The Small Field Parabolic Flow for Bosonic Many–body Models:
Part 1 — Main Results and Algebra

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the "small field" approximation to the "parabolic flow" which exhibits the formation of a "Mexican hat" potential well. Here we state the main result of this analysis, outline the strategy of the proof, which uses a renormalization group flow, and perform the first, algebraic, part of a renormalization group step.

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Contents

1 Introduction

1.1 The model and dominant contributions to the effective action

An interacting many Boson system on a three–dimensional lattice in thermodynamic equilibrium is characterized by a single particle "kinetic energy" operator $\mathbf{h}$ on this lattice[1], a translation invariant two–body potential, $\mathbf{v}$ that describes the particle–particle interaction, the temperature $T > 0$ and the chemical potential μ. This paper is a contribution to a program that is to provide a mathematically rigorous investigation of the partition and correlation functions of such a gas of bosons. For simplicity, we assume that the underlying lattice has been scaled to be the unit lattice $\mathbb{Z}^3$. We also assume that both the two–body potential, $\mathbf{v}$ and the kernel of $\mathbf{h}$ decay exponentially in the distance between their arguments, and that $\mathbf{v}$ is the kernel of a strictly positive operator.

It is a standard strategy for the investigation of such a system to consider it as a limit of the correponding systems with a periodic cutoff $L_{\rm sp}$, as this infrared cutoff tends to infinity. The system with periodic cutoff $L_{\rm sp}$ is defined on the finite lattice

$$X = \mathbb{Z}^3 / L_{\rm sp}\mathbb{Z}^3$$

and is characterized by the periodizations[2] h of $\mathbf{h}$ and v of $\mathbf{v}$.

In previous papers we started an investigation of the partition function of the periodized system

$$\mathrm{Tr} e^{-\frac{1}{kT}(H-\mu N)}$$

where H is the second quantized Hamiltonian and N is the number operator. In [3, 4], we represented this partition function in terms of coherent state functional integrals (see also [25]) and then, in [7], using "decimation", controlled the "temporal ultraviolet limit" to obtain the following representation for the partition function. (The precise hypotheses are specified in [7, §2].)

There exists a constant $\theta > 0$ and a function $I_\theta(\alpha_*, \beta)$ of two complex valued

[1]The most commonly used $\mathbf{h} = -\frac{1}{2m}\Delta$, where Δ is the lattice Laplacian.

[2]The periodization of a translation invariant function $f(\xi_1, \cdots, \xi_n)$ on $(\mathbb{Z}^3)^n$ is the function on X^n that maps $(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in X^n$ to $\sum\limits_{\substack{\xi_2,\cdots\xi_n \\ [\xi_i]=\mathbf{x}_i}} f(\xi_1, \cdots, \xi_n)$, where ξ_1 is an arbitrary point in $\mathbb{Z}^3$ whose class $[\xi_1]$ in X equals $\mathbf{x}_1$. The periodization of an operator is the operator whose kernel is the periodization of the given operator.

fields α_* and β on X such that

$$\mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)} = \int \prod_{\tau\in\theta\mathbb{Z}\cap(0,1/kT]} \Big[\prod_{\mathbf{x}\in X} \tfrac{d\alpha_\tau(\mathbf{x})^*\wedge d\alpha_\tau(\mathbf{x})}{2\pi\imath}\, e^{-\alpha_\tau(\mathbf{x})^*\alpha_\tau(\mathbf{x})}\Big] I_\theta(\alpha^*_{\tau-\theta},\alpha_\tau) \tag{1.1}$$

One can write I_θ as the sum of a dominant part $I_\theta^{(SF)}$, called the *pure small field contribution*, and terms, indexed by proper subsets of X, which are nonperturbatively small[3], exponentially in the size of the subsets. The properties of the function $I_\theta^{(SF)}$ are reviewed in §D.1 and §D.2.

We want to control the integrals in the representation (1.1) of the partition function uniformly in small temperature T and lattice size $L_{\rm sp}$ to rigorously establish the phase transition in the many particle system of bosons, when the chemical potential μ lies sufficiently above a certain critical value. This phase transition is intimately related to the formation of a "mexican hat" shaped potential well in the effective action. See, for example, [19] and [28, §19] for an introduction to symmetry breaking in general, and [1, 17, 20, 26] as general references to Bose-Einstein condensation. See [16, 18, 24, 27] for other mathematically rigorous work on the subject.

In this paper, we replace the function I_θ in (1.1) by $I_\theta^{(SF)}$, that is, we study

$$\int \prod_{\tau\in\theta\mathbb{Z}\cap(0,1/kT]} \Big[\prod_{\mathbf{x}\in X} \tfrac{d\alpha_\tau(\mathbf{x})^*\wedge d\alpha_\tau(\mathbf{x})}{2\pi\imath}\, e^{-\alpha_\tau(\mathbf{x})^*\alpha_\tau(\mathbf{x})}\Big] I_\theta^{(SF)}(\alpha^*_{\tau-\theta},\alpha_\tau) \tag{1.2}$$

Using this model, we exhibit the mechanism that leads to the onset of the potential well. A full fledged large field/small field analysis of (1.1) will be performed later.

To simplify the discussion, we assume that $L_{\rm sp}$ and $L_{\rm tp} = \frac{1}{\theta kT}$ are powers of some odd natural number $L > 2$. After rescaling the "temporal lattice" $\theta\mathbb{Z}/\frac{1}{kT}\mathbb{Z}$ to $\mathbb{Z}/\frac{1}{\theta kT}\mathbb{Z}$, (1.2) can be viewed as a functional integral over fields on the lattice

$$\mathcal{X}_0 = \big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big)\times\big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$$

For a point $x=(x_0,\mathbf{x})\in\mathcal{X}_0$, we call x_0 its time and $\mathbf{x}$ its spatial component. The "real inner product" for functions f,g on $\mathcal{X}_0$ is $\langle f,g\rangle_0 = \sum_{x\in\mathcal{X}_0} f(x)g(x)$.

In Proposition D.1, we show that, up to a normalization constant, the integral (1.2) can be written in the form

$$\int \Big[\prod_{x\in\mathcal{X}_0} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi\imath}\Big] e^{\mathcal{A}_0(\psi^*,\psi)}\chi_0(\psi) \tag{1.3}$$

[3] We call a function nonperturbatively small, if it is of order $O(e^{-1/\|v\|^\varepsilon})$ for some norm on v and some $\varepsilon>0$. A precise bound is given in [7, Theorem 2.18]

where

$$\mathcal{A}_0(\psi_*,\psi) = -\left\langle \psi_*,\ D_0\psi\right\rangle_0 - \mathcal{V}_0(\psi_*,\psi) + \mu_0\left\langle \psi_*,\ \psi\right\rangle_0 + \mathcal{R}_0(\psi_*,\psi) + \mathcal{E}_0(\psi_*,\psi) \quad (1.4)$$

and

- $D_0 = \mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0$ with $h_0 = \theta h$ and ∂_0 the forward time derivative $(\partial_0 f)(x_0,\mathbf{x}) = f(x_0+1,\mathbf{x}) - f(x_0,\mathbf{x})$,
- $\mathcal{V}_0(\psi_*,\psi) = \frac{1}{2}\sum\limits_{x_1,\cdots,x_4\in\mathcal{X}_0} V_0(x_1,x_2,x_3,x_4)\,\psi_*(x_1)\psi(x_2)\psi_*(x_3)\psi(x_4)$ is a quartic monomial whose translation invariant kernel V_0 is determined by v and h. It is invariant under $x_1 \leftrightarrow x_3$ and under $x_2 \leftrightarrow x_4$. Its average $\mathrm{v}_0 = \sum\limits_{x_2,x_3,x_4\in\mathcal{X}_0} V_0(0,x_2,x_3,x_4)$ is positive. The kernel $V_0(x_1,x_2,x_3,x_4)$ is the spatial periodization of a translation invariant, exponentially decaying kernel $\mathbf{V}_0$ on $\big((\mathbb{Z}/L_{\rm tp}\mathbb{Z})\times\mathbb{Z}^3\big)^4$.
- μ_0 is close to $\theta\mu$.
- $\mathcal{R}_0(\psi_*,\psi)$ and $\mathcal{E}_0(\psi_*,\psi)$ are perturbatively small, particle–number preserving functions. For the different characteristics and roles of $\mathcal{R}_0$ and $\mathcal{E}_0$, see Proposition D.1 and Theorem 1.17.
- $\chi_0(\psi)$ is a "small field cut off function".

h_0 acts only on the spatial variables of a function of $x = (x_0,\mathbf{x}) \in \mathcal{X}_0$. Observe that ψ^* denotes the complex conjugate of the field ψ, while ψ_* and ψ are treated as two independent complex valued fields on $\mathcal{X}_0$.
More details, including precise estimates, are given in Proposition D.1.

For a constant field $\psi(x) = \psi$, the dominant part, $-\mathcal{A}_0(\psi^*,\psi)$, of (minus) the action in (1.3), reduces to

$$\mathcal{V}_0(\psi^*,\psi) - \mu_0\left\langle \psi^*,\ \psi\right\rangle_0 = |\mathcal{X}_0|\big[\tfrac{1}{2}\mathrm{v}_0|\psi|^4 - \mu_0|\psi|^2\big] = |\mathcal{X}_0|\big[\tfrac{1}{2}\mathrm{v}_0\big(|\psi|^2 - \tfrac{\mu_0}{\mathrm{v}_0}\big)^2 - \tfrac{\mu_0^2}{2\mathrm{v}_0}\big]$$

and has a potential well. If μ_0 is of the order of v_0, this well is quite shallow. Using block spin transformations, as in [23, 2, 21], (see Definition 1.1 below) we will successively perform parts of the integral and show that the effective action after these block spin transformations has a much better developed potential well. We expect that the result of this paper will be the starting point for an analysis that is adapted to the symmetry breaking caused by the degenerate ground state in this potential well.

We believe (see the discussion before [13, Lemma A.1]) that the scenario described above holds whenever the chemical potential μ is bigger than some critical value that, to leading order in $\mathbf{v}$, should be

$$2\int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{\hat{\mathbf{v}}(\mathbf{0})+\hat{\mathbf{v}}(\mathbf{k})}{e^{\hat{\mathbf{h}}(\mathbf{k})/kT}-1} \quad (1.5)$$

where $\hat{\mathbf{h}}(\mathbf{k})$ and $\hat{\mathbf{v}}(\mathbf{k})$ are the Fourier transforms of $\mathbf{h}(\mathbf{x},\mathbf{0})$ and $\mathbf{v}(\mathbf{x},\mathbf{0})$. See (1.19), Lemma D.2 and Corollary D.3. Also observe that (1.5) converges to zero as $\beta \to \infty$. In this paper we assume $\mathbf{v}$ is small and that μ is bigger than (1.5) by a number that is at least a norm of $\mathbf{v}$ raised to a power that is a bit bigger than one. For details see (1.19).

After n block spin (and scaling) transformations, the partition function will be represented by a functional integral on the lattice[4]

$$\mathcal{X}_0^{(n)} = \big(\mathbb{Z}\times\mathbb{Z}^3\big)/\big(\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\times\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

where $L > 2$ is the odd natural number chosen above. The asymmetry in the time and spatial variables arises from the "parabolic scaling" of Definition 1.3, below.

Definition 1.1 (Blockspin Transformation)**.** Fix a nonnegative even function q in $L^1(\mathbb{Z}\times\mathbb{Z}^3)$.

(a) For a field ψ on $\mathcal{X}_0^{(n)}$ define the "averaged " field $Q\psi$ on

$$\mathcal{X}_{-1}^{(n+1)} = \big(L^2\mathbb{Z}\times L\mathbb{Z}^3\big)/\big(\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\times\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

by

$$(Q\psi)(y) = \sum_{x\in\mathbb{Z}\times\mathbb{Z}^3} q(x)\psi(y+[x])$$

where $[x]$ denotes the class of $x\in\mathbb{Z}\times\mathbb{Z}^3$ in the quotient space $\mathcal{X}_0^{(n)}$.

(b) If $F(\psi_*,\psi)$ is a function of complex valued fields ψ_*,ψ on $\mathcal{X}_0^{(n)}$, we define the *block spin transform* of F (with respect to q and a constant $a>0$) as the function

$$(\mathbb{T}F)(\theta_*,\theta) = \tfrac{1}{N_{\mathbb{T}}^{(n)}}\int\Big[\prod_{x\in\mathcal{X}_0^{(n)}}\tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big]e^{-aL^{-2}\langle\theta_*-Q\psi^*,\theta-Q\psi\rangle_{-1}}\,F(\psi^*,\psi)$$

of the fields θ_*,θ on $\mathcal{X}_{-1}^{(n+1)}$. Here, for any two fields f,g on $\mathcal{X}_{-1}^{(n+1)}$

$$\langle f,g\rangle_{-1} = L^5\sum_{y\in\mathcal{X}_{-1}^{(n+1)}} f(y)g(y)$$

and the normalization constant is $N_{\mathbb{T}}^{(n)} = \int\Big[\prod_{y\in\mathcal{X}_{-1}^{(n+1)}}\frac{d\theta(y)^*\wedge d\theta(y)}{2\pi i}\Big]e^{-aL^{-2}\langle\theta^*,\theta\rangle_{-1}}$. We choose $a=1$.

[4]We shall define a family of lattices $\mathcal{X}_j^{(n)}$ in Definition 1.5.

Remark 1.2. (a) As

$$1 = \tfrac{1}{N_{\mathbb{T}}^{(n)}} \int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \tfrac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] e^{-aL^{-2} \langle \theta^* - Q\psi^*, \theta - Q\psi \rangle_{-1}}$$

for all functions $F(\psi_*, \psi)$ fields on $\mathcal{X}_0^{(n)}$

$$\int \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \tfrac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] F(\psi^*, \psi) = \int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \tfrac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] (\mathbb{T}F)(\theta^*, \theta)$$

(b) We make a specific choices of q in Definition 1.11.d. The main result, Theorem 1.17, will apply for all sufficiently large L.

The lattice $\mathcal{X}_{-1}^{(n+1)}$ is coarser than the unit lattice $\mathcal{X}_0^{(n)}$. We choose to scale it back to a unit lattice.

Definition 1.3 (Scaling)**.** Let $\mathbb{L}$ be the linear isomorphism

$$\mathbb{L} : \mathcal{X}_0^{(n+1)} \to \mathcal{X}_{-1}^{(n+1)} \qquad\qquad (x_0, \mathbf{x}) \mapsto (L^2 x_0, L\mathbf{x})$$

For a field θ on $\mathcal{X}_{-1}^{(n+1)}$, we define the scaled field

$$(\mathbb{S}\theta)(x) = L^{3/2}\, \theta\big(\mathbb{L}x\big)$$

on $\mathcal{X}_0^{(n+1)}$. For a function $F(\theta_*, \theta)$ of fields on $\mathcal{X}_{-1}^{(n+1)}$, we define the function $(\mathbb{S}F)(\Psi_*, \Psi)$ of fields on $\mathcal{X}_0^{(n+1)}$ by

$$(\mathbb{S}F)(\Psi_*, \Psi) = F\big(\mathbb{S}^{-1}\Psi_*, \mathbb{S}^{-1}\Psi\big)$$

Remark 1.4. (a) For any function $\mathcal{F}(\theta_*, \theta)$ of fields on $\mathcal{X}_{-1}^{(n+1)}$

$$\int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \tfrac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \mathcal{F}(\theta^*, \theta) = \tfrac{1}{L^{3|\mathcal{X}_0^{(n+1)}|}} \int \Big[\prod_{x \in \mathcal{X}_0^{(n+1)}} \tfrac{d\Psi(x)^* \wedge d\Psi(x)}{2\pi i} \Big] (\mathbb{S}F)(\Psi^*, \Psi)$$

(b) The exponents $\frac{3}{2}$ of $L^{3/2}\, \theta\big(\mathbb{L}x\big)$ and 2 of $L^2 x_0$ in the definition of $\mathbb{S}$ have been chosen so that

$$\langle \psi^*, \partial_0 \psi \rangle_0 = \big\langle \mathbb{S}^{-1}\psi^*, \partial_0(\mathbb{S}^{-1}\psi) \big\rangle_{-1} \qquad \langle \psi^*, \Delta\psi \rangle_0 = \big\langle \mathbb{S}^{-1}\psi^*, \Delta\big(\mathbb{S}^{-1}\psi\big) \big\rangle_{-1}$$

That is, $\langle \psi^*, \partial_0\psi \rangle_0$ and $\langle \psi^*, \Delta\psi \rangle_0$ are "marginal". Because the time derivative ∂_0 is first order while the spatial Laplacian Δ is second order, we refer to Definition 1.3 as "parabolic scaling".

Applying Remarks 1.2.a and 1.4.a to (1.3) we see that, for any natural number n with $L^n \le \min\{\sqrt{L_{\rm tp}},\, L_{\rm sp}\}$,

$$\int \Big[\prod_{x\in\mathcal{X}_0} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi\imath} \Big]\, e^{\mathcal{A}_0(\psi^*,\psi)} = \tfrac{1}{\tilde{\mathcal{Z}}_n} \int \Big[\prod_{x\in\mathcal{X}_0^{(n)}} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i} \Big] \big[(\mathbb{S}\mathbb{T})^n \big(e^{\mathcal{A}_0} \big) \big] (\psi^*,\psi) \tag{1.6}$$

with $\tilde{\mathcal{Z}}_n = \prod_{j=1}^n L^{3|\mathcal{X}_0^{(j)}|}$. In this paper we argue that, for $n \le n_p$ (with n_p specified in Definition 1.11.b), the function $\big[(\mathbb{S}\mathbb{T})^n \big(e^{\mathcal{A}_0} \big) \big] (\psi^*,\psi)$ has — up to errors which can reasonably be expected to be nonperturbatively small (see [14] or [8, §2.2.2]) — a logarithm whose dominant term, A_n, described in Definition 1.5.b below, exhibits a much deeper potential well. See (1.8).

The representation (1.6) of the partition function is built by iterating block spin transformations. We use stationary phase to analyze each block spin integral, as in Definition 1.1.b, in the integrand of (1.6). It is then natural to express the (dominant part) of the integrand in terms of the composition of the stationary phase critical fields for the various block spin transformations. We shall call this composition the “background field”. The definition that we are about to give for the background field does not appear to have anything to do with compositions. The “composition law” Proposition 3.4.b shows that the background field is indeed a composition of critical fields.

Definition 1.5 (Background field and dominant part of the action)**.**

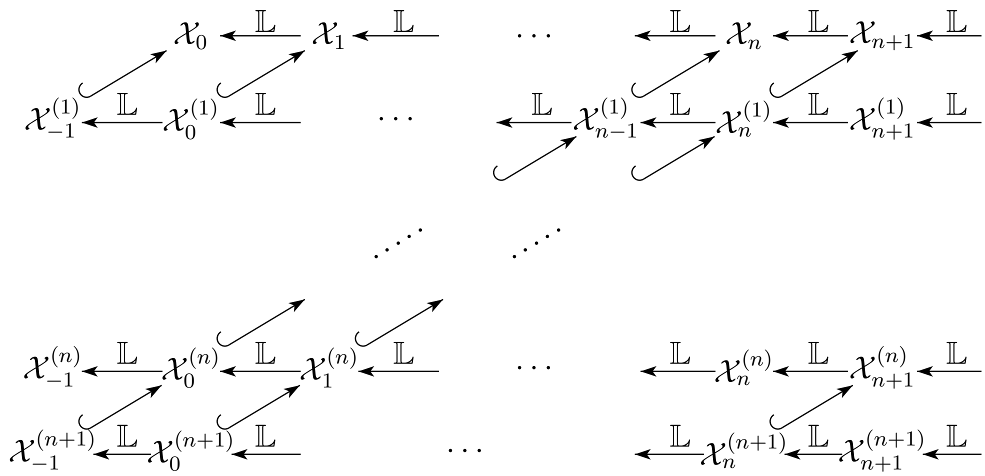
(a) For $j \ge -1$ and $n \ge 0$ define the lattices

$$\mathcal{X}_j^{(n)} = \big(\tfrac{1}{L^{2j}}\mathbb{Z} \big/ \tfrac{L_{\rm tp}}{L^{2(n+j)}}\mathbb{Z} \big) \times \big(\tfrac{1}{L^{j}}\mathbb{Z}^3 \big/ \tfrac{L_{\rm sp}}{L^{(n+j)}}\mathbb{Z}^3 \big)$$

The subscript in $\mathcal{X}_j^{(n)}$ determines the “coarseness” of the lattice — nearest neighbour points are a distance $\frac{1}{L^{2j}}$ apart in the time direction and a distance $\frac{1}{L^{j}}$ apart in spatial directions. The superscript in $\mathcal{X}_j^{(n)}$ determines the number of points in the lattice — $|\mathcal{X}_j^{(n)}| = |\mathcal{X}_0|/L^{5n}$ for all j. On $\mathcal{X}_j^{(n)}$, we use the integral notation $\int_{X_j^{(n)}} du = \frac{1}{L^{5j}} \sum_{u\in\mathcal{X}_j^{(n)}}$. The maps

$$\mathbb{L} : \mathcal{X}_j^{(n)} \to \mathcal{X}_{j-1}^{(n)} \qquad\qquad (u_0,\mathbf{u}) \mapsto (L^2 u_0, L\mathbf{u})$$

are linear isomorphisms. We routinely view $\mathcal{X}_j^{(n)}$ as a sublattice of $\mathcal{X}_{j+k}^{(n-k)}$, for each $1 \le k \le n$. We also abbreviate $\mathcal{X}_n^{(0)}$ by $\mathcal{X}_n$.



