||_{3,\Sigma} \Big), \\
|||\hat{\omega}'_{2n}|||_{3,\Sigma'} &\le const^n \left[\tfrac{\mathfrak{l}}{\mathfrak{l}'}\right]^{2n-4} |||\hat{\omega}_{2n}|||_{3,\Sigma}.
\end{aligned} \tag{A.2}$$

For this reason, we usually estimate the combination $|||\hat{\omega}_{2n}|||_{1,\Sigma} + \frac{1}{\mathfrak{l}}\,|||\hat{\omega}_{2n}|||_{3,\Sigma}$. The analog of (II.11) and (II.15) for this combination is

$$|||\hat{\omega}_{2n}|||_{1,\Sigma} + \tfrac{1}{\mathfrak{l}}\,|||\hat{\omega}_{2n}|||_{3,\Sigma} \ \text{ is of order } \ \tfrac{M^{j(n-2)}}{\mathfrak{l}^{n-1}} \ \text{ for all } n.$$

Thanks to (A.2), this bound is preserved, for $n > 2$, when the sector decomposition is refined,

$$\begin{aligned}
&M^{-(j+1)(n-2)}\mathfrak{l}'^{\,n-1}\Big(\left|\!\left|\!\left|\hat{\omega}'_{2n}\right|\!\right|\!\right|_{1,\Sigma'} + \tfrac{1}{\mathfrak{l}'} \left|\!\left|\!\left|\hat{\omega}'_{2n}\right|\!\right|\!\right|_{3,\Sigma'} \Big) \\
&\quad \le const^n\, M^{-(j+1)(n-2)}\mathfrak{l}'^{\,n-1}\Big[\tfrac{\mathfrak{l}}{\mathfrak{l}'}\Big]^{2n-3}\Big(|||\hat{\omega}_{2n}|||_{1,\Sigma} + \tfrac{1}{\mathfrak{l}}\,|||\hat{\omega}_{2n}|||_{3,\Sigma} \Big) \\
&\quad = const^n\, M^{-(n-2)}\Big[\tfrac{\mathfrak{l}}{\mathfrak{l}'}\Big]^{n-2}\ \ M^{-j(n-2)}\mathfrak{l}^{n-1}\Big(|||\hat{\omega}_{2n}|||_{1,\Sigma} + \tfrac{1}{\mathfrak{l}}\,|||\hat{\omega}_{2n}|||_{3,\Sigma} \Big) \\
&\quad = const^n\, M^{-(1-\aleph)(n-2)}\ \ M^{-j(n-2)}\mathfrak{l}^{n-1}\Big(|||\hat{\omega}_{2n}|||_{1,\Sigma} + \tfrac{1}{\mathfrak{l}}\,|||\hat{\omega}_{2n}|||_{3,\Sigma} \Big).
\end{aligned}$$

We even have a small factor, $M^{-(1-\aleph)(n-2)}$, available for eating up constants like $const^n$.

Notation

Not'n	Description	Reference
$\mathcal{E}$	counterterm space	Definition I.1
r_0	number of k_0 derivatives tracked	following (I.3)
r	number of $\mathbf{k}$ derivatives tracked	following (I.3)
M	scale parameter, $M > 1$	before Definition I.2
$\nu^{(j)}(k)$	j^{th} scale function	Definition I.2
$\nu^{(\geq j)}(k)$	$\sum_{i \geq j} \nu^{(j)}(k)$	Definition I.2
n_0	degree of asymmetry	Definition I.10
$\vert\vert\vert \cdot \vert\vert\vert_{1,\Sigma}$	no derivatives, all but 1 sector summed	(II.6)
$\vert\vert\vert \cdot \vert\vert\vert_{3,\Sigma}$	no derivatives, all but 3 sectors summed	(II.14)
J	particle/hole swap operator	(III.3)
$\Omega_S(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\mathcal{W}(\phi, \psi+\zeta)} \, d\mu_S(\zeta)$	Definition III.1
$\tilde{\Omega}_C(\mathcal{W})(\phi, \psi)$	$\log \frac{1}{Z} \int e^{\phi J \zeta} \, e^{\mathcal{W}(\phi, \psi+\zeta)} d\mu_C(\zeta)$	Definition III.1