We denote by $\mathcal{H}_j^{(n)}$ the space of fields on $\mathcal{X}_j^{(n)}$, endowed with the *real* inner product

$$\langle \alpha_1, \alpha_2 \rangle_j = \int_{X_j^{(n)}} \alpha_1(u)\, \alpha_2(u)\; du$$

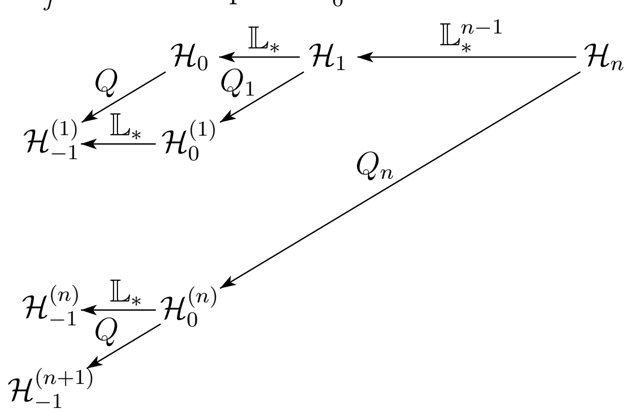
Set $\mathcal{H}_n = \mathcal{H}_n^{(0)}$. For a field $\alpha \in \mathcal{H}_j^{(n)}$, define the field $\mathbb{L}_*(\alpha) \in \mathcal{H}_{j-1}^{(n)}$ by $\mathbb{L}_*(\alpha)(\mathbb{L}u) = \alpha(u)$, and the field $Q^{(j)}\alpha \in \mathcal{H}_{j-1}^{(n+1)}$ by

$$Q^{(j)}\alpha = \mathbb{L}_*^{-j}\, Q\, \mathbb{L}_*^{j}\, \alpha$$

Set

$$Q_n = Q^{(1)} \circ \cdots \circ Q^{(n)} : \mathcal{H}_n = \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)}$$

Q_j is an iterated averaging operation that maps the space $\mathcal{H}_j^{(n-j)}$ of fields on the fine lattice $\mathcal{X}_j^{(n-j)}$ to the space $\mathcal{H}_0^{(n)}$ of fields on the unit lattice $\mathcal{X}_0^{(n)}$.



The "horizontal" operators $\mathbb{L}_*$ and $\mathbb{L}_*^{n-1}$ are isomorphisms.

The operator D_n on fields $\phi \in \mathcal{H}_n$ on $\mathcal{X}_n$ is defined by

$$D_n = L^{2n}\, \mathbb{L}_*^{-n}\, D_0\, \mathbb{L}_*^n$$

(b) For $\mu \in \mathbb{C}$, fields $\psi_*, \psi \in \mathcal{H}_0^{(n)}$ and $\phi_*, \phi \in \mathcal{H}_n$, and a quartic monomial $\mathcal{V}$ in the fields ϕ_*, ϕ, define, for $n \geq 1$,

$$\begin{aligned} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) &= \big\langle \psi_* - Q_n\phi_*, \mathfrak{Q}_n\big(\psi - Q_n\phi\big)\big\rangle_0 + \int_{\mathcal{X}_n} \phi_*(u)\,(D_n\phi)(u)\,du \\ &\qquad + \mathcal{V}(\phi_*, \phi) - \mu \int_{\mathcal{X}_n} \phi_*(u)\,\phi(u)\,du \\ &= \big\langle \psi_* - Q_n\phi_*, \mathfrak{Q}_n\big(\psi - Q_n\phi\big)\big\rangle_0 + \langle \phi_*,\, D_n\phi\rangle_n \\ &\qquad + \mathcal{V}(\phi_*, \phi) - \mu\, \langle \phi_*,\, \phi\rangle_n \end{aligned} \tag{1.7}$$

where

$$\mathfrak{Q}_n = \begin{cases} a\big(\mathbb{1} + \sum\limits_{j=1}^{n-1} \frac{1}{L^{2j}} Q_j Q_j^*\big)^{-1} & \text{if } n \geq 2 \\ a\mathbb{1} & \text{if } n = 1 \end{cases}$$

and Q_j^* denotes the transpose of Q_j with respect to the "real" inner product $\langle f_1, f_2\rangle_j$.

For $n = 0$, set

$$A_0(\psi_*, \psi, \mu, \mathcal{V}) = \langle \psi_*,\, D_0\psi\rangle_0 + \mathcal{V}(\psi_*, \psi) - \mu\, \langle \psi_*,\, \psi\rangle_0$$

In the case $n = 0$, we shall use $\mathcal{V} = \mathcal{V}_0$ and $\mu = \mu_0$ so that A_0 is the dominant part of the action $\mathcal{A}_0$ of (1.4). For $n \geq 0$ we shall use, in A_n, a quartic monomial $\mathcal{V}$ that is a perturbation of

$$\mathcal{V}_n^{(u)}(\phi_*, \phi) = \tfrac{1}{2} \int_{\mathcal{X}_n^4} du_1 \cdots du_4\; V_n^{(u)}(u_1, u_2, u_3, u_4)\, \phi_*(u_1)\phi(u_2)\phi_*(u_3)\phi(u_4)$$

where

$$V_n^{(u)}(u_1, u_2, u_3, u_4) = L^{14n}\, V_0(\mathbb{L}^n u_1, \mathbb{L}^n u_2, \mathbb{L}^n u_3, \mathbb{L}^n u_4)$$

is the kernel V_0 rescaled[5] to scale n. For μ we will use a "renormalized chemical potential" μ_n which will be described Theorem 1.17.

[5] For the origin of the L^{14n} see [13, Lemma C.2].

(c) In Proposition 1.14 and [15, Proposition 2.1], we solve, for $n \ge 1$, the *background field equations*

$$\tfrac{\partial}{\partial \phi_*} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) = \tfrac{\partial}{\partial \phi} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) = 0$$

We show that, for ψ_*, ψ sufficiently near $0 \in \mathcal{H}_0^{(n)}$, $\mu \in \mathbb{C}$ sufficiently small, and $\mathcal{V}$ sufficiently small, there are fields

$$\phi_{*n}(\psi_*, \psi, \mu, \mathcal{V}),\ \phi_n(\psi_*, \psi, \mu, \mathcal{V})$$

such that $\phi_* = \phi_{*n}(\psi_*, \psi, \mu, \mathcal{V})$, $\phi = \phi_n(\psi_*, \psi, \mu, \mathcal{V})$ solves these equations. The maps[6] $(\psi_*, \psi, \mu, \mathcal{V}) \mapsto \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})$ are analytic and are uniquely determined by this property. They are called the *background fields.*

We shall show that for all $0 \le n \le n_p$, with the number n_p specified in Definition 1.11.b, below, the dominant contribution to $\big[(\mathbb{ST})^n\big(e^{\mathcal{A}_0}\big)\big](\psi^*, \psi)$ is of the form

$$\exp\Big\{-A_n\big(\psi_*, \psi,\ \phi_{*n}(\psi_*, \psi, \mu_n, \mathcal{V}_n), \phi_n(\psi_*, \psi, \mu_n, \mathcal{V}_n),\ \mu_n, \mathcal{V}_n\big) + p_n(\psi^*, \psi)\Big\}\chi_n(\psi^*, \psi)$$

with a number μ_n, which we call the *renormalized chemical potential at scale* n, a quartic monomial $\mathcal{V}_n$ close to the monomial $\mathcal{V}_n^{(u)}$ of Definition 1.5.b, which we call the *renormalized interaction at scale* n, a "perturbative correction" $p_n(\psi^*, \psi)$, and a "small field cut off function" χ_n which is discussed in [14]. The renormalized chemical potential μ_n will grow with n like $L^{2n}(\mu_0 - \mu_*)$; see Theorem 1.17.

For constant, not too big, fields $\psi(x) = \psi$ and $\psi_*(x) = \psi^*$, the background field ϕ_n is again constant, again obeys $\phi_{*n} = \phi_n^*$ and is approximately equal to ψ. See [15, Remark 1.1]. So $\big\langle \psi_* - Q_n\phi_*, \mathfrak{Q}_n\big(\psi - Q_n\phi\big)\big\rangle_0 \approx 0$, and, in this case, the dominant part of the effective action $A_n(\psi_*, \psi,\ \phi_{*n}, \phi_n,\ \mu_n, \mathcal{V}_n)$

$$\tfrac{1}{L^{-5n}|\mathcal{X}_0|} A_n \approx \tfrac{1}{2}\tfrac{\mathrm{v}_0}{L^n}|\psi|^4 - \mu_n|\psi|^2 = \tfrac{1}{2}\tfrac{\mathrm{v}_0}{L^n}\big(|\psi|^2 - L^n\tfrac{\mu_n}{\mathrm{v}_0}\big)^2 - L^n\tfrac{\mu_n^2}{2\mathrm{v}_0} \tag{1.8}$$

has a much better developed potential well.

1.2 The stationary phase approximation

We want to argue that for all $1 \le n \le n_p$, and small fields ψ, a good approximation to $\big[(\mathbb{ST})^n\big(e^{\mathcal{A}_0}\chi_0\big)\big](\psi^*, \psi)$ is of the form

$$\exp\Big\{-A_n\big(\psi^*, \psi, \phi_{*n}, \phi_n,\ \mu_n, \mathcal{V}_n\big) + p_n(\psi^*, \psi)\Big\}\chi_n(\psi^*, \psi) \tag{1.9}$$

[6] We routinely use the "optional $*$" notation $\alpha_{(*)}$ to denote "α_* or α".

with the background fields

$$\phi_{*n} = \phi_{*n}(\psi^*, \psi, \mu_n, \mathcal{V}_n) \qquad \phi_n = \phi_n(\psi^*, \psi, \mu_n, \mathcal{V}_n)$$

as in Definition 1.5, with the renormalized chemical potential μ_n and the renormalized interaction $\mathcal{V}_n$ as above, and with a "perturbative correction" $p_n(\psi_*, \psi)$ which is an analytic function of the small fields ψ_*, ψ. To substantiate this claim, we will prove, that up to errors which can be expected to be nonperturbatively small,

$$(\mathbb{ST})\Big(\exp\Big\{ - A_n\big(\psi^*, \psi, \phi_{*n}, \phi_n,\, \mu_n, \mathcal{V}_n\big) + p_n(\psi^*, \psi)\Big\}\chi_n(\psi^*, \psi)\Big)$$

is again of the form (1.9), with n replaced by $n+1$.

When $n \geq 1$, application of the block spin transformation to the function (1.9) leads to the integral

$$\tfrac{1}{N_{\mathbb{T}}^{(n)}}\Big[\prod_{x\in\mathcal{X}_0^{(n)}} \int \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big]\; e^{-aL^{-2}\langle\theta_*-Q\psi^*,\theta-Q\psi\rangle_{-1} - A_n(\psi^*,\psi,\phi_{*n},\phi_n,\mu_n,\mathcal{V}_n)+p_n(\psi^*,\psi)}\chi_n \tag{1.10.a}$$

Similarly, when $n = 0$, application of the block spin transformation to the function $e^{\mathcal{A}_0}\chi_0$ leads to the integral

$$\tfrac{1}{N_{\mathbb{T}}^{(0)}}\Big[\prod_{x\in\mathcal{X}_0} \int \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big]\; e^{-aL^{-2}\langle\theta_*-Q\psi^*,\theta-Q\psi\rangle_{-1} - A_0(\psi^*,\psi,\mu_0,\mathcal{V}_0)+\mathcal{R}_0(\psi^*,\psi)+\mathcal{E}_0(\psi^*,\psi)}\chi_0 \tag{1.10.b}$$

We compute the dominant contributions to the integrals (1.10.a,b) by a "stationary phase" type calculation. The first step is to calculate the approximate critical point of the integrand. In Proposition 1.15, below, and Proposition 3.4.a, we prove that the critical field equations[7]

$$\begin{aligned} \nabla_{\psi_*}\big\{\tfrac{a}{L^2}\,\langle\theta_* - Q\psi_*, \theta - Q\psi\rangle_{-1} + A_n(\psi_*, \psi, \phi_{*n}, \phi_n, \mu_n, \mathcal{V}_n)\big\} = 0 \\ \nabla_{\psi}\big\{\tfrac{a}{L^2}\,\langle\theta_* - Q\psi_*, \theta - Q\psi\rangle_{-1} + A_n(\psi_*, \psi, \phi_{*n}, \phi_n, \mu_n, \mathcal{V}_n)\big\} = 0 \end{aligned} \tag{1.11}$$

have a solution

$$\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n)\ , \qquad \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)$$

for (θ_*, θ) in a neighbourhood of the origin in $\mathcal{H}_{-1}^{(n+1)} \times \mathcal{H}_{-1}^{(n+1)}$.

[7] When, $n = 0$, drop the arguments ϕ_{*n}, ϕ_n from A_n.

Typically, $\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n)$ is *not* the complex conjugate of $\psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)$, even when $\theta_* = \theta^*$. Therefore we consider the integral (1.10) as the integral of the holomorphic differential form

$$\frac{1}{N_{\mathbb{T}}^{(n)}} e^{-aL^{-2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\mu_n,\mathcal{V}_n)+p_n(\psi_*,\psi)} \prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\psi_*(x)\wedge d\psi(x)}{2\pi i} \tag{1.12}$$

over part of the real $2|\mathcal{X}_0^{(n)}|$–dimensional set $\big\{ (\psi_*, \psi) \bigm| \psi_* = \psi^* \big\}$ in the complex space $\mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)}$. The change of variables

$$\psi_* = \psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n) + \delta\psi_* \qquad \psi = \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n) + \delta\psi \tag{1.13}$$

maps the domain of integration to an appropriate subset, $I_n(\theta_*, \theta)$ of

$$\big\{(\delta\psi_*, \delta\psi) \in \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)} \bigm| \delta\psi_* = \delta\psi^* + \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)^* - \psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n)\big\}$$

We write the integral (1.10) as

$$\int_{I_n(\theta_*,\theta)} \tilde{\omega}_n$$

where $\tilde{\omega}_n$ is the holomorphic differential form obtained from (1.12) through the substitution (1.13). The leading part of $\tilde{\omega}_n$ is $e^{-\left\langle \delta\psi_*, (\frac{a}{L^2}Q^*Q+\Delta^{(n)})\delta\psi\right\rangle} \prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\delta\psi_*(x)\wedge d\delta\psi(x)}{2\pi i}$ where

$$\Delta^{(n)} = \begin{Bmatrix} \big(\mathbb{1} + \mathfrak{Q}_n Q_n D_n^{-1} Q_n^*\big)^{-1}\mathfrak{Q}_n & \text{if } n \ge 1 \\ D_0 & \text{if } n = 0 \end{Bmatrix} : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)} \tag{1.14}$$

See Lemma 4.1 and [12, Lemma 12]. In [10, Corollary 4.5] we show that the operator $(\frac{a}{L^2}Q^*Q + \Delta^{(n)})$ is invertible. To diagonalize the quadratic form in the resulting Gaussian integral, let $D^{(n)}$ be an operator square root of

$$C^{(n)} = (\tfrac{a}{L^2}Q^*Q + \Delta^{(n)})^{-1} \tag{1.15}$$

Denote by $\omega_n(\theta_*, \theta, \mu_n; p_n)$ the differential form (in the fields ζ_*, ζ on $\mathcal{X}_0^{(n)}$) obtained from $\tilde{\omega}_n$ through the second substitution

$$\delta\psi_* = D^{(n)*}\zeta_* \qquad \delta\psi = D^{(n)}\zeta$$

Note, again, that $D^{(n)*}$ is the transpose of $D^{(n)}$. As in [7], [6, Appendix A] and [8, §2.2.1] we construct a $(2|\mathcal{X}_0^{(n)}| + 1)$–dimensional set $\mathcal{Y}$ whose boundary consists of

- $\left\{ \ (\zeta_*, \zeta) \ \middle| \ (D^{(n)*}\zeta_*, D^{(n)}\zeta) \in I_n(\theta_*, \theta) \ \right\}$
- $B_n = \left\{ \ (\zeta_*, \zeta) \ \middle| \ \zeta_* = \zeta^* \ , \ |\zeta(x)| \le r_n \text{ for all } x \in \mathcal{X}_0^{(n)} \ \right\}$
- components on which we would expect $\omega_n(\theta_*, \theta, \mu_n; p_n)$ to be nonperturbatively small.

Here, r_n behaves like one over a very small power of a norm of $\mathcal{V}_n$. See Definition 1.11.c. Applying Stokes Theorem to the holomorphic – and hence closed – differential form, we expect, as in part (c) of [8, §2.2.2], that the difference between $\mathbb{T}\Big(e^{-A_n(\psi^*,\psi,\phi_{*n},\phi_n,\mu_n,\mathcal{V}_n)+p_n(\psi^*,\psi)}\chi_n\Big)$ and $\int_{B_n} \omega_n(\theta_*, \theta, \mu_n; p_n)$ is nonperturbatively small. See [14, Step 3].

Definition 1.6 (Approximate Blockspin Transformation)**.** Let $F(\psi_*, \psi)$ be an analytic function of complex valued fields ψ_*, ψ on $\mathcal{X}_0^{(n)}$. The approximate blockspin transform at scale n of F (with respect to q, the constant $a > 0$ and the radius r_n and the chemical potential μ and quartic interaction $\mathcal{V}$) is

$$
\begin{aligned}
&(\mathbb{T}_n^{(SF)} F)(\theta_*, \theta; \mu, \mathcal{V}) \\
&= \tfrac{1}{\tilde N_{\mathbb{T}}^{(n)}} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \int_{|\zeta(x)| \le r_n} \tfrac{d\zeta(x)^* \wedge d\zeta(x)}{2\pi i} \Big] e^{-aL^{-2}\langle \theta_* - Q\psi_*, \theta - Q\psi\rangle_{-1}} \, F(\psi_*, \psi) \Big|_{\substack{\psi_* = \psi_{*n}(\theta_*,\theta,\mu,\mathcal{V}) + D^{(n)*}\zeta^* \\ \psi = \psi_n(\theta_*,\theta,\mu,\mathcal{V}) + D^{(n)}\zeta}}
\end{aligned}
$$

where $\tilde N_{\mathbb{T}}^{(n)} = \frac{1}{\det C^{(n)}} N_{\mathbb{T}}^{(n)}$.

As said above, we expect the difference between

$$
\mathbb{T}\Big(e^{-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\mu_n,\mathcal{V}_n)+p_n(\psi_*,\psi)}\chi_n\Big) \quad \text{and} \quad \mathbb{T}_n^{(SF)}\Big(e^{-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\mu_n,\mathcal{V}_n)+p_n(\psi_*,\psi)}\Big)
$$

to be nonperturbatively small.

Our main result, Theorem 1.17, is a representation for

$$
\Big((\mathbb{S}\mathbb{T}_{n-1}^{(SF)}) \circ (\mathbb{S}\mathbb{T}_{n-2}^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big) \tag{1.16}
$$

where the starting point $e^{\mathcal{A}_0}$ is the output (1.3) of the ultraviolet flow, and $n \le n_p$.

1.3 The perturbative corrections

As said before, we shall show that, for $n \le n_p$, (1.16) has a logarithm, whose dominant term is of the form $-A_n(\psi^*, \psi, \phi_{*n}, \phi_n, \mu_n, \mathcal{V}_n)$ with renormalized chemical potential μ_n and a renormalized quartic interaction $\mathcal{V}_n$ close to $\mathcal{V}_n^{(u)}$. We will write the (perturbative) correction to it in the form

$$\mathcal{R}_n\big(\phi_{*n}(\psi_*, \psi, \mu_n, \mathcal{V}_n), \phi_n(\psi_*, \psi, \mu_n, \mathcal{V}_n)\big) + \mathcal{E}_n(\psi_*, \psi)$$

where ϕ_{*n}, ϕ_n are the background fields of Definition 1.5, $\mathcal{R}_n(\phi_*, \phi)$ is a low degree polynomial in fields on the fine lattice $\mathcal{X}_n$, and $\mathcal{E}_n(\psi_*, \psi)$ is an analytic function on a neighborhood of the origin in $\mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)}$. Some motivation for the need to distinguish between "high degree" and "low degree" monomials and for our choice of the particular form of the "low degree monomials" is provided in [13, Remark 5.4].

Remark 1.7. We choose to express the "low degree" parts, A_n and $\mathcal{R}_n$, of the effective action as functions of the background field $\phi_{(*)n}$, which are in turn functions of $\psi_{(*)}$, rather than directly as functions of $\psi_{(*)}$. Here is a brief motivation. During the course of each renormalization group step we perform an integral over ψ. To do so, we make the change of variables, $\psi = \psi_n + \delta\psi$, (see (1.13)), where ψ_n is a critical field. The leading part of the critical field is a linear operator, which is not particularly small, acting on the field "ψ of the next scale" (which is a scaled version of θ). See Proposition 1.15, below. If we simply substitute this leading part into a monomial in $\psi_{(*)}$ of degree p, we again get a monomial of degree p, but our bound on the kernel of the monomial can grow because of the linear operator. On the other hand if we substitute the full critical field into a monomial in $\phi_{(*)n}(\psi_*, \psi)$, we get, by the composition law Proposition 3.4, followed by the appropriate scaling, the monomial in $\phi_{(*)n+1}(\psi_*, \psi)$ with the identical kernel.

On the other hand, we choose to express the "high degree" part, $\mathcal{E}_n$, of the effective action directly as a function of $\psi_{(*)}$. If we were to express it, instead, through the background field $\phi_{(*)n}$, it would be defined on the fine lattice $\mathcal{X}_n$ but would only be translation invariant with respect to the unit sublattice $\mathcal{X}_0^{(n)}$. This would complicate the process of localization and renormalization.

The functions $\mathcal{E}_n(\psi_*, \psi)$ and $\mathcal{R}_n(\phi_*, \phi)$ will depend on the fields in their arguments both directly and through partial derivatives of the fields. To make this precise, we write

$$\tilde{\mathcal{H}}_j^{(n)} = \big\{\ \tilde{\alpha} = \big(\alpha, \{\alpha_\nu\}_{\nu=0,1,2,3}\big)\ \big|\ \alpha, \alpha_\nu \in \mathcal{H}_j^{(n)}\ \big\} \tag{1.17}$$

We shall write

$$\mathcal{E}_n(\psi_*, \psi) = \tilde{\mathcal{E}}_n\big((\psi_*, \{\partial_\nu \psi_*\}), (\psi, \{\partial_\nu \psi\})\big)$$

with an analytic function $\tilde{\mathcal{E}}_n$ on a neighbourhood of the origin in $\tilde{\mathcal{H}}_0^{(n)} \times \tilde{\mathcal{H}}_0^{(n)}$. In the next subsection we describe how we measure the size of $\tilde{\mathcal{E}}_n$. Similarly we shall write

$$\mathcal{R}_n(\phi_*, \phi) = \tilde{\mathcal{R}}_n\big((\phi_*, \{\partial_\nu \phi_*\}), (\phi, \{\partial_\nu \phi\})\big)$$

with a polynomial $\tilde{\mathcal{R}}_n$ on $\tilde{\mathcal{H}}_n^{(0)} \times \tilde{\mathcal{H}}_n^{(0)}$. Here, for a field α on $\mathcal{X}_j^{(n)}$ and $\nu = 0, 1, 2, 3$ we define the forward derivative by

$$(\partial_\nu \alpha)(x) = \begin{cases} L^{2j}\big(\alpha\big(x + \frac{1}{L^{2j}} e_0\big) - \alpha(x)\big) & \text{if } \nu = 0 \\ L^{j}\big(\alpha\big(x + \frac{1}{L^{j}} e_\nu\big) - \alpha(x)\big) & \text{if } \nu = 1, 2, 3 \end{cases}$$

where e_ν is a unit vector in the ν^{th} direction. To make this precise we use the

Definition 1.8 (Monomial type). For a vector $\vec{p} = (p_u, p_0, p_{\rm sp})$ of nonnegative integers, a monomial of type $\vec{p}$ in the fields $\tilde{\alpha}_*, \tilde{\alpha} \in \tilde{\mathcal{H}}_j^{(n-j)}$ is a function of the form

$$\int_{\mathcal{X}_j^{(n-j)}} du_1 \cdots du_p \; M(u_1, \cdots, u_p) \prod_{\ell=1}^{p} \alpha_{\sigma_\ell}(u_\ell)$$

where each α_{σ_ℓ} is one of $\alpha_*, \alpha, \big\{\alpha_{*\nu}, \alpha_\nu\big\}_{\nu=0}^{3}$ but with

- the number of α_{σ_ℓ}'s that are α_* or α being p_u and
- the number of α_{σ_ℓ}'s that are α_{*0} or α_0 being p_0 and
- the number of α_{σ_ℓ}'s that are $\alpha_{*\nu}$ or α_ν for some $1 \le \nu \le 3$ being $p_{\rm sp}$.

In the monomial above there are p_u undifferentiated fields, p_0 fields corresponding to time derivatives and $p_{\rm sp}$ fields corresponding to space derivatives. The subscript u stands for "undifferentiated" and the subscript sp stands for "spatial".
A polynomial of type $\vec{p}$ is a sum of monomials of type $\vec{p}$.

$\tilde{\mathcal{R}}_n$ will be a sum, over $\vec{p} \in \mathfrak{D}$, of polynomials, $\tilde{\mathcal{R}}_n^{(\vec{p})}$, of type $\vec{p}$ in the fields $\tilde{\phi}_*, \tilde{\phi} \in \tilde{\mathcal{H}}_n^{(0)}$ where

$$\mathfrak{D} = \big\{(1,1,0)\,,\,(0,1,1)\,,\,(0,0,2)\,,\,(6,0,0)\big\} \tag{1.18}$$

The motivation for this choice of $\mathfrak{D}$ is provided in [13, Remark 5.4].

In the next subsection we describe how we measure the size of the kernels in this representation.

1.4 Norms for measuring the size of the perturbative corrections

Let $\mathcal{X}$ be any lattice that is equipped with a metric d and a "cell volume" vol. As an example, the lattice $\mathcal{X}_j^{(n-j)}$ has vol $= \frac{1}{L^{5j}}$. The following Definition describes how we measure the size of the kernels $R_n^{(\vec{p})}$ as above.

Definition 1.9. Let $f(u_1, \cdots, u_r)$ be a function on $\mathcal{X}^r$. For a mass $\mathfrak{m} \ge 0$ we set

$$\|f\|_{\mathfrak{m}} = \max_{i=1\cdots,r} \max_{u_i} \int du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_r \, |f(u_1, \cdots, u_r)|\, e^{\mathfrak{m}\tau(u_1,\cdots,u_r)}$$

where *the tree length* $\tau(u_1, \cdots, u_r)$ is the minimal length of a tree in $\mathcal{X}$ that has $u_1, \cdots, u_r$ among its vertices, and $\int du\ g(u) = \mathrm{vol} \sum_{u\in\mathcal{X}} g(u)$.

As in [22], our perturbative corrections are analytic functions of the fields. The following definitions describe how we measure the size of complex valued analytic functions of fields, like $\mathcal{E}_n(\psi_*, \psi)$ and $\tilde{\mathcal{E}}_n\big((\psi_*, \{\psi_{*\nu}\}), (\psi, \{\psi_\nu\})\big)$. The norms in the following definition are special cases of the norms in [5, Definition 2.6].

Definition 1.10.

(a) For a field α on $\mathcal{X}$ and $\vec{x} = (x_1, \cdots, x_r) \in \mathcal{X}^r$ we set $\alpha(\vec{x}) = \prod_{i=1}^r \alpha(x_i)$.

(b) A power series $\mathcal{F}$ in the fields $\alpha_1, \cdots, \alpha_s$, on $\mathcal{X}$ has a unique expansion

$$\mathcal{F}(\alpha_1, \cdots, \alpha_s) = \sum_{r_1,\cdots,r_s\ge 0} \mathrm{vol}^{r_1+\cdots+r_s} \sum_{\substack{\vec{x}_i\in\mathcal{X}^{r_i} \\ 1\le i\le s}} f_{r_1,\cdots,r_s}\big(\vec{x}_1, \cdots, \vec{x}_s\big) \prod_{i=1}^s \alpha_i(\vec{x}_i)$$

where the coefficients $f\big(\vec{x}_1, \cdots, \vec{x}_s\big)$ are invariant under permutations of the components of each vector $\vec{x}_i$.

(c) For each choice of "weights" $\kappa_1, \cdots, \kappa_s > 0$, for the fields $\alpha_1, \cdots, \alpha_s$, we define *the norm of* $\mathcal{F}$ *with mass* $\mathfrak{m}$ *and weights* $\kappa_1, \cdots, \kappa_s > 0$ to be

$$\sum_{r_1,\cdots,r_s\ge 0} \big\| f_{r_1,\cdots,r_s}\big(\vec{x}_1, \cdots, \vec{x}_s\big)\big\|_{\mathfrak{m}} \prod_{i=1}^s \kappa_i^{r_i}$$

Similarly, Definition A.3 describes how we measure the size of analytic maps like the background field map $(\psi_*, \psi) \mapsto \phi_n(\psi_*, \psi, \mu_n, \mathcal{V}_n)$.

1.5 The Starting Point Setup

We shall state our results in terms of an abstraction of the output of the temporal ultraviolet limit outlined following (1.3) and (1.4). We assume that we are given a mass $m > 0$, positive odd integers L_{tp} and L_{sp}, a small real number $\epsilon > 0$, and

- a kinetic energy operator
$$\mathbf{h}_0 = \nabla^* \mathbf{H} \nabla$$
where $\mathbf{H}$ is a real, translation invariant, reflection invariant, strictly positive definite operator on the space, $L^2\big((\mathbb{Z}^3)^*\big)$, of functions on the set, $(\mathbb{Z}^3)^*$, of nearest neighbor bonds of the lattice $\mathbb{Z}^3$. The operator $\nabla : L^2\big(\mathbb{Z}^3\big) \to L^2\big((\mathbb{Z}^3)^*\big)$ is given by
$$\big(\nabla f\big)\big(\,\langle \mathbf{x}, \mathbf{y}\rangle\,\big) = f(\mathbf{y}) - f(\mathbf{x})$$
We assume that the kernel of $\mathbf{H}$ is exponentially decaying with $\|\mathbf{h}_0\|_{2m}$ finite.
- a kernel $\mathbf{V}_0(x_1, x_2, x_3, x_4)$ on $\big((\mathbb{Z}/L_{\rm tp}\mathbb{Z})\times\mathbb{Z}^3\big)^4$ that is invariant under $x_1 \leftrightarrow x_3$ and under $x_2 \leftrightarrow x_4$ and under the symmetry group $\mathfrak{S}$ of Definition B.1. We assume that its average
$$\mathrm{v}_0 = \sum_{x_2,x_3,x_4\in\mathbb{Z}^3} \mathbf{V}_0(0, x_2, x_3, x_4) > 0 \qquad \text{and} \qquad \mathfrak{v}_0 = 2\|\mathbf{V}_0\|_{2m}$$
are sufficiently small.
- a real chemical potential μ_0 obeying
$$\mu_* + \mathfrak{v}_0^{\frac{4}{3}-16\epsilon} \le \mu_0 \le \mathfrak{v}_0^{\frac{8}{9}+\epsilon}$$
where[8]
$$\mu_* = 2\int_{((\mathbb{Z}/L_{\rm tp}\mathbb{Z})\times\mathbb{Z}^3)^3} dx_1 \cdots dx_3\ \mathbf{V}_0(0, x_1, x_2, x_3)\ \mathbf{D}_0^{-1}(x_3, 0) \tag{1.19}$$
with $\mathbf{D}_0 = \mathbb{1} - e^{-\mathbf{h}_0} - e^{-\mathbf{h}_0}\partial_0$.

The periodized versions, on the lattice $\mathcal{X}_0 = \big(\mathbb{Z}\times\mathbb{Z}^3\big)/\big(L_{\rm tp}\mathbb{Z}\times L_{\rm sp}\mathbb{Z}^3\big)$, of $\mathbf{h}_0$, $\mathbf{V}_0$, $\mathbf{D}_0$ are denoted h_0, V_0 and D_0, respectively. We also assume that we are given

- an $\mathfrak{S}$ invariant polynomial $\tilde{\mathcal{R}}_0(\tilde\psi_*, \tilde\psi) = \sum_{\vec p\in\mathfrak{D}} \tilde{\mathcal{R}}_0^{(\vec p)}(\tilde\psi_*, \tilde\psi)$ on $\tilde{\mathcal{H}}_0 \times \tilde{\mathcal{H}}_0$. Each $\tilde{\mathcal{R}}_0^{(\vec p)}$ is a polynomial of type $\vec p$ with a real valued kernel[9] that obeys the bound
$$\big\|\tilde{\mathcal{R}}_0^{(\vec p)}\big\|_m \le \mathfrak{r}_{\vec p}(0) = \begin{cases} \mathfrak{v}_0^{2-\epsilon} & \text{if } \vec p = (6,0,0) \\ \mathfrak{v}_0^{1-4\epsilon} & \text{otherwise} \end{cases}$$
- $\mathcal{E}_0(\psi_*, \psi)$ is an $\mathfrak{S}$ invariant, particle–number preserving function with real valued kernels and with $\mathcal{E}_0(0,0) = 0$, that

[8]We show, in Lemma D.2, that, to leading order, μ_* is θ times (1.5).

[9]That is, $\tilde{\mathcal{R}}_0^{(\vec p)}((\psi_*, \{\psi_{*\nu}\}), (\psi, \{\psi_\nu\})) = \tilde{\mathcal{R}}_0^{(\vec p)}((\psi_*^*, \{\psi_{*\nu}^*\}), (\psi^*, \{\psi_\nu\}^*))$.

◦ is of degree at least four both in ψ_* and in ψ, and
◦ whose norm

$$\|\mathcal{E}_0\|^{(0)} \le \mathfrak{v}_0^{\epsilon}$$

where $\|\cdot\|^{(0)}$ is the norm with mass m which associates the weight $\kappa(0) = \frac{1}{\mathfrak{v}_0^{1/3-\epsilon}}$ to the fields ψ_*, ψ .

We set

$$\mathcal{A}_0(\psi_*,\psi) = -\left\langle \psi_*,\, D_0\psi\right\rangle_0 - \mathcal{V}_0(\psi_*,\psi) + \mu_0\left\langle \psi_*,\, \psi\right\rangle_0 + \mathcal{R}_0(\psi_*,\psi) + \mathcal{E}_0(\psi_*,\psi) \quad (1.20)$$

where

$$\begin{aligned} \mathcal{V}_0(\psi_*,\psi) &= \tfrac{1}{2}\int_{\mathcal{X}_0^4} dx_1\cdots dx_4\; V_0(x_1,x_2,x_3,x_4)\,\psi_*(x_1)\psi(x_2)\psi_*(x_3)\psi(x_4) \\ \mathcal{R}_0(\psi_*,\psi) &= \tilde{\mathcal{R}}_0\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big) \end{aligned}$$

Under reasonable conditions on the various parameters, the small field part of the output of the temporal ultraviolet limit, described following (1.2), satisfies these conditions. This is stated in more detail and proven in Corollary D.3.

1.6 The main results

We start by defining a number of parameters that will be used in the statement of the main results.

Definition 1.11.

(a) Set

$$\begin{aligned} \kappa(n) &= \tfrac{L^{\eta n}}{\mathfrak{v}_0^{1/3-\epsilon}} &&\text{with} && \eta = \tfrac{1}{2} + \tfrac{1}{3}\tfrac{\log \mathfrak{v}_0}{\log(\mu_0-\mu_*)} \\ \kappa'(n) &= \tfrac{L^{\eta' n}}{\mathfrak{v}_0^{1/3-\epsilon}} &&\text{with} && \eta' = \tfrac{3}{2} - \tfrac{\log \mathfrak{v}_0}{\log(\mu_0-\mu_*)} - \epsilon \\ \mathfrak{e}_{\mathfrak{l}}(n) &= L^{\eta_{\mathfrak{l}} n}\, \mathfrak{v}_0^{\frac{1}{3}-2\epsilon} &&\text{with} && \eta_{\mathfrak{l}} = \big(\frac{2}{3} - 4\epsilon\big)\,\tfrac{\log \mathfrak{v}_0}{\log(\mu_0-\mu_*)} \end{aligned}$$

With the notation of Definition 1.10, we define the norm $\|\tilde{\mathcal{E}}\|^{(n)}$, of an analytic function $\tilde{\mathcal{E}}(\tilde{\psi}_*,\tilde{\psi})$, as the norm with mass m which associates the weight $\kappa(n)$ to the fields ψ_*, ψ , and the weight $\kappa'(n)$ to the fields $\psi_{\nu*}, \psi_\nu$, $\nu = 0,\cdots,3$. Similarly, we define $\|\tilde{\mathcal{E}}\|_m$ as the norm with mass m which associates the weight 1 to all fields. The parameter $\mathfrak{e}_{\mathfrak{l}}(n)$ will be used as an upper bound on the size of the output of the fluctuation integral in the n^{th} step.

(b) The number of steps we perform in this paper, using the "parabolic flow", is the largest integer n_p such that

$$L^{2n_p}(\mu_0 - \mu_*) \le \big(\tfrac{\mathfrak{v}_0}{L^{n_p}}\big)^{5\epsilon} \iff L^{(2+5\epsilon)n_p}(\mu_0 - \mu_*) \le \mathfrak{v}_0^{5\epsilon}$$

(c) For the radius of integration of the fluctuation variables in Definition 1.6, we choose

$$r_n = \tfrac{1}{4}\kappa_{\mathfrak{l}}(n+1) \qquad \text{where } \kappa_{\mathfrak{l}}(n) = \big(\tfrac{L^n}{\mathfrak{v}_0}\big)^{\epsilon/2}$$

(d) To specify the averaging profile q of Definition 1.1.a we fix[10] a (small) even natural number $\mathfrak{q} \ge 4$ and denote by $1_{\square}(x)$ the characteristic function of the rectangle $\big[-\frac{L^2-1}{2}, \frac{L^2-1}{2}\big] \times \big[-\frac{L-1}{2}, \frac{L-1}{2}\big]^3$ in $\mathbb{Z} \times \mathbb{Z}^3$. Set

$$q = \tfrac{1}{L^{5\mathfrak{q}}} \overbrace{1_{\square} * 1_{\square} * \cdots * 1_{\square}}^{\mathfrak{q} \text{ times}}$$

to be the convolution of $1_{\square}$ with itself $\mathfrak{q}$ times, normalized to have integral one. Properties of q are discussed in [10, §2].

Remark 1.12. By construction

$$\tfrac{3}{4} + 8\epsilon < \tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)} < \tfrac{9}{8} - \epsilon \qquad \tfrac{3}{4} + 2\epsilon < \eta < \tfrac{7}{8} - \tfrac{\epsilon}{3} \qquad \tfrac{3}{8} < \eta' < \tfrac{3}{4} - 8\epsilon$$

Also, by Definition 1.11.b and the condition $\mu_0 - \mu_* \ge \mathfrak{v}_0^{\frac{4}{3} - 16\epsilon}$ of §1.5,

$$L^{n_p} \le \frac{1}{\mathfrak{v}_0^{\frac{2}{3} - 8\epsilon}}$$

For the approximate block spin transformations to be well defined, we need to make sure that the background fields of Definition 1.5.c and the critical fields of (1.11) are defined for small fields. The main technical work here is to show that the linearized equations for the background field (see [12, Definition 3.a and (1')]) are solvable. This is guaranteed by

Theorem 1.13 (Green's Functions)**.** *There are constants $\mu_{\rm up}, m_0 > 0$ and $\Gamma_{\rm op}$, that are independent of n and L, such that the following hold. Let $0 \le n \le n_p$, $0 \le m \le \frac{m_0}{2}$ and $|\mu| \le \mu_{\rm up}$. The operators $D_n + Q_n^* \mathfrak{Q}_n Q_n$ and $D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu$ on $\mathcal{H}_0^{(n)}$ are invertible. We set*

$$S_n = (D_n + Q_n^* \mathfrak{Q}_n Q_n)^{-1} \qquad S_n(\mu) = (D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu)^{-1}$$

Then

$$\|S_n\|_{2m}\,,\ \|S_n(\mu)\|_{2m} \le \Gamma_{\rm op}$$

[10] The reasons for requiring that $\mathfrak{q} \ge 4$ are discussed in [10, Remark 2.7].

This theorem is proven in [10, Proposition 5.1].

Proposition 1.14 (Background fields)**.** *Let $1 \le n \le n_p$. Let μ be a complex number and $\mathcal{V}(\phi_*, \phi)$ be a quartic monomial with $\|\mathcal{V}\|_{2m}\kappa(n)^2 + |\mu|$ sufficiently small. Then there are analytic maps*

$$(\psi_*, \psi) \mapsto \phi_{*n}(\psi_*, \psi, \mu, \mathcal{V}),\ \ \phi_n(\psi_*, \psi, \mu, \mathcal{V})$$

to $\mathcal{H}_n$, that are defined for all $(\psi_*, \psi) \in \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)}$ obeying $\big|\psi_*(x)\big|,\ \big|\psi(x)\big| \le \kappa(n)$ for all $x \in \mathcal{X}_0^{(n)}$, and that fulfill the background field equations

$$\tfrac{\partial}{\partial \phi_*} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) = \tfrac{\partial}{\partial \phi} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) = 0$$

Furthermore

$$\begin{aligned} \phi_{*n}(\psi_*, \psi, \mu, \mathcal{V}) &= S_n(\mu)^* Q_n^* \mathfrak{Q}_n\, \psi_* + \phi_{*n}^{(\ge 3)}(\psi_*, \psi, \mu, \mathcal{V}) \\ \phi_n(\psi_*, \psi, \mu, \mathcal{V}) &= S_n(\mu) Q_n^* \mathfrak{Q}_n\, \psi + \phi_n^{(\ge 3)}(\psi_*, \psi, \mu, \mathcal{V}) \end{aligned}$$

with analytic maps $\phi_{*n}^{(\ge 3)}$ and $\phi_n^{(\ge 3)}$ that are of degree at least three in (ψ_*, ψ).