References

[A1] Anderson, P.W.: Luttinger-liquid behavior of the normal metallic state of the 2D Hubbard model. Phys. Rev. Lett. **64**, 1839–1841 (1990)

[A2] Anderson, P.W.: Singular forward scattering in the 2D Hubbard model and a renormalized Bethe Ansatz ground state. Phys. Rev. Lett. **65**, 2306–2308 (1990)

[AGD] Abrikosov, A.A., Gorkov, L.P., Dzyaloshinski, I.E.: *Methods of Quantum Field Theory in Statistical Physics*. New York: Dover Publications, 1963

[AM] Ashcroft, N.W., Mermin, N.D.: *Solid State Physics*, Philadelphia, PA: Saunders College, 1976

[BG] Benfatto, G., Gallavotti, G.: *Renormalization Group*. Physics Notes, Vol. **1**, Princeton, NJ: Princeton University Press, 1995

[C] Caianello, E.R.: Number of Feynman Diagrams and Convergence. Nuovo Cimento **3**, 223–225 (1956)

[DMR] Disertori, M., Magnen, J. Rivasseau, V.: Interacting Fermi liquid in three dimensions at finite temperature: Part I: Convergent Contributions. Ann. Henri Poincaré **2**, 733–806 (2001)

[DR1] Disertori, M., Rivasseau, V.: Interacting Fermi liquid in two dimensions at finite temperature. Part I: Convergent Attributions. Commun. Math. Phys. **215**, 251–290 (2000)

[DR2] Disertori, M., Rivasseau, V.: Interacting Fermi liquid in two dimensions at finite temperature. Part II: Renormalization. Commun. Math. Phys. **215**, 291–341 (2000)

[FKLT1] Feldman, J., Knörrer, H., Lehmann, D., Trubowitz, E.: Fermi Liquids in Two-Space Dimensions. In: *Constructive Physics*, V. Rivasseau, ed., Springer Lecture Notes in Physics **446**, Berlin-Heidelberg-New York: Springer Verlag, 1995, pp. 267–300

[FKLT2] Feldman, J., Knörrer, H., Lehmann, D., Trubowitz, E.: Are There Two Dimensional Fermi Liquids?. In: *Proceedings of the XIth International Congress of Mathematical Physics*, D. Iagolnitzer, ed., Cambridge, MA: International Press, 1995, pp. 440–444

[FKLT3] Feldman, J., Knörrer, H., Lehmann, D., Trubowitz, E.: A Class of Fermi Liquids. In: *Particles and Fields*, G. Semenoff, L. Vinet, eds., CRM Series in Mathematical Physics, New York: Springer-Verlag, 1999, pp. 35–62

[FKTa] Feldman, J., Knörrer, H., Trubowitz, E.: Asymmetric Fermi Surfaces for Magnetic Schrödinger Operators. Commun. Partial Differ.Eq. **25**, 319–336 (2000)

[FKTcf] Feldman, J., Knörrer, H., Trubowitz, E.: A Nonperturbative Representation for Fermionic Correlation Functions. Commun. Math. Phys. **195**, 465–493 (1998)

[FKTf2] Feldman, J., Knörrer, H., Trubowitz, E.: A Two Dimensional Fermi Liquid, Part 2: Convergence. Commun. Math. Phys. **247**, 49–111 (2004)

[FKTf3] Feldman, J., Knörrer, H., Trubowitz, E.: A Two Dimensional Fermi Liquid, Part 3: The Fermi Surface. Commun. Math. Phys. **247**, 113–177 (2004)

[FKTffi] Feldman, J., Knörrer, H., Trubowitz, E.: Fermionic Functional Integrals and the Renormalization Group, CRM Monograph Series 16, American Mathematical Society (2002)

[FKTl] Feldman, J., Knörrer, H., Trubowitz, E.: Particle–Hole Ladders. Commun. Math. Phys. **247**, 179–194 (2004)