This Proposition, with more details and bounds, is proven in [15, Proposition 2.1], using a contraction mapping argument.

Proposition 1.15 (Critical fields)**.** *Let $0 \le n < n_p$. Let μ be a complex number and $\mathcal{V}(\phi_*, \phi)$ be a quartic monomial with $\|\mathcal{V}\|_{2m}\kappa(n+1)^2 + |\mu|$ sufficiently small. Then there are analytic maps*

$$(\theta_*, \theta) \mapsto \psi_{*n}(\theta_*, \theta, \mu, \mathcal{V}),\ \ \psi_n(\theta_*, \theta, \mu, \mathcal{V})$$

to $\mathcal{H}_0^{(n)}$, that are defined for all $(\theta_*, \theta) \in \mathcal{H}_{-1}^{(n+1)} \times \mathcal{H}_{-1}^{(n+1)}$ obeying $\big|\theta_*(y)\big|,\ \big|\theta(y)\big| < \frac{\kappa(n+1)}{L^{3/2}}$ for all $y \in \mathcal{X}_{-1}^{(n+1)}$, and that fulfill the the critical field equations

$$\left.\begin{aligned} \nabla_{\psi_*}\Big\{ \tfrac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_n\big(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}\big)\Big|_{\phi_{(*)} = \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})} \Big\} = 0 \\ \nabla_{\psi}\Big\{ \tfrac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_n\big(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}\big)\Big|_{\phi_{(*)} = \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})} \Big\} = 0 \end{aligned}\right\} \text{if } n \ge 1$$

$$\left.\begin{aligned} \nabla_{\psi_*}\Big\{ \tfrac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_0\big(\psi_*, \psi, \mu, \mathcal{V}\big) \Big\} = 0 \\ \nabla_{\psi}\Big\{ \tfrac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_0\big(\psi_*, \psi, \mu, \mathcal{V}\big) \Big\} = 0 \end{aligned}\right\} \text{if } n = 0$$

Furthermore

$$\begin{aligned} \psi_{*n}(\theta_*, \theta, \mu, \mathcal{V}) &= \tfrac{a}{L^2} C^{(n)}(\mu)^* Q^*\, \theta_* + \psi_{*n}^{(\ge 3)}(\theta_*, \theta, \mu, \mathcal{V}) \\ \psi_n(\theta_*, \theta, \mu, \mathcal{V}) &= \tfrac{a}{L^2} C^{(n)}(\mu) Q^*\, \theta + \psi_n^{(\ge 3)}(\theta_*, \theta, \mu, \mathcal{V}) \end{aligned}$$

where[11]

$$C^{(n)}(\mu) = \left(\tfrac{a}{L^2} Q^* Q + \Delta^{(n)}(\mu)\right)^{-1}$$
$$\Delta^{(n)}(\mu) = \begin{cases} \mathfrak{Q}_n - \mathfrak{Q}_n\, Q_n S_n(\mu) Q_n^* \mathfrak{Q}_n & \text{if } n \geq 1 \\ D_0 - \mu & \text{if } n = 0 \end{cases}$$

are well defined operators. Both $\psi_{*n}^{(\geq 3)}$ *and* $\psi_n^{(\geq 3)}$ *are of degree at least three in* (θ_*, θ).

This Proposition, with more details and bounds, is proven in Proposition 3.4 and [15, Proposition 5.1and Remark 5.3].

Definition 1.16. The "scaling/weight relevant" monomials are those of type

$$\begin{aligned} \vec{p} \in \mathfrak{D}_{\rm rel} = \mathfrak{D} \cup \Big\{ (2,0,0)\ ,\ (1,0,1)\ ,\ (3,0,1),\ (4,0,0) \Big\} \\ = \Big\{ (2,0,0)\,, (1,1,0)\,, (1,0,1)\,, (0,1,1)\,, (0,0,2)\,, (4,0,0)\,, (3,0,1)\,, (6,0,0) \Big\} \end{aligned}$$

The motivation for this choice of $\mathfrak{D}_{\rm rel}$ is provided in [13, Remark 5.4].

Theorem 1.17. *Assume that the parameter* ϵ *of §1.5 is sufficiently small, that the parameter* L *of Definition 1.1 is sufficiently large, and that* $\mathfrak{v}_0$ *is sufficiently small, depending on* ϵ *and* L*. There exists, for each* $1 \leq n \leq n_p$,

- *a number* μ_n *with* $\big|\mu_n - L^{2n}(\mu_0 - \mu_*)\big| \leq \mathfrak{v}_0^{1-\epsilon} + L^{2n} \mathfrak{v}_0^{\frac{4}{3} - 15\epsilon}$
- *a quartic interaction* $\mathcal{V}_n(\phi_*, \phi)$ *with* $\|\mathcal{V}_n - \mathcal{V}_n^{(u)}\|_{2m} \leq \frac{1}{L^n} \mathfrak{v}_0^{\frac{5}{3} - 7\epsilon}$
- *a* $\mathfrak{S}$ *invariant polynomial* $\tilde{\mathcal{R}}_n(\tilde{\phi}_*, \tilde{\phi}) = \sum_{\vec{p} \in \mathfrak{D}} \tilde{\mathcal{R}}_n^{(\vec{p})}(\tilde{\phi}_*, \tilde{\phi})$ *on* $\tilde{\mathcal{H}}_n \times \tilde{\mathcal{H}}_n$ *. Each* $\tilde{\mathcal{R}}_n^{(\vec{p})}$ *is a polynomial of type* $\vec{p}$ *that obeys the bound*

$$\big\|\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_m \leq \begin{cases} \mathfrak{v}_0^{2-6\epsilon}\, L^{-4n} & \text{if } \vec{p} = (6,0,0) \\ \mathfrak{v}_0^{1-6\epsilon} & \text{if } \vec{p} = (1,1,0),\ (0,1,1),\ (0,0,2) \end{cases}$$

- *an* $\mathfrak{S}$ *invariant analytic function* $\tilde{\mathcal{E}}_n(\tilde{\psi}_*, \tilde{\psi})$ *with* $\tilde{\mathcal{E}}_n(0,0) = 0$*, whose power series expansion does not contain scaling/weight relevant monomials and which has norm*

$$\|\tilde{\mathcal{E}}_n\|^{(n)} \leq \mathfrak{v}_0^{\epsilon}$$

- *and a normalization constant* $\mathcal{Z}_n$

[11] By [12, Remark 10.a] $\Delta^{(n)}(0)$ is the $\Delta^{(n)}$ of (1.14) and consequently $C^{(n)}(0)$ is the $C^{(n)}$ of (1.15).

such that

$$\Big((\mathbb{ST}_{n-1}^{(SF)})\circ(\mathbb{ST}_{n-2}^{(SF)})\circ\cdots\circ(\mathbb{ST}_{0}^{(SF)})\Big)\Big(e^{\mathcal{A}_0}\Big)$$
$$= \tfrac{1}{\mathcal{Z}_n}\exp\Big\{-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\,\mu_n,\mathcal{V}_n)+\mathcal{R}_n(\phi_{*n},\phi_n)+\mathcal{E}_n(\psi_*,\psi)\Big\}$$

with the A_n *and the background fields*

$$\phi_{*n}=\phi_{*n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)\qquad \phi_n=\phi_n(\psi_*,\psi,\mu_n,\mathcal{V}_n)$$

as in Definition 1.5, and

$$\mathcal{R}_n(\phi_*,\phi)=\tilde{\mathcal{R}}_n\big((\phi_*,\{\partial_\nu\phi_*\}),(\phi,\{\partial_\nu\phi\})\big)$$
$$\mathcal{E}_n(\psi_*,\psi)=\tilde{\mathcal{E}}_n\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big)$$

Here, for each $0\le j\le n-1$, $\mathbb{T}_j^{(SF)}$ *is the approximate block spin transformation of Definition 1.6 with chemical potential* $\mu=\mu_j$ *and quartic interaction* $\mathcal{V}_j$.

Remark 1.18. Theorem 1.17 will be proven by induction on n. The proof runs over §4–[13, §6], and is completed at the end of [13, §6]. In the inductive proof for the estimates on μ_n, $\mathcal{V}_n$ and $\tilde{\mathcal{R}}_n$, we will prove slightly stronger estimates that are more suited for the induction. Set $\mu_0^*=\mu_0$ and, for $n\ge 1$,

$$\mu_n^*=L^{2n}\mu_0-\tfrac{2}{|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_n^4}du_1\cdots du_4\;V_n^{(u)}(u_1,u_2,u_3,u_4)\;S_n(u_4,u_1)$$

Then we will show that there are constants $C_{\delta\mathcal{V}}$ and $C_{\mathcal{R}}$ such that, for all $0\le n\le n_p$,

$$\begin{aligned}|\mu_n-\mu_n^*|&\le L^{2n}\,\mathfrak{v}_0^{1-8\epsilon}\sum_{\ell=1}^{n}\tfrac{1}{L^{(2-3\epsilon)\ell}}\big[\mathfrak{v}_0^{\frac{1}{3}-5\epsilon}+L^{2\ell}(\mu_0-\mu_*)\big]\\ \|\mathcal{V}_n-\mathcal{V}_n^{(u)}\|_{2m}&\le\tfrac{C_{\delta\mathcal{V}}}{L^n}\sum_{\ell=1}^{n}\tfrac{L^\ell}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1)\end{aligned}\tag{1.21}$$

and

$$\big\|\tilde{\mathcal{R}}_n^{(\vec p)}\big\|_m\le\mathfrak{r}_{\vec p}(n,C_{\mathcal{R}})\tag{1.22}$$

where

$$\mathfrak{r}_{\vec p}(n,C)=\begin{cases}\Pi_0^n(C)\;\mathfrak{r}_{\vec p}(0)+C\sum_{\ell=1}^{n}\frac{\mathfrak{e}_{\mathfrak{l}}(\ell-1)}{\kappa(\ell)\kappa'(\ell)}\Pi_\ell^n(C) & \text{if } \vec p=(1,1,0)\\ \frac{1}{L^n}\Pi_0^n(C)\;\mathfrak{r}_{\vec p}(0)+\frac{C}{L^n}\sum_{\ell=1}^{n}L^\ell\frac{\mathfrak{e}_{\mathfrak{l}}(\ell-1)}{\kappa'(\ell)^2}\Pi_\ell^n(C) & \text{if } \vec p=(0,1,1)\\ \Pi_0^n(C)\;\mathfrak{r}_{\vec p}(0)+C\sum_{\ell=1}^{n}\frac{\mathfrak{e}_{\mathfrak{l}}(\ell-1)}{\kappa'(\ell)^2}\Pi_\ell^n(C) & \text{if } \vec p=(0,0,2)\\ \frac{1}{L^{4n}}\Pi_0^n(C)\;\mathfrak{r}_{\vec p}(0)+\frac{C}{L^{4n}}\sum_{\ell=1}^{n}L^{4\ell}\frac{\mathfrak{e}_{\mathfrak{l}}(\ell-1)}{\kappa(\ell)^6}\Pi_\ell^n(C) & \text{if } \vec p=(6,0,0)\end{cases}$$

with

$$\Pi_\ell^n(C) = \prod_{j=\ell+1}^{n} \left(1 + C^{\frac{\mathfrak{e}_\mathfrak{l}(j-1)}{\kappa(j)^2}}\right)$$

More precisely, we shall show that there are constants $C_{\delta\mathcal{V}}$ and $C_{\mathcal{R}}$ such that if (1.21) and (1.22) are valid for some $n \ge 0$, they are also valid for $n+1$. Observe that (1.21) and (1.22) are trivially satisfied when $n = 0$. See [13, Lemma 6.7], for $|\mu_n - \mu_n^*|$, [13, Lemma 6.4], for the construction of $C_{\delta\mathcal{V}}$, and [13, Lemma 6.6], for the construction of $C_{\mathcal{R}}$.
That (1.21) and (1.22) imply the bounds on μ_n, $\mathcal{V}_n$ and $\tilde{\mathcal{R}}_n$ of Theorem 1.17 is proven in Corollary C.4 and Lemma C.1.d.

We are grateful for some very useful comversations with Martin Lohmann and Serena Cenatiempo.

1.7 Outline

The rest of this paper contains the more algebraic parts of the proof of Theorem 1.17. The more analytical part of the proof is given in [13]. Here is a more detailed outline of these two papers, as well as an indication of their connections to [14, 15, 10].

- In Appendix D, we review the results of [7] and rewrite the main output of that paper in the form of §1.5.
- §2 provides a number of simple preparatory results regarding the interaction of the scaling operation of Definition 1.3 with objects that will be encountered during the course of the construction.
- Various algebraic properties of the background and critical fields, such as the "composition rule", are reviewed in §3.
- The algebraic steps of the application of one block spin transformation, $\mathbb{T}_n^{(SF)}$, and subsequent scaling, $\mathbb{S}$, leading up to the formulation of the "fluctuation integral", are performed in §4.
- The fluctuation integral is evaluated and bounded in [13, §5].
- In [13, §6], the output of the fluctuation integral is reorganized to complete the inductive proof of Theorem 1.17. Part of the reorganization is the renormalization of the chemical potential and the interaction.
- The translation and reflection symmetries we use are discussed in Appendix B.
- A large number of inequalities relating our weight factors and various other parameters are proven in Appendix C. In particular we prove that the detailed inequalities of Remark 1.18 imply the simple bounds of Theorem 1.17.

- Localization operations and decompositions, that are used in the renormalization of the chemical potential, are discussed in [13, Appendix B].
- In [13, Appendix A] we identify the μ_* of (1.5) as the limit of the μ_n^*'s of Remark 1.18.
- The effect of scaling on the norms used in this paper is discussed in [13, Appendix C]. This is used to identify "scaling relevant and irrelevant" monomials. See Definition 1.16 and [13, Remark 5.4].
- In [14], we give reasons, on a handwaving level, why we expect that the errors introduced by approximating the blockspin transformation $\mathbb{T}$ by the small field blockspin transformation $\mathbb{T}^{(SF)}$ are nonperturbatively small.
- Estimates on the background and critical fields are crucial for our construction. They are proven in [15]. The upper bounds of that paper involve a number of constants $K_1, K_2, \cdots$ that are all independent of L and the scale index n. In [15, Convention 1.2], we define $K_{\rm bg}$ to be the maximum of the K_j's. We shall refer only to $K_{\rm bg}$, as opposed to the K_j's, in the main body of this paper.
- The estimates in this paper, and in particular, the bounds on the background and critical fields, depend heavily on bounds on various linear operators like the averaging operators of Definitions 1.5.a and 1.11.d, the covariances 1.12, and the Green's functions of Theorem 1.13. Such bounds have been proven in [10]. They involve constants $\Gamma_1, \Gamma_2, \cdots$ that are all independent of L and n. In [10, Convention 1.2], we define $\Gamma_{\rm op}$ to be the maximum of the Γ_j's. Again, we shall refer only to $\Gamma_{\rm op}$, as opposed to the Γ_j's, in this paper.

Here are the conventions that we use in naming the various constants that appear in this paper.

- The constants $\Gamma_{\rm op}$ and $\mu_{\rm up}$ were defined in [10, Convention 1.2 and Proposition 5.1], respectively. They are independent of n and L.
- The constants K_j's and their maximum $K_{\rm bg}$ and the constants ρ_j, and their minimum $\rho_{\rm bg}$ are defined in [15]. They are independent of n and L.
- The constants $C_{\delta\mathcal{V}}$, $C_{\mathcal{R}}$, $C_{\mathfrak{l}}$, and $C_{\rm ren}$ are the more important n and L independent constants of the main body of this paper. They depend only on $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m.
- The constants $\Lambda_{\delta\mu}$, Λ_j and Λ_j' are independent of n, but depend on L.
- The constants $c_{\rm loc}$, c_A, c_Ω, $c_{\delta\mathcal{V}}$, K_Φ, $c_{\rm gar}$, c_{μ_*} and c_j are the less important n and L independent constants. They depend only on $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m.

2 Scaling

We extend Definition 1.3 to

Definition 2.1 (Scaling)**.**

(a) As in Definition 1.5.a, let $\mathbb{L}$ be the linear isomorphism

$$\mathbb{L} : \mathcal{X}_j^{(k)} \to \mathcal{X}_{j-1}^{(k)} \qquad , \qquad (u_0, \mathbf{u}) \mapsto (L^2 u_0, L\mathbf{u})$$

For a field α on $\mathcal{X}_{j-1}^{(k)}$, we define the scaled fields

$$(\mathbb{S}\alpha)(u) = L^{3/2}\,\alpha\big(\mathbb{L}u\big) \qquad (\mathbb{S}_\nu\alpha)(u) = \left\{\begin{matrix} L^{7/2} & \text{if } \nu = 0 \\ L^{5/2} & \text{if } \nu \in \{1,2,3\}\end{matrix}\right\}\alpha\big(\mathbb{L}u\big)$$

on $\mathcal{X}_j^{(k)}$. For $\tilde\alpha = \big(\alpha, \{\alpha_\nu\}_{\nu=0}^3\big) \in \tilde{\mathcal{H}}_{j-1}^{(k)}$ as in (1.17), we define

$$\mathbb{S}\tilde\alpha = \big(\mathbb{S}\alpha, \{\mathbb{S}_\nu\alpha_\nu\}_{\nu=0}^3\big) \in \tilde{\mathcal{H}}_j^{(k)}$$

(b) For a complex valued function $F(\alpha_*, \alpha)$ of fields on $\mathcal{X}_{j-1}^{(k)}$, we define the function $(\mathbb{S}F)(\beta_*, \beta)$ of fields on $\mathcal{X}_j^{(k)}$ by

$$(\mathbb{S}F)(\beta_*, \beta) = F\big(\mathbb{S}^{-1}\beta_*, \mathbb{S}^{-1}\beta\big)$$

Similarly, for a function $\tilde F(\tilde\alpha_*, \tilde\alpha)$ on subset of $\tilde{\mathcal{H}}_{j-1}^{(k)} \times \tilde{\mathcal{H}}_{j-1}^{(k)}$, we define the function $(\mathbb{S}\tilde F)(\tilde\beta_*, \tilde\beta)$ on a corresponding subset of $\tilde{\mathcal{H}}_j^{(k)} \times \tilde{\mathcal{H}}_j^{(k)}$ by

$$(\mathbb{S}\tilde F)(\tilde\beta_*, \tilde\beta) = \tilde F\big(\mathbb{S}^{-1}\tilde\beta_*, \mathbb{S}^{-1}\tilde\beta\big)$$

Remark 2.2.

(a) The definition of $\mathbb{S}$, acting on $\mathcal{H}_{j-1}^{(k)}$, can be rephrased, using the notation $\mathbb{L}_*$ of Definition 1.5.a, as $\mathbb{S} = L^{3/2}\mathbb{L}_*^{-1}$. In particular conjugation with $\mathbb{S}$ is the same as conjugation with $\mathbb{L}_*^{-1}$.

(b) The definition of $\mathbb{S}_\nu$ is motivated by

$$\mathbb{S}_\nu\partial_\nu = \partial_\nu\mathbb{S} \qquad 0 \le \nu \le 3$$

If $\tilde F(\tilde\alpha_*, \tilde\alpha)$ is a function on a subset of $\tilde{\mathcal{H}}_{j-1}^{(k)} \times \tilde{\mathcal{H}}_{j-1}^{(k)}$ and

$$F(\alpha_*, \alpha) = \tilde F\big((\alpha_*, \{\partial_\nu\alpha_*\})\,,\,(\alpha, \{\partial_\nu\alpha\})\big)$$

then

$$(\mathbb{S}F)(\beta_*, \beta) = (\mathbb{S}\tilde F)\big((\beta_*, \{\partial_\nu\beta_*\})\,,\,(\beta, \{\partial_\nu\beta\})\big)$$

(c) For $\alpha, \alpha' \in \mathcal{H}_{j-1}^{(k)}$,

$$\langle \mathbb{S}\alpha, \mathbb{S}\alpha' \rangle_j = L^{-2} \langle \alpha, \alpha' \rangle_{j-1}$$

(d) The inverse map $\mathbb{S}^{-1} : \mathcal{H}_j^{(k)} \to \mathcal{H}_{j-1}^{(k)}$ is given by $(\mathbb{S}^{-1}\beta)(u) = L^{-3/2}\beta(\mathbb{L}^{-1}u)$. The adjoint $\mathbb{S}^* = L^{-2}\mathbb{S}^{-1}$, by part (c).

(e) By Definition 2.1.b,

$$\int \Big[\prod_{u \in \mathcal{X}_j^{(k)}} \tfrac{d\beta(u)^* \wedge d\beta(u)}{2\pi i} \Big] (\mathbb{S}F)(\beta^*, \beta) = N_{\mathbb{S}}^{(k)} \int \Big[\prod_{v \in \mathcal{X}_{j-1}^{(k)}} \tfrac{d\alpha(v)^* \wedge d\alpha(v)}{2\pi i} \Big] F(\alpha^*, \alpha)$$

where the normalization constant $N_{\mathbb{S}}^{(k)} = \big(L^3\big)^{|\mathcal{X}_j^{(k)}|} = \big(L^3\big)^{|\mathcal{X}_{j-1}^{(k)}|}$.

(f) For a complex valued function $F(\alpha_*, \alpha)$ of fields on $\mathcal{X}_{j-1}^{(k)}$,

$$\tfrac{\partial}{\partial \beta(u)} (\mathbb{S}F)(\beta_*, \beta) = L^{-3/2} \tfrac{\partial F}{\partial \alpha(\mathbb{L}u)} \big(\mathbb{S}^{-1}\beta_*, \mathbb{S}^{-1}\beta\big)$$

(g) Let $A : \mathcal{H}_{j-1}^{(k)} \to \mathcal{H}_{j-1}^{(k)}$ be a linear operator with kernel $A(\ \cdot\ ,\ \cdot\ )$. Then the kernel of $\mathbb{S}A\mathbb{S}^{-1} : \mathcal{H}_j^{(k)} \to \mathcal{H}_j^{(k)}$ is

$$\big(\mathbb{S}A\mathbb{S}^{-1}\big)(u, u') = L^5\, A\big(\mathbb{L}u, \mathbb{L}u'\big)$$

(h) Let

$$\tilde{\mathcal{M}}\big(\, (\alpha_*, \{\alpha_{*\nu}\})\, ,\, (\alpha, \{\alpha_\nu\})\, \big) = \int_{\mathcal{X}_{j-1}^{(k)}} dv_1 \cdots dv_n\ M(v_1, \cdots, v_n) \prod_{\ell=1}^n \alpha_{\sigma_\ell}(v_\ell)$$

be a monomial of degree n. Here each α_{σ_ℓ} is one of $\alpha_*, \alpha, \big\{\alpha_{*\nu}, \alpha_\nu\big\}_{\nu=0}^3$. We denote by

- n_u, the number of α_{σ_ℓ}'s that is either α_* or α and
- n_0, the number of α_{σ_ℓ}'s that is either α_{*0} or α_0 and
- $n_{\rm sp}$, the number of α_{σ_ℓ}'s that is one of $\big\{\alpha_{*\nu}, \alpha_\nu\big\}_{\nu=1}^3$.

Then

$$\big(\mathbb{S}\tilde{\mathcal{M}}\big)\big(\, (\beta_*, \{\beta_{*\nu}\})\, ,\, (\beta, \{\beta_\nu\})\, \big) = \int_{\mathcal{X}_j^{(k)}} du_1 \cdots du_n\ M^{(s)}(u_1, \cdots, u_n) \prod_{\ell=1}^n \beta_{\sigma_\ell}(u_\ell)$$

has kernel

$$M^{(s)}(u_1, \cdots, u_n) = L^{\frac{7}{2}n_u + \frac{3}{2}n_0 + \frac{5}{2}n_{\rm sp}} M(\mathbb{L}u_1, \cdots, \mathbb{L}u_n)$$

Definition 2.3. Let $n \geq 1$. The dominant contribution, A_n, to the effective action was defined in Definition 1.5.b. Its scaled version is

$$\check{A}_n(\theta_*, \theta, \check{\phi}_*, \check{\phi}, \mu, \mathcal{V}) = (\mathbb{S}^{-1} A_n)(\theta_*, \theta, \check{\phi}_*, \check{\phi}, L^2\mu, \mathbb{S}\mathcal{V}) = A_n(\mathbb{S}\theta_*, \mathbb{S}\theta, \mathbb{S}\check{\phi}_*, \mathbb{S}\check{\phi}, L^2\mu, \mathbb{S}\mathcal{V})$$

where $\theta_*, \theta \in \mathcal{H}_{-1}^{(n)}$, $\check{\phi}_*, \check{\phi} \in \mathcal{H}_{n-1}^{(0)}$, $\mu \in \mathbb{C}$ and $\mathcal{V}$ is a quartic monomial in the fields ϕ_*, ϕ.

Lemma 2.4.

(a) *For each* $n \geq 0$, $\mathcal{V}_n^{(u)} = \mathbb{S}^n \mathcal{V}_0$, $D_n = L^{2n}\, \mathbb{S}^n D_0 \mathbb{S}^{-n}$ *and* $Q^{(n)} = \mathbb{S}^n Q \mathbb{S}^{-n}$.

(b) *Set, for each* $n \geq 1$, $\check{Q}_n = \mathbb{S}^{-1} Q_n \mathbb{S}$ *and* $\check{\mathfrak{Q}}_n = \frac{1}{L^2}\mathbb{S}^{-1}\mathfrak{Q}_n\mathbb{S}$. *Then* $\check{Q}_n = Q Q_{n-1}$ *(with* $Q_0 = \mathbb{1}$*) and*

$$\check{\mathfrak{Q}}_n = \begin{cases} \frac{a}{L^2}\mathbb{1} & \text{if } n = 1 \\ \left(\frac{L^2}{a}\mathbb{1} + Q\, \mathfrak{Q}_{n-1}^{-1} Q^*\right)^{-1} & \text{if } n \geq 2 \end{cases}$$

(c) *For all* $n \geq 1$,

$$\begin{aligned} \check{A}_n(\theta_*, \theta, \check{\phi}_*, \check{\phi}, \mu, \mathcal{V}) = \big\langle \theta_* - QQ_{n-1}\check{\phi}_*\,,\, \check{\mathfrak{Q}}_n\big(\theta - QQ_{n-1}\check{\phi}\big)\big\rangle_{-1} + \big\langle \check{\phi}_*,\, D_{n-1}\check{\phi}\big\rangle_{n-1} \\ + \mathcal{V}(\check{\phi}_*, \check{\phi}) - \mu\, \big\langle \check{\phi}_*,\, \check{\phi}\big\rangle_{n-1} \end{aligned}$$

In particular $\check{A}_1(\theta_*, \theta, \psi_*, \psi, \mu, \mathcal{V}) = \frac{a}{L^2}\, \langle \theta_* - Q\psi_*, \theta - Q\psi\rangle_{-1} + A_0(\psi_*, \psi, \mu, \mathcal{V})$

Proof. (a) By part (h) of Remark 2.2, the kernel of $\mathbb{S}^n \mathcal{V}_0$ is

$$L^{14n}\, V_0(\mathbb{L}^n u_1, \mathbb{L}^n u_2, \mathbb{L}^n u_3, \mathbb{L}^n u_4) = V_n^{(u)}(u_1, u_2, u_3, u_4)$$

by Definition 1.5.a. The remaining two claims follow immediately from Remark 2.2.a and Definition 1.5.a.

(b) The first part follows immediately from part (a) and Definition 1.5.a. By Definition 1.5.b, when $n \geq 2$,

$$\begin{aligned} \mathbb{S}^{-1}\mathfrak{Q}_n^{-1}\mathbb{S} &= \tfrac{1}{a}\Big(\mathbb{1} + \sum_{j=1}^{n-1} \tfrac{1}{L^{2j}} Q Q_{j-1} Q_{j-1}^* Q^*\Big) \\ &= \tfrac{1}{a}\Big(\mathbb{1} + \tfrac{1}{L^2} Q\Big[\mathbb{1} + \sum_{j=1}^{n-2} \tfrac{1}{L^{2j}} Q_j Q_j^*\Big] Q^*\Big) \\ &= \tfrac{1}{L^2}\Big(\tfrac{L^2}{a}\mathbb{1} + Q\mathfrak{Q}_{n-1}^{-1} Q^*\Big) \end{aligned}$$

(c) By definition

$$\begin{aligned}
\check{A}_n(\theta_*, \theta, \check{\phi}_*, \check{\phi}, \mu, \mathcal{V}) &= \big\langle \mathbb{S}\theta_* - Q_n \mathbb{S}\check{\phi}_* \,,\, \mathfrak{Q}_n\big(\mathbb{S}\theta - Q_n\mathbb{S}\check{\phi}\big)\big\rangle_0 + \big\langle \mathbb{S}\check{\phi}_*,\, D_n\mathbb{S}\check{\phi}\big\rangle_n \\
&\quad + (\mathbb{S}\mathcal{V})(\mathbb{S}\check{\phi}_*, \mathbb{S}\check{\phi}) - L^2\mu\, \big\langle \mathbb{S}\check{\phi}_*,\, \mathbb{S}\check{\phi}\big\rangle_n \\
&= L^{-2}\big\langle \theta_* - \mathbb{S}^{-1}Q_n\mathbb{S}\check{\phi}_* \,,\, \mathbb{S}^{-1}\mathfrak{Q}_n\mathbb{S}\big(\theta - \mathbb{S}^{-1}Q_n\mathbb{S}\check{\phi}\big)\big\rangle_{-1} + L^{-2}\big\langle \check{\phi}_*,\, \mathbb{S}^{-1}D_n\mathbb{S}\check{\phi}\big\rangle_{n-1} \\
&\quad + \mathcal{V}(\check{\phi}_*, \check{\phi}) - \mu\, \big\langle \check{\phi}_*,\, \check{\phi}\big\rangle_{n-1} \\
&= \big\langle \theta_* - QQ_{n-1}\check{\phi}_* \,,\, \check{\mathfrak{Q}}_n\big(\theta - QQ_{n-1}\check{\phi}\big)\big\rangle_{-1} + \big\langle \check{\phi}_*,\, D_{n-1}\check{\phi}\big\rangle_{n-1} \\
&\quad + \mathcal{V}(\check{\phi}_*, \check{\phi}) - \mu\, \big\langle \check{\phi}_*,\, \check{\phi}\big\rangle_{n-1}
\end{aligned}$$

□

3 The Background Field and its Variations

Let $1 \le n \le n_p$. If $|\mu|$ and $\|\mathcal{V}\|_{2m}$ are small enough, the background fields

$$\phi_{(*)n}(\,\cdot\,,\,\cdot\,,\mu,\mathcal{V}) : \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)} \to \mathcal{H}_n$$

were defined in Proposition 1.14. They are solutions of the background field equations

$$\tfrac{\partial}{\partial \phi_*} A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V}) = \tfrac{\partial}{\partial \phi} A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V}) = 0$$

Putting in the action A_n of Definition 1.5.b, we get

$$\begin{aligned} S_n^{*-1}(\mu)\phi_* + \nabla_\phi \mathcal{V}(\phi_*,\phi) &= Q_n^* \mathfrak{Q}_n \psi_* \\ S_n^{-1}(\mu)\phi + \nabla_{\phi_*} \mathcal{V}(\phi_*,\phi) &= Q_n^* \mathfrak{Q}_n \psi \end{aligned} \tag{3.1}$$

with the $S_n(\mu) = (D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu)^{-1}$ of Theorem 1.13. See [12, Remark 8]. To evaluate the gradients of $\mathcal{V}$ we use

Definition 3.1. Let

$$\mathcal{M}(\phi_*,\phi) = \tfrac{1}{2} \int_{\mathcal{X}_n^4} du_1 \cdots du_4\ M(u_1,u_2,u_3,u_4)\, \phi_*(u_1)\phi(u_2)\phi_*(u_3)\phi(u_4)$$

be a quartic monomial whose kernel $M(u_1,u_2,u_3,u_4)$ is invariant under $u_1 \leftrightarrow u_3$ and under $u_2 \leftrightarrow u_4$. We denote its gradients by

$$\begin{aligned} \mathcal{M}'_*(u;\zeta_{*1},\zeta,\zeta_{*2}) &= \int du_1 du_2 du_3\ M(u_1,u_2,u_3,u)\, \zeta_{*1}(u_1)\zeta(u_2)\zeta_{*2}(u_3) \\ \mathcal{M}'(u;\zeta_1,\zeta_*,\zeta_2) &= \int du_2 du_3 du_4\ M(u,u_2,u_3,u_4)\, \zeta_1(u_2)\zeta_*(u_3)\zeta_2(u_4) \end{aligned}$$

Using this notation, the background field equations become

$$\begin{aligned} S_n^{*-1}(\mu)\phi_* + \mathcal{V}'_*(\phi_*,\phi,\phi_*) &= Q_n^* \mathfrak{Q}_n \psi_* \\ S_n^{-1}(\mu)\phi + \mathcal{V}'(\phi,\phi_*,\phi) &= Q_n^* \mathfrak{Q}_n \psi \end{aligned} \tag{3.2}$$

In [15, Proposition 2.1] we prove that these equations have a solution $\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})$ which is analytic on the set of all (ψ_*,ψ)'s obeying $|\psi_*(x)|, |\psi(x)| < \kappa(n)$.

Definition 3.2. The scaled versions

$$\check{\phi}_{(*)n}(\,\cdot\,,\,\cdot\,,\mu,\mathcal{V}) : \mathcal{H}_{-1}^{(n)} \times \mathcal{H}_{-1}^{(n)} \to \mathcal{H}_{n-1}$$

of $\phi_{(*)n}$ are

$$\check\phi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) = \mathbb{S}^{-1}\big[\phi_{(*)n}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V})\big]$$

That is

$$\check\phi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V})(v) = L^{-3/2}\phi_{(*)n}\big(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V}\big)(\mathbb{L}^{-1}v)$$

They are analytic on the set of all (θ_*,θ)'s obeying $|\theta_*(x)|,|\theta(x)| < \frac{\kappa(n)}{L^{3/2}}$.

Remark 3.3. By Definition 2.3 and Remark 2.2.f,

$$\begin{aligned}\tfrac{\partial\,A_n}{\partial\phi(u)}\big(\mathbb{S}\theta_*,\mathbb{S}\theta,\phi_*,\phi,L^2\mu,\mathbb{S}\mathcal{V}\big) &= \tfrac{\partial\,\mathbb{S}\check A_n}{\partial\phi(u)}\big(\mathbb{S}\theta_*,\mathbb{S}\theta,\phi_*,\phi,\mu,\mathcal{V}\big)\\ &= L^{-3/2}\tfrac{\partial\,\check A_n}{\partial\check\phi(\mathbb{L}u)}\big(\theta_*,\theta,\mathbb{S}^{-1}\phi_*,\mathbb{S}^{-1}\phi,\mu,\mathcal{V}\big)\end{aligned}$$

Consequently, by Definition 1.5.c,

$$\check\phi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) = \mathbb{S}^{-1}\big[\phi_{(*)n}\big(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V}\big)\big]$$

are critical fields for $\check A_n(\theta_*,\theta,\check\phi_*,\check\phi,\mu,\mathcal{V})$.

Proposition 3.4. *Define, $\psi_{(*)0}(\theta_*,\theta,\mu,\mathcal{V}) = \check\phi_{1(*)}(\theta_*,\theta,\mu,\mathcal{V})$ and, for $n\ge 1$,*

$$\psi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) = \big(\tfrac{a}{L^2}Q^*Q+\mathfrak{Q}_n\big)^{-1}\big\{\tfrac{a}{L^2}Q^*\theta_{(*)}+\mathfrak{Q}_n\,Q_n\,\check\phi_{(*)n+1}(\theta_*,\theta,\mu,\mathcal{V})\big\}$$

- *(a) The $\psi_{(*)n}$'s solve the critical field equations of Proposition 1.15. They are analytic on the set of all (θ_*,θ)'s obeying $|\theta_*(x)|,|\theta(x)| < \frac{\kappa(n+1)}{L^{3/2}}$.*
- *(b) For $n\ge 1$, we have the composition rule*

$$\check\phi_{(*)n+1}(\theta_*,\theta,\mu,\mathcal{V}) = \phi_{(*)n}\big(\,\psi_{*n}(\theta_*,\theta,\mu,\mathcal{V})\,,\,\psi_n(\theta_*,\theta,\mu,\mathcal{V})\,,\,\mu,\mathcal{V}\big)$$

- *(c) For all $n\ge 1$,*

$$\begin{aligned}&\check A_{n+1}(\theta_*,\theta,\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V}),\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V}),\mu,\mathcal{V})\\ &\quad= \tfrac{a}{L^2}\,\langle\theta_*-Q\psi_{*n}(\theta_*,\theta,\mu,\mathcal{V}),\theta-Q\psi_n(\theta_*,\theta,\mu,\mathcal{V})\rangle_{-1}\\ &\qquad+A_n\big(\psi_{*n}(\theta_*,\theta,\mu,\mathcal{V}),\psi_n(\theta_*,\theta,\mu,\mathcal{V}),\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V}),\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V}),\mu,\mathcal{V}\big)\end{aligned}$$

Proof. Case $n \geq 1$: We apply the strategy of [12, Remark 5.c] with

$$\begin{array}{lll} \mathcal{H} = \mathcal{H}_0^{(n)} & \mathcal{H}_- = \mathcal{H}_n & \mathcal{H}_+ = \mathcal{H}_{-1}^{(n+1)} \\ Q_- = Q_n & \mathfrak{Q} = \mathfrak{Q}_n & b = \frac{a}{L^2} \\ D = D_n - \mu & P(\phi_*, \phi) = \mathcal{V}(\phi_*, \phi) & \\ \multicolumn{3}{l}{\mathfrak{A}(\phi_*, \phi) = \langle \phi_*, D_n \phi \rangle_n + \mathcal{V}(\phi_*, \phi) - \mu \langle \phi_*, \phi \rangle_n} \end{array} \tag{3.3.a}$$

and the background/next scale background fields

$$\phi_{(*)\mathrm{bg}}(\psi_*, \psi) = \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V}) \qquad \check{\phi}_{(*)\mathrm{bg}}(\theta_*, \theta) = \check{\phi}_{(*)n+1}(\theta_*, \theta, \mu, \mathcal{V}) \tag{3.3.b}$$

Then [12, Proposition 9] applies, and, in particular, gives the proof of part (a) for $n \geq 1$. Part (b) follows by the uniqueness provision of [12, Proposition 9] and part (c) of [12, Proposition 4]. Then part (c) follows by [12, Proposition 4.b].

Case $n = 0$: It suffices to observe that, by Lemma 2.4.c, the fields $\check{\phi}_{1(*)}(\theta_*, \theta, \mu, \mathcal{V})$ are critical (with respect to $\psi_{(*)}$) for $\frac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_0(\psi_*, \psi, \mu)$. □

The main part of the action, A_n, is expressed in terms of the background field $\phi_{(*)n}(\psi_*, \psi, \mu_n, \mathcal{V}_n)$. (See Theorem 1.17.) In the fluctuation integral we make a change of variables $\psi_{(*)} = \psi_{(*)n}(\theta_*, \theta, \mu_n, \mathcal{V}_n) + \delta\psi_{(*)}$. (Set $D^{(n)(*)}\zeta^{(*)} = \delta\psi_{(*)}$ in Definition 1.6.) So we must study the impact of this change of variables on $\phi_{(*)n}$.

Definition 3.5. Let $1 \leq n \leq n_p$, and let $\|\mathcal{V}\|_{2m}\kappa(n)^2 + |\mu|$ be sufficiently small as in Proposition 1.14.

(a) Define $\delta\phi_{*n}\big(\psi_*, \psi, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$ and $\delta\phi_n\big(\psi_*, \psi, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$ by

$$\phi_{(*)n}\big(\psi_* + \delta\psi_*, \psi + \delta\psi, \mu, \mathcal{V}\big) = \phi_{(*)n}\big(\psi_*, \psi, \mu, \mathcal{V}\big) + \delta\phi_{(*)n}\big(\psi_*, \psi, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$$

and set

$$\begin{aligned} &\delta\check{\phi}_{(*)n+1}\big(\theta_*, \theta, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big) \\ &\qquad = \delta\phi_{(*)n}\big(\psi_{*n}(\theta_*, \theta, \mu, \mathcal{V})\,,\, \psi_n(\theta_*, \theta, \mu, \mathcal{V})\,,\, \delta\psi_*\,,\, \delta\psi, \mu, \mathcal{V}\big) \end{aligned}$$

(b) Define $\delta\check{\phi}^{(+)}_{(*)n+1}\big(\theta_*, \theta; \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$ by

$$\delta\check{\phi}_{(*)n+1}\big(\theta_*, \theta; \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big) = S_n^{(*)} Q_n^* \mathfrak{Q}_n\, \delta\psi_{(*)} + \delta\check{\phi}^{(+)}_{(*)n+1}\big(\theta_*, \theta; \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$$

where $S_n = (D_n + Q_n^* \mathfrak{Q}_n Q_n)^{-1}$ as in Theorem 1.13.