[FKTo1] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 1: Insulators. Rev. Math. Phys. **15**, 949–993 (2003)
[FKTo2] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 2: The First Scale. Rev. Math. Phys. **15**, 995–1037 (2003)
[FKTo3] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 3: Sectorized Norms. Rev. Math. Phys. **15**, 1039–1120 (2003)
[FKTo4] Feldman, J., Knörrer, H., Trubowitz, E.: Single Scale Analysis of Many Fermion Systems, Part 4: Sector Counting. Rev. Math. Phys. **15**, 1121–1169 (2003)
[FKTr1] Feldman, J., Knörrer, H., Trubowitz, E.: Convergence of Perturbation Expansions in Fermionic Models, Part 1: Nonperturbative Bounds. Commun. Math. Phys. **247**, 195–242 (2004)
[FKTr2] Feldman, J., Knörrer, H., Trubowitz, E.: Convergence of Perturbation Expansions in Fermionic Models, Part 2: Overlapping Loops. Commun. Math. Phys. **247**, 243–319 (2004)
[FMRT] Feldman, J., Magnen, J., Rivasseau, V., Trubowitz, E.: An Infinite Volume Expansion for Many Fermion Green's Functions. Helv. Phys. Acta. **65**, 679–721 (1992)
[FST1] Feldman, J., Salmhofer, M., Trubowitz, E.: Perturbation Theory around Non–Nested Fermi Surfaces, I. Keeping the Fermi Surface fixed. J. Stat. Phys. **84**, 1209–1336 (1996)
[FST2] Feldman, J., Salmhofer, M., Trubowitz, E.: Regularity of the Moving Fermi Surface: RPA Contributions. Commun. Pure and Appl. Math. **LI**, 1133–1246 (1998)
[FST3] Feldman, J., Salmhofer, M., Trubowitz, E.: Regularity of Interacting Nonspherical Fermi Surfaces: The Full Self–Energy. Commun. Pure and Appl. Math. **LII**, 273–324 (1999)
[FST4] Feldman, J., Salmhofer, M., Trubowitz, E.: An inversion theorem in Fermi surface theory. Commun. Pure and Appl. Math. **LIII**, 1350–1384 (2000)
[FW] Fetter, A.L., Walecka, J.D.: *Quantum Theory of Many-Particle Systems*. New York: McGraw-Hill, 1971
[KL] Kohn, W., Luttinger, J.M.: New Mechanism for Superconductivity. Phys. Rev. Lett. **15**, 524–526 (1965)
[L] Luttinger, J.M.: New Mechanism for Superconductivity. Phys. Rev. **150**, 202–214 (1966)
[L1] Landau, L.D.: The Theory of a Fermi Liquid. Sov. Phys. JETP **3**, 920 (1956)
[L2] Landau, L.D.: Oscillations in a Fermi Liquid. Sov. Phys. JETP **5**, 101 (1957)
[L3] Landau, L.D.: On the Theory of the Fermi Liquid. Sov. Phys. JETP **8**, 70 (1959)
[MR] Magnen, J., Rivasseau, V.: A Single Scale Infinite Volume Expansion for Three-Dimensional Many Fermion Green's Functions. Math. Phys. Electronic Journal **1**, 3 (1995)
[MCD] Metzner, W., Castellani, C., Di Castro, C.: Fermi Systems with Strong Forward Scattering. Adv. Phys. **47**, 317–445 (1998)
[N] Nozières, P.: *Theory of Interacting Fermi Systems*. New York: Benjamin, 1964
[PS] Pedra, W., Salmhofer, M.: Fermi Systems in Two Dimensions and Fermi Surface Flows. To appear in the *Proceedings of the 14th International Congress of Mathematical Physics*. Lisbon, 2003
[PT] Pöschel, J., Trubowitz, E.: *Inverse Spectral Theory*. New York: Academic Press, 1987
[S] Salmhofer, M.: Continuous renormalization for Fermions and Fermi liquid theory. Commun. Math. Phys. **194**, 249–295 (1998)

Communicated by J.Z. Imbrie