Remark 3.6. (a) By the composition rule Proposition 3.4.b,

$$\begin{aligned}\phi_{(*)n}\big(\psi_{*n}(\theta_*,\theta,\mu,\mathcal{V})+\delta\psi_*\,,\,\psi_n(\theta_*,\theta,\mu,\mathcal{V})+\delta\psi,\mu,\mathcal{V}\big)\\ =\check\phi_{(*)n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{(*)n+1}\big(\theta_*,\theta;\delta\psi_*,\delta\psi,\mu,\mathcal{V}\big)\end{aligned}$$

(b) The quantities $\delta\phi_{(*)n}$, $\delta\check\phi_{(*)n+1}$, $\delta\check\phi^{(+)}_{(*)n+1}$, $S_n(\mu)$ correspond to the quantities $\delta\phi_{(*)\mathrm{bg}}$, $\delta\check\phi_{(*)\mathrm{bg}}$, $\delta\check\phi^{(+)}_{(*)}$, S in [12] under the substitution (3.3.a,b). Hence, by [12, Remark 11], the fields $\delta\check\phi_{(*)n+1}$ obey

$$\delta\check\phi_{*n+1}=S_n(\mu)^*Q_n^*\mathfrak{Q}_n\,\delta\psi_*-S_n^*\nabla_\phi\mathcal{V}(\phi_*,\phi)\Big|^{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{*n+1}\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{n+1}}}_{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})}}$$

$$\delta\check\phi_{n+1}=S_n(\mu)Q_n^*\mathfrak{Q}_n\,\delta\psi-S_n\nabla_{\phi_*}\mathcal{V}(\phi_*,\phi)\Big|^{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{*n+1}\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{n+1}}}_{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})}}$$

(c) Since $S_n(\mu)=\big[\mathbb{1}-\mu S_n\big]^{-1}S_n$, the equations of part (b) may be rewritten

$$\delta\check\phi_{*n+1}=S_n^*Q_n^*\mathfrak{Q}_n\,\delta\psi_*+\mu S_n^*\delta\check\phi_{*n+1}-S_n^*\nabla_\phi\mathcal{V}(\phi_*,\phi)\Big|^{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{*n+1}\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{n+1}}}_{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})}}$$

$$\delta\check\phi_{n+1}=S_nQ_n^*\mathfrak{Q}_n\,\delta\psi+\mu S_n\delta\check\phi_{n+1}-S_n\nabla_{\phi_*}\mathcal{V}(\phi_*,\phi)\Big|^{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{*n+1}\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})+\delta\check\phi_{n+1}}}_{\substack{\phi_*=\check\phi_{*n+1}(\theta_*,\theta,\mu,\mathcal{V})\\ \phi=\check\phi_{n+1}(\theta_*,\theta,\mu,\mathcal{V})}}$$

In particular, if $\mu=\mathcal{V}=0$, then $\delta\check\phi_{(*)n+1}=S_n^{(*)}Q_n^*\mathfrak{Q}_n\,\delta\psi_{(*)}$. This is the motivation for the definition of $\delta\check\phi^{(+)}_{(*)n+1}$ in Definition 3.5.b.

4 One Block Spin Transformation — The Algebra

In this section we consider the output of the approximate block spin transformation $\mathbb{T}_n^{(SF)}$ acting on $e^{\mathcal{A}_0}$, with the $\mathcal{A}_0$ of (1.20) in the case $n = 0$, and on $e^{-A_n+\mathcal{R}_n+\mathcal{E}_n}$, in the case $n \geq 1$ (see Theorem 1.17). The main result of this section is Proposition 4.2, which provides a representation of this output that will be used in the (inductive) proof of Theorem 1.17. If the conclusion of Theorem 1.17 holds for some $1 \leq n < n_p$, then Proposition 4.2 gives a representation for

$$\Big(\mathbb{T}_n^{(SF)} \circ (\mathbb{S}\mathbb{T}_{n-1}^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)})\Big) \Big(e^{\mathcal{A}_0(\psi_*,\psi)}\Big)(\theta_*, \theta)$$

which, up to a multiplicative constant, is of the form

$$e^{\check{\mathcal{C}}_n(\theta_*,\theta)}\ \check{\mathcal{F}}_n(\theta_*, \theta)$$

where

- the "contribution from the critical field" is

$$\begin{aligned}\check{\mathcal{C}}_n(\theta_*, \theta) = -\check{A}_{n+1}(\theta_*, \theta, \check{\phi}_{*n+1}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \check{\phi}_{n+1}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \mu_n, \mathcal{V}_n)\\ + \mathcal{R}_n\big(\check{\phi}_{*n+1}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \check{\phi}_{n+1}(\theta_*, \theta, \mu_n, \mathcal{V}_n)\big) + \check{\mathcal{E}}_{n+1,1}(\theta_*, \theta)\end{aligned}$$

  with

$$\check{\mathcal{E}}_{n+1,1}(\theta_*, \theta) = \mathcal{E}_n\big(\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)\big)$$

  and the μ_n, $\mathcal{V}_n$, $\mathcal{R}_n$ and $\mathcal{E}_n$ of Theorem 1.17 for $n \geq 1$ and of §1.5 for $n = 0$,
- and the "fluctuation integral" is

$$\begin{aligned}\check{\mathcal{F}}_n(\theta_*, \theta) = \Big[\prod_{x\in\mathcal{X}_0^{(n)}} \int_{|\zeta(x)|\leq r_n} \tfrac{d\zeta(x)^*\wedge d\zeta(x)}{2\pi i} e^{-|\zeta(x)|^2}\Big]\\ \exp\Big\{-\delta\check{A}_n(\theta_*, \theta, \delta\psi_*, \delta\psi) + \delta\check{\mathcal{R}}_n\big(\theta_*, \theta, \delta\psi_*, \delta\psi\big) + \delta\check{\mathcal{E}}_n\big(\theta_*, \theta, \delta\psi_*, \delta\psi\big)\Big\}\end{aligned}$$

  with $\delta\psi_* = D^{(n)*}\zeta^*$, $\delta\psi = D^{(n)}\zeta$, $D^{(n)}$ being an operator square root of $C^{(n)}$, as in (1.15), and
  - for $n \geq 0$

$$\begin{aligned}\delta\check{\mathcal{E}}_n(\theta_*, \theta, \delta\psi_*, \delta\psi) = \mathcal{E}_n\big(\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n) + \delta\psi_*, \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n) + \delta\psi\big)\\ - \mathcal{E}_n\big(\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)\big)\end{aligned}$$

◦ for $n \ge 0$

$$\delta\check{\mathcal{R}}_n(\theta_*,\theta,\delta\psi_*,\delta\psi) = \Big[\mathcal{R}_n\big(\phi_*+\delta\phi_*,\phi+\delta\phi\big)-\mathcal{R}_n\big(\phi_*,\phi\big)\Big]_{\substack{\phi_{(*)}=\check{\phi}_{(*)n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)\\ \delta\phi_{(*)}=\delta\check{\phi}_{(*)n+1}(\theta_*,\theta;\delta\psi_*,\delta\psi,\mu_n,\mathcal{V}_n)}}$$

where, for $n \ge 1$, $\delta\check{\phi}_{(*)n+1}$ was defined in Definition 3.5.a and, for $n=0$, $\delta\check{\phi}_{1(*)} = \delta\psi_{(*)}$ and,

◦ for $n \ge 1$,

$$\delta\check{A}_n(\theta_*,\theta,\delta\psi_*,\delta\psi) = -\int_0^1 dt\,\big\langle\delta\psi_*,\mathfrak{Q}_n\,Q_n\,\delta\check{\phi}^{(+)}_{n+1}\big(\theta_*,\theta;t\,\delta\psi_*,t\,\delta\psi,\,\mu_n,\mathcal{V}_n\big)\big\rangle_0 - \int_0^1 dt\,\big\langle\mathfrak{Q}_n\,Q_n\,\delta\check{\phi}^{(+)}_{*n+1}\big(\theta_*,\theta;t\,\delta\psi_*,t\,\delta\psi,\,\mu_n,\mathcal{V}_n\big),\,\delta\psi\big\rangle_0$$

and, for $n=0$,

$$\delta\check{A}_0(\theta_*,\theta,\delta\psi_*,\delta\psi) = \int_0^1(1-t)\,\tfrac{d^2}{dt^2}\mathcal{V}_0\big(\psi_{*0}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+t\delta\psi_*\,,\,\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)+t\delta\psi\big)\,dt - \mu_0\,\langle\delta\psi_*,\,\delta\psi\rangle_0$$

The integral in $\delta\check{A}_0$ is the part of $\mathcal{V}_0\big(\psi_{*0}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi_*\,,\,\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi\big)$ that is of degree at least two in $\delta\psi_{(*)}$.

The significance of $\delta\check{A}_n$ may be seen in

Lemma 4.1.

(a) For all $n \ge 1$,

$$\Big[\tfrac{a}{L^2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}+A_n\big(\psi_*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n\big)\Big]_{\substack{\psi_{(*)}=\psi_{(*)n}(\theta_*,\theta,\mu_n,\mathcal{V}_n)+\delta\psi_{(*)}\\ \phi_{(*)}=\check{\phi}_{n+1(*)}(\theta_*,\theta,\mu_n,\mathcal{V}_n)+\delta\check{\phi}_{(*)n+1}}} - \Big[\tfrac{a}{L^2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}+A_n\big(\psi_*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n\big)\Big]_{\substack{\psi_{(*)}=\psi_{(*)n}(\theta_*,\theta,\mu_n,\mathcal{V}_n)\\ \phi_{(*)}=\check{\phi}_{(*)n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)}}$$

$$= \big\langle\delta\psi_*,C^{(n)^{-1}}\,\delta\psi\big\rangle_0+\delta\check{A}_n(\theta_*,\theta,\delta\psi_*,\delta\psi)$$

(b) For $n=0$,

$$\Big[\tfrac{a}{L^2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}+A_0\big(\psi_*,\psi,\mu_0,\mathcal{V}_0\big)\Big]^{\psi_{(*)}=\psi_{0(*)}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi_{(*)}}_{\psi_{(*)}=\psi_{0(*)}(\theta_*,\theta,\mu_0,\mathcal{V}_0)} = \big\langle\delta\psi_*,C^{(0)^{-1}}\,\delta\psi\big\rangle_0+\delta\check{A}_0(\theta_*,\theta,\delta\psi_*,\delta\psi)$$

Proof. (a) We use [12] with the substitutions

$$\begin{array}{lll} \mathcal{H} = \mathcal{H}_0^{(n)} & \mathcal{H}_- = \mathcal{H}_n & \mathcal{H}_+ = \mathcal{H}_{-1}^{(n+1)} \\ Q_- = Q_n & \mathfrak{Q} = \mathfrak{Q}_n & b = \frac{a}{L^2} \\ D = D_n & P(\phi_*, \phi) = \mathcal{V}_n(\phi_*, \phi) - \mu_n \langle \phi_*, \phi \rangle_n & \\ \mathfrak{A}(\phi_*, \phi) = \langle \phi_*, D_n \phi \rangle_n + \mathcal{V}_n(\phi_*, \phi) - \mu_n \langle \phi_*, \phi \rangle_n & & \end{array}$$

They give

$$\begin{aligned} \mathcal{A}_{\text{eff}}(\theta_*, \theta; \psi_*, \psi; \phi_*, \phi) &= \tfrac{a}{L^2} \langle \theta_* - Q\psi_*\,,\, \theta - Q\psi \rangle_{-1} + \langle \psi_* - Q_n\, \phi_*\,,\, \mathfrak{Q}_n(\psi - Q_n\, \phi) \rangle_0 \\ &\qquad + \langle \phi_*, D_n \phi \rangle_n + \mathcal{V}_n(\phi_*, \phi) - \mu_n \langle \phi_*, \phi \rangle_n \\ &= \tfrac{a}{L^2} \langle \theta_* - Q\psi_*\,,\, \theta - Q\psi \rangle_{-1} + A_n(\psi_*, \psi, \phi_*, \phi, \mu_n, \mathcal{V}_n) \end{aligned}$$

Comparing [12, (12), (14) and Remark 10.a] with Theorem 1.13, (1.15) and (1.14), we have $C = C^{(n)}$ and $S = S_n$. Also

$$\begin{aligned} \phi_{(*)\text{bg}}(\psi_*, \psi) &= \phi_{(*)n}(\psi_*, \psi, \mu_n, \mathcal{V}_n) \\ \psi_{(*)\text{cr}}(\theta_*, \theta) &= \psi_{(*)n}(\theta_*, \theta, \mu_n, \mathcal{V}_n) \\ \check{\phi}_{(*)\text{bg}}(\theta_*, \theta) &= \check{\phi}_{(*)n+1}(\theta_*, \theta, \mu_n, \mathcal{V}_n) \end{aligned}$$

are the background, critical and next scale background fields, respectively, in the sense of [12, Definition 3].
The claim now follows from [12, Lemma 12].

(b) Observe that

$$\tfrac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_0\big(\psi_*, \psi, \mu_0, \mathcal{V}_0\big)$$

is the sum of the quadratic form $\frac{a}{L^2} \langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + \langle \psi_*,\, D_0\psi \rangle_0 - \mu_0 \langle \psi_*,\, \psi \rangle_0$ and $\mathcal{V}_0(\psi_*, \psi)$. Now imagine substituting in $\psi_{(*)} = \psi_{0(*)}(\theta_*, \theta, \mu_0, \mathcal{V}_0) + \delta\psi_{(*)}$ and expanding in powers of $\delta\psi_{(*)}$. The total contribution that is of degree precisely one in $\delta\psi_{(*)}$ vanishes by the criticality of $\psi_{(*)0}$. By Taylor's theorem with remainder,

$$\begin{aligned} &\Big[\tfrac{a}{L^2}\langle \theta_* - Q\psi_*, \theta - Q\psi \rangle_{-1} + A_0\big(\psi_*, \psi, \mu_0, \mathcal{V}_0\big)\Big]_{\psi_{(*)}=\psi_{0(*)}(\theta_*,\theta,\mu_0,\mathcal{V}_0)}^{\psi_{(*)}=\psi_{0(*)}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi_{(*)}} \\ &\quad = \tfrac{a}{L^2} \langle Q\delta\psi_*, Q\delta\psi \rangle_{-1} + \langle \delta\psi_*,\, D_0\delta\psi \rangle_0 - \mu_0 \langle \delta\psi_*,\, \delta\psi \rangle_0 \\ &\qquad + \int_0^1 (1-t)\, \tfrac{d^2}{dt^2} \mathcal{V}_0\big(\psi_{0*}(\theta_*, \theta, \mu_0, \mathcal{V}_0) + t\delta\psi_*\,,\, \psi_0(\theta_*, \theta, \mu_0, \mathcal{V}_0) + t\delta\psi\big)\, dt \end{aligned}$$

□

Proposition 4.2.

(a) Let $1 \le n < n_p$. *Let* μ_n, $\mathcal{V}_n$, $\mathcal{R}_n$ *and* $\mathcal{E}_n$ *be as in Theorem 1.17. Set*

$$\mathcal{A}_n(\psi_*,\psi) = \Big[-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\,\mu_n,\mathcal{V}_n) + \mathcal{R}_n(\phi_{*n},\phi_n)\Big]_{\phi_{(*)n}=\phi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)} + \mathcal{E}_n(\psi_*,\psi)$$

Then

$$\mathbb{T}_n^{(SF)}\Big(e^{\mathcal{A}_n(\psi_*,\psi)}\Big)(\theta_*,\theta;\mu_n,\mathcal{V}_n) = \tfrac{1}{\tilde N_{\mathbb{T}}^{(n)}}\ e^{\check{\mathcal{C}}_n(\theta_*,\theta)}\ \check{\mathcal{F}}_n(\theta_*,\theta)$$

(b) For $n=0$, $\quad \mathbb{T}_0^{(SF)}\Big(e^{\mathcal{A}_0(\psi_*,\psi)}\Big)(\theta_*,\theta;\mu_0,\mathcal{V}_0) = \tfrac{1}{\tilde N_{\mathbb{T}}^{(0)}}\ e^{\check{\mathcal{C}}_0(\theta_*,\theta)}\ \check{\mathcal{F}}_0(\theta_*,\theta).$

Proof. Case $n \ge 1$: By Definition 1.6,

$$\begin{aligned}&\mathbb{T}_n^{(SF)}\Big(e^{\mathcal{A}_n(\psi_*,\psi)}\Big)(\theta_*,\theta;\mu_n,\mathcal{V}_n)\\ &\quad= \tfrac{1}{\tilde N_{\mathbb{T}}^{(n)}}\Big[\prod_{x\in\mathcal{X}_0^{(n)}}\int_{|\zeta(x)|\le r_n}\tfrac{d\zeta(x)^*\wedge d\zeta(x)}{2\pi i}\Big] e^{-\frac{a}{L^2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}} e^{\mathcal{A}_n(\psi_*,\psi,\mu_n,\mathcal{V}_n)}\Big|_{\psi_{(*)}=\psi_{(*)n}+\delta\psi_{(*)}}\end{aligned} \tag{4.1}$$

where $\psi_{(*)n}=\psi_{(*)n}(\theta_*,\theta,\mu_n,\mathcal{V}_n)$, $\delta\psi_* = D^{(n)*}\zeta^*$, $\delta\psi = D^{(n)}\zeta$, and

$$\begin{aligned}\mathcal{A}_n(\psi_*,\psi,\mu_n,\mathcal{V}_n) = -A_n(\psi_*,\psi,\phi_*,\phi,\,\mu_n,\mathcal{V}_n) + \mathcal{R}_n(\phi_*,\phi)&\\ + \mathcal{E}_n(\psi_*,\psi)\Big|_{\substack{\phi_*=\phi_{*n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)\\ \phi=\phi_n(\psi_*,\psi,\mu_n,\mathcal{V}_n)}}&\end{aligned}$$

When $\zeta=0$ the exponent of the integral (4.1) reduces to

$$\begin{aligned}
&-\tfrac{a}{L^2}\langle\theta_*-Q\psi_{*n}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\theta-Q\psi_n(\theta_*,\theta,\mu_n,\mathcal{V}_n)\rangle_{-1}\\
&\quad - A_n\big(\psi_{*n},\psi_n,\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\mu_n,\mathcal{V}_n\big)\Big|_{\psi_{(*)n}=\psi_{*n}(\theta_*,\theta,\mu_n,\mathcal{V}_n)}\\
&\qquad + \mathcal{R}_n\big(\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)\\
&\qquad + \mathcal{E}_n\big(\psi_{*n}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\psi_n(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)\\
&= -\check A_{n+1}(\theta_*,\theta,\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\mu_n,\mathcal{V}_n)\\
&\qquad + \mathcal{R}_n\big(\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)\\
&\qquad + \mathcal{E}_n\big(\psi_{*n}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\psi_n(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)
\end{aligned}$$

by Proposition 3.4.b,c.

The part of the exponent of the integral (4.1) that is of degree at least one in $\delta\psi$ is

$$- \big\langle \delta\psi_*, C^{(n)^{-1}}\,\delta\psi \big\rangle_0 - \delta\check{A}_n(\theta_*,\theta,\delta\psi_*,\delta\psi) + \delta\check{\mathcal{R}}_n\big(\theta_*,\theta,\delta\psi_*,\delta\psi\big) + \delta\check{\mathcal{E}}_n\big(\theta_*\,,\,\theta\,,\,\delta\psi_*\,,\,\delta\psi\big)$$

by Lemma 4.1.a. Since $D^{(n)}$ is an operator square root of $C^{(n)}$

$$\big\langle \delta\psi_*, C^{(n)^{-1}}\,\delta\psi \big\rangle_0\Big|_{\substack{\delta\psi_*=D^{(n)*}\zeta^*\\ \delta\psi=D^{(n)}\zeta}} = \langle \zeta^*, \zeta\rangle_0$$

Case $n=0$: By Definition 1.6,

$$\begin{aligned}&(\mathbb{T}_0^{(SF)}e^{\mathcal{A}_0})(\theta_*,\theta;\mu_0,\mathcal{V}_0)\\ &\quad= \tfrac{1}{\tilde{N}_{\mathbb{T}}^{(0)}}\Big[\prod_{x\in\mathcal{X}_0}\int_{|\zeta(x)|\le r_0}\tfrac{d\zeta(x)^*\wedge d\zeta(x)}{2\pi i}\Big] e^{-aL^{-2}\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_{-1}}\, e^{\mathcal{A}_0(\psi_*,\psi)}\Big|_{\substack{\psi_*=\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi_*\\ \psi=\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi}}\end{aligned} \tag{4.2}$$

where, again, $\delta\psi_*=D^{(0)*}\zeta^*$, $\delta\psi=D^{(0)}\zeta$. By (1.20) and Definition 1.5.b, when $\zeta=0$ the exponent of the integral (4.2) reduces to

$$\begin{aligned}&- aL^{-2}\,\langle \theta_*-Q\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\theta-Q\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)\rangle\\ &\qquad - A_0\big(\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0),\mu_0,\mathcal{V}_0\big)\\ &\qquad\qquad + \mathcal{R}_0\big(\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)\big)\\ &\qquad\qquad\qquad + \mathcal{E}_0\big(\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)\big)\\ &= -\check{A}_1(\theta_*,\theta,\check{\phi}_{1*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\check{\phi}_1(\theta_*,\theta,\mu_0),\mu_0,\mathcal{V}_0)\\ &\qquad + \mathcal{R}_0\big(\check{\phi}_{1*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\check{\phi}_1(\theta_*,\theta,\mu_0,\mathcal{V}_0)\big)\\ &\qquad\qquad + \mathcal{E}_0\big(\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0),\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)\big)\end{aligned}$$

by Lemma 2.4.c and Proposition 3.4.b.

The part of the exponent of the integral (4.2) that is of degree at least one in $\delta\psi$ is

$$- \big\langle \delta\psi_*, C^{(0)^{-1}}\,\delta\psi \big\rangle_0 - \delta\check{A}_0(\theta_*,\theta,\delta\psi_*,\delta\psi) + \delta\check{\mathcal{R}}_0\big(\theta_*,\theta,\delta\psi_*,\delta\psi\big) + \delta\check{\mathcal{E}}_0\big(\theta_*\,,\,\theta\,,\,\delta\psi_*\,,\,\delta\psi\big)$$

by Lemma 4.1.b.

□

Corollary 4.3, below, gives a representation for

$$\Big((\mathbb{S}\mathbb{T}_n^{(SF)})\circ\cdots\circ(\mathbb{S}\mathbb{T}_0^{(SF)})\Big)\Big(e^{\mathcal{A}_0}\Big)(\psi_*,\psi)$$
$$= \Big(\mathbb{T}_n^{(SF)}\circ(\mathbb{S}\mathbb{T}_{n-1}^{(SF)})\circ\cdots\circ(\mathbb{S}\mathbb{T}_0^{(SF)})\Big)\Big(e^{\mathcal{A}_0}\Big)(\mathbb{S}^{-1}\psi_*,\mathbb{S}^{-1}\psi)$$

which, up to a multiplicative constant, is of the form

$$e^{\mathcal{C}_n(\psi_*,\psi)}\ \mathcal{F}_n(\psi_*,\psi)$$

where

- the "contribution from the critical field" is

$$\begin{aligned}\mathcal{C}_n(\psi_*,\psi) = &-A_{n+1}(\psi_*,\psi,\phi_{*n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n),\phi_{n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n),L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ &+(\mathbb{S}\mathcal{R}_n)\big(\phi_{*n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n),\phi_{n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\big)+\mathcal{E}_{n+1,1}(\psi_*,\psi)\end{aligned}$$

  with

$$\begin{aligned}\mathcal{E}_{n+1,1}(\psi_*,\psi) &= (\mathbb{S}\mathcal{E}_n)\big(\hat\psi_{*n}(\psi_*,\psi,\mu_n,\mathcal{V}_n),\hat\psi_n(\psi_*,\psi,\mu_n,\mathcal{V}_n)\big)\\ \hat\psi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) &= \mathbb{S}\big[\psi_{*n}(\mathbb{S}^{-1}\psi_*,\mathbb{S}^{-1}\psi,\mu,\mathcal{V})\big]\end{aligned}\tag{4.3}$$

  and the μ_n, $\mathcal{V}_n$, $\mathcal{R}_n$ and $\mathcal{E}_n$ of Theorem 1.17 for $n\ge 1$ and of §1.5 for $n=0$,
- and the "fluctuation integral" is

$$\begin{aligned}\mathcal{F}_n(\psi_*,\psi) = \Big[\prod_{w\in\mathcal{X}_1^{(n)}}\int_{|z(w)|\le r_n}&\tfrac{dz(w)^*\wedge dz(w)}{2\pi i}e^{-|z(w)|^2}\Big]\\ &\exp\Big\{-\delta A_n(\psi_*,\psi,z_*,z)+\delta\mathcal{R}_n(\psi_*,\psi,z_*,z)+\delta\mathcal{E}_n\big(\psi_*,\psi,z_*,z\big)\Big\}\end{aligned}\tag{4.4}$$

  with
  - for $n\ge 0$

$$\delta\mathcal{E}_n(\psi_*,\psi,z_*,z) = (\mathbb{S}\mathcal{E}_n)(\Psi_*,\Psi)\Big|^{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}}_{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)}\tag{4.5}$$

  - for $n\ge 0$

$$\delta\mathcal{R}_n(\psi_*,\psi,z_*,z) = (\mathbb{S}\mathcal{R}_n)(\Phi_*,\Phi)\Big|^{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)+\delta\hat\phi_{(*)n+1}(\psi_*,\psi,z_*,z)}_{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}\tag{4.6}$$

where,

$$\delta\hat\phi_{(*)n+1}(\psi_*,\psi,z_*,z) = \begin{cases} \mathbb{S}\big[\delta\check\phi_{(*)n+1}\big(\mathbb{S}^{-1}\psi_*\,,\ \mathbb{S}^{-1}\psi_*\,,\ D^{(n)*}\mathbb{L}_* z_*\,,\ D^{(n)}\mathbb{L}_* z\,,\ \mu_n,\mathcal{V}_n\big)\big] & \text{if } n\ge 1 \\ L^{3/2}\mathbb{S}D^{(0)(*)}\mathbb{S}^{-1}z_{(*)} & \text{if } n=0 \end{cases} \tag{4.7}$$

and

- for $n\ge 1$,

$$\begin{aligned}\delta A_n(\psi_*,\psi,z_*,z) = &-L^{7/2}\int_0^1 dt\,\big\langle z_*,\mathbb{S}D^{(n)}\mathfrak{Q}_n\,Q_n\mathbb{S}^{-1}\,[\delta\hat\phi^{(+)}_{n+1}\big(\psi_*,\psi;t\,z_*,t\,z\big)]\big\rangle_1 \\ &-L^{7/2}\int_0^1 dt\,\big\langle\mathbb{S}D^{(n)*}\mathfrak{Q}_n\,Q_n\mathbb{S}^{-1}\,[\delta\hat\phi^{(+)}_{*n+1}\big(\psi_*,\psi;t\,z_*,t\,z\big)],\ z\big\rangle_1\end{aligned} \tag{4.8.a}$$

and, for $n=0$,

$$\begin{aligned}\delta A_0(\psi_*,\psi,z_*,z) &= \int_0^1 (1-t)\,\tfrac{d^2}{dt^2}(\mathbb{S}\mathcal{V}_0)\big(\hat\psi_*+t\delta\psi_*\,,\ \hat\psi+t\delta\psi\big)\,dt\Big|_{\substack{\hat\psi_{(*)}=\hat\psi_{0(*)}(\psi_*,\psi,\mu_0,\mathcal{V}_0)\\ \delta\psi_{(*)}=L^{3/2}\mathbb{S}D^{(0)(*)}\mathbb{S}^{-1}z_{(*)}}} \\ &\quad -\mu_0 L^5\big\langle z_*,\ \mathbb{S}C^{(0)}\mathbb{S}^{-1}z\big\rangle_1\end{aligned} \tag{4.8.b}$$

where, for $n\ge 1$,

$$\delta\hat\phi^{(+)}_{(*)n+1}(\psi_*,\psi,z_*,z) = \delta\hat\phi_{(*)n+1}(\psi_*,\psi,z_*,z) - L^{3/2}\mathbb{S}S^{(*)}_n Q^*_n\mathfrak{Q}_n D^{(n)(*)}\mathbb{S}^{-1}z_{(*)} \tag{4.9}$$

Corollary 4.3.

(a) Let $1\le n<n_p$. Let μ_n, $\mathcal{R}_n$ and $\mathcal{E}_n$ be as in Theorem 1.17. Set

$$\begin{aligned}\mathcal{A}_n(\psi_*,\psi) = &\Big[-A_n(\psi_*,\psi,\phi_{*n},\phi_n,\ \mu_n,\mathcal{V}_n)+\mathcal{R}_n(\phi_{*n},\phi_n)\Big]_{\phi_{(*)n}=\phi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)} \\ &+\mathcal{E}_n(\psi_*,\psi)\end{aligned}$$

Then

$$\big(\mathbb{S}\mathbb{T}^{(SF)}_n\big)\Big(e^{\mathcal{A}_n}\Big)(\psi_*,\psi;\mu_n,\mathcal{V}_n) = \tfrac{1}{\tilde N^{(n)}_{\mathbb{T}}}\ e^{\mathcal{C}_n(\psi_*,\psi)}\ \mathcal{F}_n(\psi_*,\psi)$$

(b) For $n=0$, $\big(\mathbb{S}\mathbb{T}_0^{(SF)}\big)\Big(e^{\mathcal{A}_0}\Big)(\psi_*,\psi;\mu_0,\mathcal{V}_0)=\frac{1}{\tilde N_{\mathbb{T}}^{(0)}}\ e^{\mathcal{C}_0(\psi_*,\psi)}\ \mathcal{F}_0(\psi_*,\psi)$

Proof. By Proposition 4.2, it suffices to verify that

$$(\mathbb{S}\check{\mathcal{C}}_n)(\psi_*,\psi)=\mathcal{C}_n(\psi_*,\psi)\qquad(\mathbb{S}\check{\mathcal{F}}_n)(\psi_*,\psi)=\mathcal{F}_n(\psi_*,\psi)$$

for all $n\ge 0$. That we have $\mathbb{S}\check{\mathcal{C}}_n=\mathcal{C}_n$ follows immediately from

$$\begin{aligned}&\check{A}_{n+1}(\theta_*,\theta,\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\mu_n,\mathcal{V}_n)\Big|_{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}}\\&\qquad=A_{n+1}(\psi_*,\psi,\phi_{*n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n),\phi_{n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n),L^2\mu_n,\mathbb{S}\mathcal{V}_n)\end{aligned}$$

by Definition 2.3 and Definition 3.2, and

$$\begin{aligned}&\mathcal{R}_n\big(\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)\Big|_{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}}\\&\qquad=(\mathbb{S}\mathcal{R}_n)\big(\,\mathbb{S}[\check\phi_{*n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)]\,,\,\mathbb{S}[\check\phi_{n+1}(\theta_*,\theta,\mu_n,\mathcal{V}_n)]\,\big)\Big|_{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}}\\&\qquad=(\mathbb{S}\mathcal{R}_n)\big(\,\phi_{*n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\,,\,\phi_{n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\,\big)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{E}_n\big(\psi_{*n}(\theta_*,\theta,\mu_n,\mathcal{V}_n),\psi_n(\theta_*,\theta,\mu_n,\mathcal{V}_n)\big)\Big|_{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}}\\&\qquad=(\mathbb{S}\mathcal{E}_n)\big(\hat\psi_{*n}(\psi_*,\psi,\mu_n,\mathcal{V}_n),\hat\psi_n(\psi_*,\psi,\mu_n,\mathcal{V}_n)\big)\end{aligned}$$

by (4.3).

Under the substitution $\zeta(x)=(\mathbb{L}_*z)(x)=z(\mathbb{L}^{-1}x)$

$$\begin{aligned}&\Big[\prod_{x\in\mathcal{X}_0^{(n)}}\int_{|\zeta(x)|\le r_n}\tfrac{d\zeta(x)^*\wedge d\zeta(x)}{2\pi i}e^{-|\zeta(x)|^2}\Big]\mathcal{G}(\zeta_*,\zeta)\\&\qquad=\Big[\prod_{w\in\mathcal{X}_1^{(n)}}\int_{|z(w)|\le r_n}\tfrac{dz(w)^*\wedge dz(w)}{2\pi i}e^{-|z(w)|^2}\Big]\mathcal{G}(\mathbb{L}_*z_*,\mathbb{L}_*z)\end{aligned}\tag{4.10}$$

By (4.3) and Remark 2.2.a,

$$\begin{aligned}\delta\check{\mathcal{E}}_n(\theta_*,\theta,\delta\psi_*,\delta\psi)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}}&=\mathcal{E}_n(\hat\psi_*,\hat\psi)\Big|^{\hat\psi_{(*)}=\mathbb{S}^{-1}\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)+D^{(n)(*)}\mathbb{L}_*z_{(*)}}_{\hat\psi_{(*)}=\mathbb{S}^{-1}\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)}\\&=(\mathbb{S}\mathcal{E}_n)(\Psi_*,\Psi)\Big|^{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}}_{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)}\end{aligned}\tag{4.11}$$

and, by Definition 3.2 and (4.7)

$$\begin{aligned} \delta\check{\mathcal{R}}_n(\theta_*,\theta,\delta\psi_*,\delta\psi)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}} &= \mathcal{R}_n(\phi_*,\phi)\Big|^{\phi_{(*)}=\mathbb{S}^{-1}[\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)]+\mathbb{S}^{-1}[\delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z)]}_{\phi_{(*)}=\mathbb{S}^{-1}[\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)]} \\ &= (\mathbb{S}\mathcal{R}_n)(\Phi_*,\Phi)\Big|^{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)+\delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z)}_{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)} \end{aligned} \tag{4.12}$$

Since, for $n \ge 1$,

$$\begin{aligned} \delta\check{\phi}^{(+)}_{n+1}\big(\theta_*,\theta;\delta\psi_*,\delta\psi,\mu_n,\mathcal{V}_n\big)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}} &= \delta\check{\phi}_{n+1}\big(\theta_*,\theta;\delta\psi_*,\delta\psi,\mu_n,\mathcal{V}_n\big)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}} - S_nQ_n^*\mathfrak{Q}_nD^{(n)}\mathbb{L}_*z \\ &= \mathbb{S}^{-1}[\delta\hat{\phi}_{n+1}(\psi_*,\psi,z_*,z)] - L^{3/2}S_nQ_n^*\mathfrak{Q}_nD^{(n)}\mathbb{S}^{-1}z \\ &= \mathbb{S}^{-1}[\delta\hat{\phi}^{(+)}_{n+1}(\psi_*,\psi,z_*,z)] \end{aligned}$$

by Definition 3.5.b, (4.7) and Remark 2.2.a, we have

$$\begin{aligned} \big\langle\delta\psi_*,\mathfrak{Q}_n\,Q_n\,\delta\check{\phi}^{(+)}_{n+1}\big(\theta_*,\theta;\delta\psi_*,\delta\psi,\mu_n,\mathcal{V}_n\big)\big\rangle_0\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}} &= L^2\big\langle\mathbb{S}D^{(n)*}\mathbb{L}_*z_*\,,\,\mathbb{S}\mathfrak{Q}_n\,Q_n\,\mathbb{S}^{-1}[\delta\hat{\phi}^{(+)}_{n+1}(\psi_*,\psi,z_*,z)]\big\rangle_1 \\ &= L^{7/2}\big\langle\mathbb{S}D^{(n)*}\mathbb{S}^{-1}z_*\,,\,\mathbb{S}\mathfrak{Q}_n\,Q_n\,\mathbb{S}^{-1}[\delta\hat{\phi}^{(+)}_{n+1}(\psi_*,\psi,z_*,z)]\big\rangle_1 \\ &= L^{7/2}\big\langle z_*\,,\,\mathbb{S}D^{(n)}\mathfrak{Q}_n\,Q_n\,\mathbb{S}^{-1}[\delta\hat{\phi}^{(+)}_{n+1}(\psi_*,\psi,z_*,z)]\big\rangle_1 \end{aligned}$$

by Remark 2.2.a,c,d. Consequently, for $n \ge 1$,

$$\delta\check{A}_n(\theta_*,\theta,\delta\psi_*,\delta\psi)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(n)(*)}\mathbb{L}_*z_{(*)}}} = \delta A_n(\psi_*,\psi,z_*,z) \tag{4.13}$$

For $n = 0$,

$$\begin{aligned} &\mathcal{V}_0\big(\psi_{0*}(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi_*\,,\,\psi_0(\theta_*,\theta,\mu_0,\mathcal{V}_0)+\delta\psi\big)\Big|_{\substack{\theta_{(*)}=\mathbb{S}^{-1}\psi_{(*)}\\ \delta\psi_{(*)}=D^{(0)(*)}\mathbb{L}_*z_{(*)}}} \\ &\quad= (\mathbb{S}\mathcal{V}_0)\big(\hat{\psi}_{0*}(\psi_*,\psi,\mu_0,\mathcal{V}_0)+L^{3/2}\mathbb{S}D^{(0)*}\mathbb{S}^{-1}z_*\,,\,\hat{\psi}_0(\psi_*,\psi,\mu_0,\mathcal{V}_0)+L^{3/2}\mathbb{S}D^{(0)}\mathbb{S}^{-1}z\big) \end{aligned}$$

and

$$\begin{aligned}\left\langle \delta\psi_*,\, \delta\psi \right\rangle_0 \Big|_{\delta\psi_{(*)} = D^{(0)(*)}\mathbb{L}_* z_{(*)}} &= L^2 \left\langle \mathbb{S}D^{(0)*}\mathbb{L}_* z_*,\, \mathbb{S}D^{(0)}\mathbb{L}_* z \right\rangle_1 \\ &= L^5 \left\langle \mathbb{S}D^{(0)*}\mathbb{S}^{-1} z_*,\, \mathbb{S}D^{(0)}\mathbb{S}^{-1} z \right\rangle_1 \\ &= L^5 \left\langle z_*,\, \mathbb{S}C^{(0)}\mathbb{S}^{-1} z \right\rangle_1\end{aligned}$$

by (4.3) and Remark 2.2.a,c,d. Therefore (4.13) also holds for $n = 0$. That $\mathbb{S}\check{\mathcal{F}}_n = \mathcal{F}_n$ now follows from (4.10), (4.11), (4.12) and (4.13). □

A Compendium of Definitions

A.1 Lattices

We use many different lattices. Our initial system is a finite volume (continuous) spin system having one (complex valued) spin at each site of the lattice[12]

$$\mathcal{X}_0 = \big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$$

where $L_{\rm tp} \in L^2\mathbb{N}$ and $L_{\rm sp} \in L\mathbb{N}$ are the temporal and spatial sizes of this initial, finite volume, lattice and $L \geq 3$ is a fixed odd natural number. $\mathcal{X}_0$ is a unit lattice in the sense that the distance between nearest neighbours in the lattice is 1. During each renormalization group step this lattice is scaled down. In each of the first n_p steps, which are the steps considered in this paper and in [9, 13], we use (anisotropic) "parabolic scaling" and decrease the lattice spacing in the temporal direction by a factor of L^2 and in the spatial directions by a factor of L. So after n renormalization group steps the lattice spacing in the spatial directions is $\varepsilon_n = \frac{1}{L^n}$ and in the temporal direction is $\varepsilon_n^2 = \frac{1}{L^{2n}}$ and the torus $\mathcal{X}_0$ has been scaled down to

$$\mathcal{X}_n = \big(\tfrac{1}{L^{2n}}\mathbb{Z}\big/\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\tfrac{1}{L^n}\mathbb{Z}^3\big/\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

We call $\mathcal{X}_n$ the "ε_n–lattice" and denote by

$$\mathcal{H}_n = \mathbb{C}^{\mathcal{X}_n}$$

the space of all complex valued functions on $\mathcal{X}_n$. We endow $\mathcal{H}_n$ with the norm and bilinear[13] form

$$\|f\|_n = \mathrm{vol}_n \sum_{x\in\mathcal{X}_n} |f(x)|^2 \qquad\qquad \langle f, g\rangle_n = \mathrm{vol}_n \sum_{x\in\mathcal{X}_n} f(x)g(x)$$

where

$$\mathrm{vol}_n = \varepsilon_n^5$$

is the volume of a cell in $\mathcal{X}_n$. We view $\mathcal{H}_n$ as the Hilbert space $L^2(\mathcal{X}_n)$ with (positive definite) inner product $\langle f^*, g\rangle_n$ and norm $\|f\|_n$. Many of the operators acting on $\mathcal{H}_n$ that we consider are periodizations of operators acting on L^2 of the "universal cover"

$$\mathcal{Z}_n = \varepsilon_n^2\mathbb{Z} \times \varepsilon_n\mathbb{Z}^3$$

[12]Of course $\mathcal{X}_0$ is a finite set and so is perhaps more accurately described as a discrete torus, rather than a lattice.

[13]Note that the form is not sesquilinear. We will explicitly write complex conjugates when we want them.

of $\mathcal{X}_n$.

We implement each renormalization group step by performing a block averaging of the spins. During each of the steps in the parabolic regime, we average[14] over blocks of $L^2 \times L \times L \times L$ sites. So a k–block, that is a block of sites averaged over in each of the first k renormalization group steps, consists of $L^{2k} \times L^k \times L^k \times L^k$ sites when $k \le n_p$. The number of sites averaged over is $\frac{1}{\mathrm{vol}_k}$.

It is necessary to repeatedly compose critical point field configurations. For this purpose, we introduce an array of intermediate sublattices. For each $0 \le k \le n+1$, define the sublattice $\mathcal{X}_{n-k}^{(k)}$ of (centres of) k–blocks in the ε_n–lattice $\mathcal{X}_n$ and the corresponding Hilbert space, bilinear form and norm, to be

$$\mathcal{X}_{n-k}^{(k)} = \big(\varepsilon_{n-k}^2\mathbb{Z}/\varepsilon_n^2 L_{\mathrm{tp}}\mathbb{Z}\big) \times \big(\varepsilon_{n-k}\mathbb{Z}^3/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}^3\big) \quad \mathcal{H}_{n-k}^{(k)} = L^2(\mathcal{X}_{n-k}^{(k)})$$

and

$$\langle f, g\rangle_{n-k}^{(k)} = \tfrac{\mathrm{vol}_n}{\mathrm{vol}_k} \sum_{x\in\mathcal{X}_{n-k}^{(k)}} f(x)g(x) \qquad\qquad \|f\|_{n-k}^{(k)} = \tfrac{\mathrm{vol}_n}{\mathrm{vol}_k} \sum_{x\in\mathcal{X}_{n-k}^{(k)}} |f(x)|^2$$

The lower index gives the "scale" of the lattice. That is, the distance between nearest neighbour points of the lattice. The upper index gives the block size and determines the number of points in the sublattice (the number of points in $\mathcal{X}_n$ divided by the number of points in a k–block). The sum of the upper and lower indices gives the number of the renormalization group step. For example, $\mathcal{X}_j^{(k)}$

- has the lattice spacing ε_j^2 in temporal directions and
- has the lattice spacing ε_j in spatial directions and
- has $(\varepsilon_k^2 L_{\mathrm{tp}})(\varepsilon_k L_{\mathrm{sp}})^3$ points and
- has volume [volume of single cell] $\times$ [number of points] $= \varepsilon_j^5(\varepsilon_k^2 L_{\mathrm{tp}})(\varepsilon_k L_{\mathrm{sp}})^3$

Observe that $\mathcal{X}_n^{(0)} = \mathcal{X}_n$ and $\mathcal{H}_n^{(0)} = \mathcal{H}_n$ and that $\mathcal{X}_0^{(n)} = \big(\mathbb{Z}/\varepsilon_n^2 L_{\mathrm{tp}}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}^3\big)$ is a unit lattice in $\mathbb{Z}^4$.

A.2 Scaling

Scaling is performed by the linear isomorphisms

$$\mathbb{L} : \mathcal{X}_j^{(k)} \to \mathcal{X}_{j-1}^{(k)} \qquad\qquad (u_0, \mathbf{u}) \mapsto (L^2 u_0, L\mathbf{u})$$

For a function $\alpha \in \mathcal{H}_j^{(k)}$, define the function $\mathbb{L}_*(\alpha) \in \mathcal{H}_{j-1}^{(k)}$ by $\mathbb{L}_*(\alpha)(\mathbb{L}u) = \alpha(u)$. Set $\mathbb{S} = L^{3/2}\mathbb{L}_*^{-1}$. That is, for a field θ on $\mathcal{X}_{j-1}^{(k)}$,

$$(\mathbb{S}\theta)(x) = L^{3/2}\,\theta\big(\mathbb{L}x\big)$$

[14] Technically, this might be modified by smoothing.

is a field on $\mathcal{X}_j^{(k)}$.

A.3 Block Spin Operators

The block spin averaging operator $Q : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)}$, which averages fields $\psi(x)$, indexed by points x of the unit lattice $\mathcal{X}_0^{(n)}$, over blocks centered on the points of the L–lattice $\mathcal{X}_{-1}^{(n+1)}$ is defined by

$$(Q\psi)(y) = \sum_{x\in\mathbb{Z}\times\mathbb{Z}^3} q(x)\psi(y+[x]) \tag{A.1}$$

where $[x]$ denotes the class of $x \in \mathbb{Z}\times\mathbb{Z}^3$ in the quotient space $\mathcal{X}_0^{(n)}$. The averaging profile q is the $\mathfrak{q}$–fold convolution of the characteristic function, $1_\square(x)$, of the rectangle $\big[-\tfrac{L^2-1}{2}, \tfrac{L^2-1}{2}\big] \times \big[-\tfrac{L-1}{2}, \tfrac{L-1}{2}\big]^3$, normalized to have integral one. That is,

$$q = \tfrac{1}{L^{5\mathfrak{q}}} \overbrace{1_\square * 1_\square * \cdots * 1_\square}^{\mathfrak{q}\ \text{times}}$$

For bounds on Q, see [10, Lemma 2.3].

The block spin averaging operator $Q_n : \mathcal{H}_n \to \mathcal{H}_0^{(n)}$, which averages fields $\phi(u)$, indexed by points u of the fine lattice $\mathcal{X}_n$, over blocks centered on the points of the unit–lattice $\mathcal{X}_0^{(n)}$ is defined by

$$Q_n = \big(\mathbb{L}_*^{-1} Q\big)^n \mathbb{L}_*^n \tag{A.2}$$

For bounds on Q_n, see [10, Remark 2.1.a and Lemma 2.2].

The operator

$$\mathfrak{Q}_n = a\Big(\mathbb{1} + \sum_{j=1}^{n-1} \tfrac{1}{L^{2j}} Q_j Q_j^*\Big)^{-1}$$

appears in the term $\langle \psi^* - Q_n\,\phi_*\,,\,\mathfrak{Q}_n(\psi - Q_n\,\phi)\rangle_0$ of the dominant part of the action. For bounds on $\mathfrak{Q}_n$, see [10, Remark 2.1.c and Proposition 2.4]. See [12, Remark 1] for the recursion relation that builds $\mathfrak{Q}_n$.

A.4 Differential and Related Operators

The forward derivatives of $\alpha \in \mathcal{H}_j^{(n)}$ are defined by

$$(\partial_\nu \alpha)(x) = \tfrac{1}{\varepsilon_{j,\nu}}\big[\alpha(x+\varepsilon_{j,\nu}e_\nu) - \alpha(x)\big] \tag{A.3}$$

where e_ν is a unit vector in the $\nu^{\rm th}$ direction and

$$\varepsilon_{j,\nu} = \begin{cases} \varepsilon_j^2 = \frac{1}{L^{2j}} & \text{for } \nu = 0 \\ \varepsilon_j = \frac{1}{L^j} & \text{for } \nu = 1,2,3 \end{cases}$$

We associate to an operator h_0 on $L^2\big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$ the operators

$$D_n = L^{2n}\, \mathbb{L}_*^{-n}\big(\mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0\big)\mathbb{L}_*^n \tag{A.4}$$

Here ∂_0 is the forward time derivative of (A.3). We assume that h_0 is the periodization of a translation invariant operator $\mathbf{h}_0$ on $L^2\big(\mathbb{Z}^3\big)$ whose Fourier transform $\hat{\mathbf{h}}_0(\mathbf{p})$

- is entire in $\mathbf{p}$ and invariant under $\mathbf{p}_\nu \to -\mathbf{p}_\nu$ for each $1 \le \nu \le 3$
- is nonnegative when $\mathbf{p}$ is real and is strictly positive when $\mathbf{p} \in \mathbb{R}^3 \setminus 2\pi\mathbb{Z}^3$
- obeys $\hat{\mathbf{h}}_0(\mathbf{0}) = \frac{\partial \hat{\mathbf{h}}_0}{\partial \mathbf{p}_\nu}(\mathbf{0}) = 0$ for $1 \le \nu \le 3$ and has strictly positive Jacobian matrix $\Big[\frac{\partial^2 \hat{\mathbf{h}}_0}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu}(\mathbf{0})\Big]_{1\le\mu,\nu\le 3}$.

Think of $\mathbf{h}_0$, which is (a constant times) the single particle "kinetic energy" operator, as being essentially a positive constant times the discrete spatial laplacian. The operator D_n is studied in [10, §3].

A number of important operators are built from D_n. One is the covariance for the fluctuation integral in [13]. It is

$$C^{(n)} = (\tfrac{a}{L^2} Q^* Q + \Delta^{(n)})^{-1}$$

where

$$\Delta^{(n)} = \left.\begin{cases} \big(\mathbb{1} + \mathfrak{Q}_n Q_n D_n^{-1} Q_n^*\big)^{-1} \mathfrak{Q}_n & \text{if } n \ge 1 \\ D_0 & \text{if } n = 0 \end{cases}\right\} : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$$

It is bounded in [10, Corollary 4.5].

Another family of important operators built from D_n are the Green's functions

$$S_n(\mu) = \big[D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu\big]^{-1}$$

They are bounded in [10, Proposition 5.1].

A.5 Norms

Let $\mathcal{X}$ be any lattice that is equipped with a metric d and a "cell volume" vol. As an example, the lattice $\mathcal{X}_j^{(n-j)}$ has vol $= \frac{1}{L^{5j}}$. The following Definition describes how we measure the size of the kernels whose arguments run over $\mathcal{X}$.

Definition A.1. Let $f(u_1, \cdots, u_r)$ be a function on $\mathcal{X}^r$. For a mass $\mathfrak{m} \geq 0$ we set

$$\|f\|_{\mathfrak{m}} = \max_{i=1\cdots,r} \max_{u_i} \int du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_r \ |f(u_1, \cdots, u_r)|\, e^{\mathfrak{m}\tau(u_1,\cdots,u_r)}$$

where *the tree length* $\tau(u_1, \cdots, u_r)$ is the minimal length of a tree in $\mathcal{X}$ that has $u_1, \cdots, u_r$ among its vertices, and $\int du\ g(u) = \mathrm{vol} \sum_{u\in\mathcal{X}} g(u)$.

The following definitions describe how we measure the size of complex valued analytic functions of fields. The following norms are special cases of the norms in [5, Definition 2.6].

Definition A.2.

(a) For a field α on $\mathcal{X}$ and $\vec{x} = (x_1, \cdots, x_r) \in \mathcal{X}^r$ we set $\alpha(\vec{x}) = \prod_{i=1}^r \alpha(x_i)$.

(b) A power series $\mathcal{F}$ in the fields $\alpha_1, \cdots, \alpha_s$, on $\mathcal{X}$ has a unique expansion

$$\mathcal{F}(\alpha_1, \cdots, \alpha_s) = \sum_{r_1,\cdots,r_s\geq 0} \mathrm{vol}^{r_1+\cdots+r_s} \sum_{\substack{\vec{x}_i\in\mathcal{X}^{r_i} \\ 1\leq i\leq s}} f_{r_1,\cdots,r_s}\big(\vec{x}_1, \cdots, \vec{x}_s\big) \prod_{i=1}^s \alpha_i(\vec{x}_i)$$

where the coefficients $f_{r_1,\cdots,r_s}\big(\vec{x}_1, \cdots, \vec{x}_s\big)$ are invariant under permutations of the components of each vector $\vec{x}_i$.

(c) For each choice of "weights" $\kappa_1, \cdots, \kappa_s > 0$, for the fields $\alpha_1, \cdots, \alpha_s$, we define *the norm of* $\mathcal{F}$ *with mass* $\mathfrak{m}$ *and weights* $\kappa_1, \cdots, \kappa_s > 0$ to be

$$\sum_{r_1,\cdots,r_s\geq 0} \big\|f_{r_1,\cdots,r_s}\big(\vec{x}_1, \cdots, \vec{x}_s\big)\big\|_{\mathfrak{m}} \prod_{i=1}^s \kappa_i^{r_i}$$

The following definition describes how we measure the size of analytic maps like the background field map $(\psi_*, \psi) \mapsto \phi_n(\psi_*, \psi, \mu_n, \mathcal{V}_n)$. The norms in the following definition are special cases of the norms in [11, Definition 2.3].

Definition A.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be sublattices of a common finite lattice having metric d, with $\mathcal{X}$ having a "cell volume" vol and with $\mathcal{Y}$ having a "cell volume" $\mathrm{vol}_{\mathcal{Y}}$. Write[15] $\mathcal{X}^{(s)} = \bigcup_{r_1,\cdots,r_s\geq 0} \mathcal{X}^{r_1} \times \cdots \times \mathcal{X}^{r_s}$.

[15] If $(\vec{x}_1, \cdots, \vec{x}_{s-1}) \in \mathcal{X}^{r_1} \times \cdots \times \mathcal{X}^{r_{s-1}}$ then $(\vec{x}_1, \cdots, \vec{x}_{s-1}, -)$ denotes the corresponding element of $\mathcal{X}^{r_1} \times \cdots \times \mathcal{X}^{r_{s-1}} \times \mathcal{X}^0$ with "no s^{th} entry". In particular, $\mathcal{X}^0 = \{-\}$ and $\alpha(-) = 1$.

(a) An s–field map kernel is a function

$$A : (y; \vec{x}_1, \cdots, \vec{x}_s) \in \mathcal{Y} \times \mathcal{X}^{(s)} \mapsto A(y; \vec{x}_1, \cdots, \vec{x}_s) \in \mathbb{C}$$

which obeys $A(y; -, \cdots, -) = 0$ for all $y \in \mathcal{Y}$.

(b) If A is an s–field map kernel, we define the "field map"

$$(\alpha_1, \cdots, \alpha_s) \to A(\alpha_1, \cdots, \alpha_s)$$

by

$$A(\alpha_1, \cdots, \alpha_s)(y) = \sum_{r_1, \cdots, r_s \geq 0} \mathrm{vol}^{r_1 + \cdots + r_s} \sum_{\substack{\vec{x}_i \in \mathcal{X}^{r_i} \\ 1 \leq i \leq s}} A(y; \vec{x}_1, \cdots, \vec{x}_s)\ \alpha_1(\vec{x}_1) \cdots \alpha_s(\vec{x}_s) \tag{A.5}$$

(c) We define the norm $|||A|||$, with mass $\mathfrak{m}$ and weight factors $\kappa_1, \cdots, \kappa_s$, of the s–field map A by

$$|||A||| = \sum_{\substack{r_1, \cdots, r_s \geq 0 \\ r_1 + \cdots + r_s \geq 1}} \big\|A\big\|_{r_1, \cdots, r_s}$$

where

$$\big\|A\big\|_{r_1, \cdots, r_s} = \max\big\{L(A; r_1, \cdots, r_s)\,,\ R(A; r_1, \cdots, r_s)\big\}$$

and

$$L(A; r_1, \cdots, r_s) = \max_{y \in \mathcal{Y}}\ \mathrm{vol}^{r_1 + \cdots + r_s} \sum_{\substack{\vec{x}_\ell \in \mathcal{X}^{r_\ell} \\ 1 \leq \ell \leq s}} \big|A(y; \vec{x}_1, \cdots, \vec{x}_s)\big| \kappa_1^{r_1} \cdots \kappa_s^{r_s} e^{\mathfrak{m}\tau(y, \vec{x}_1, \cdots, \vec{x}_s)}$$

$$R(A; r_1, \cdots, r_s) = \max_{x' \in \mathcal{X}} \max_{\substack{1 \leq j \leq s \\ r_j \neq 0 \\ 1 \leq i \leq r_j}} \mathrm{vol}_{\mathcal{Y}} \sum_{y \in \mathcal{Y}} \mathrm{vol}^{r_1 + \cdots + r_s - 1} \sum_{\substack{\vec{x}_\ell \in \mathcal{X}^{r_\ell} \\ 1 \leq \ell \leq s \\ (\vec{x}_j)_i = x'}} \big|A(y; \vec{x}_1, \cdots, \vec{x}_s)\big| \kappa_1^{r_1} \cdots \kappa_s^{r_s} e^{\mathfrak{m}\tau(y, \vec{x}_1, \cdots, \vec{x}_s)}$$

Remark A.4. Denote by $\mathbb{C}^{\mathcal{X}}$ and $\mathbb{C}^{\mathcal{Y}}$ the spaces of fields on $\mathcal{X}$ and $\mathcal{Y}$, respectively. If A is an s–field map kernel whose norm, $|||A|||$, is finite, then $(\alpha_1, \cdots, \alpha_s) \mapsto A(\alpha_1, \cdots, \alpha_s)$ is an analytic map from the polydisc

$$\big\{\ (\alpha_1, \cdots, \alpha_s) \in \mathbb{C}^{\mathcal{X}} \times \cdots \times \mathbb{C}^{\mathcal{X}}\ \big|\ \|\alpha_j\|_{L^\infty} < \kappa_j,\ 1 \leq j \leq s\ \big\}$$

to the polydisc

$$\big\{\ \beta \in \mathbb{C}^{\mathcal{Y}}\ \big|\ \|\beta\|_{L^\infty} < |||A|||\ \big\}$$

Most operators we deal with are bounded with respect to a norm of the following kind.

Definition A.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be sublattices of a common lattice having metric d, with $\mathcal{X}$ having a "cell volume" $\mathrm{vol}_{\mathcal{X}}$ and with $\mathcal{Y}$ having a "cell volume" $\mathrm{vol}_{\mathcal{Y}}$. For any operator $A : \mathbb{C}^{\mathcal{X}} \to \mathbb{C}^{\mathcal{Y}}$, with kernel $A(y,x)$, and for any mass $m \geq 0$, we define the norm

$$\|A\|_m = \max\Big\{ \sup_{y\in\mathcal{Y}} \sum_{x\in\mathcal{X}} \mathrm{vol}_{\mathcal{X}}\ e^{m|y-x|}|A(y,x)|\ ,\ \sup_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} \mathrm{vol}_{\mathcal{Y}}\ e^{m|y-x|}|A(y,x)| \Big\}$$

In the special case that $m = 0$, this is just the usual ℓ^1–ℓ^∞ norm of the kernel.

B Symmetries

Fix any integers $j \geq 0$ and $n \geq j$. We discuss the natural symmetries of the lattice $\mathcal{X}_j^{(n-j)}$ (see Definition 1.5.a) and the corresponding symmetries induced on $\mathcal{H}_j^{(n-j)}$ and on functions on $\mathcal{H}_j^{(n-j)}$. Define $\varepsilon_j = \frac{1}{L^j}$.

Definition B.1.

(a) We define (unit) translation and reflection operators, acting on the ε_j–lattice $\mathcal{X}_j^{(n-j)}$, by

$$T_x u = u + x \qquad \big(R_\nu u\big)_i = \begin{cases} u_i & \text{if } i \neq \nu \\ -u_i & \text{if } i = \nu \end{cases}$$

for all $x \in \mathcal{X}_0^{(n)}$, $u \in \mathcal{X}_j^{(n-j)}$ and $0 \leq \nu \leq 3$.

(b) We next define translation operators, acting on the field $\alpha : \mathcal{X}_j^{(n-j)} \to \mathbb{C}$, by

$$\big(T_x\alpha\big)(u) = \alpha\big(T_{-x}u\big) = \alpha(u-x)$$

and reflection operators, acting on the fields $\alpha_{(*)}$ and $\alpha_{\nu(*)}$, by

$$\begin{aligned} \big(R_{\nu'}\alpha_{(*)}\big)(u) &= \alpha_{(*)}(R_{\nu'}u) \\ \big(R_{\nu'}\alpha_{\nu(*)}\big)(u) &= \begin{cases} -\alpha_{\nu(*)}(R_{\nu'}u - \varepsilon_j^2 e_{\nu'}) & \text{if } \nu = \nu' = 0 \\ -\alpha_{\nu(*)}(R_{\nu'}u - \varepsilon_j e_{\nu'}) & \text{if } \nu = \nu' \neq 0 \\ \alpha_{\nu(*)}(R_{\nu'}u) & \text{if } \nu \neq \nu' \end{cases} \end{aligned}$$

For the fields $\tilde\alpha = \big(\alpha, \{\alpha_\nu\}\big) \in \tilde{\mathcal{H}}_j^{(n-j)}$, as in (1.17), define

$$T_x\tilde\alpha = \big(T_x\alpha, \{T_x\alpha_\nu\}\big) \qquad R_{\nu'}\tilde\alpha = \big(R_{\nu'}\alpha, \{R_{\nu'}\alpha_\nu\}\big)$$

(c) We next define translation and reflection operators, acting on functions of the fields by

$$
\begin{aligned}
(T_x\mathcal{F})(\tilde\alpha_*,\tilde\alpha) &= \mathcal{F}\big(T_x^{-1}\tilde\alpha_*, T_x^{-1}\tilde\alpha\big)\\
(R_0\mathcal{F})(\tilde\alpha_*,\tilde\alpha) &= \overline{\mathcal{F}\big(R_0^{-1}\tilde\alpha^*, R_0^{-1}\tilde\alpha_*^*\big)}\\
(R_{\nu'}\mathcal{F})(\tilde\alpha_*,\tilde\alpha) &= \mathcal{F}\big(R_{\nu'}^{-1}\tilde\alpha_*, R_{\nu'}^{-1}\tilde\alpha\big),\ 1\le\nu'\le 3
\end{aligned}
$$

(d) We denote by $\mathfrak{S}$ the symmetry group generated by translations and reflections acting on functions $\mathcal{F}\big(\tilde\alpha_*,\tilde\alpha\big)$. Since

$$
R_\nu^2 = \mathbb{1} \qquad R_{\nu'}R_\nu = R_\nu R_{\nu'} \qquad R_\nu T_x = T_{R_\nu x}R_\nu
$$

the group is finite. We denote by $\mathfrak{S}_{\rm spatial}$ the subgroup of $\mathfrak{S}$ generated by translations and spatial reflections. It is of index two, meaning that every element $g\in\mathfrak{S}$ is of one of the forms $g=g'$ or $g=g'R_0$, with $g'\in\mathfrak{S}_{\rm spatial}$.

(e) A function $\mathcal{F}(\tilde\alpha_*,\tilde\alpha)$ is said to preserve particle number if

$$
\mathcal{F}\big(e^{-i\theta}\tilde\alpha_*, e^{i\theta}\tilde\alpha\big) = \mathcal{F}(\tilde\alpha_*,\tilde\alpha)
$$

for all $\theta\in\mathbb{R}$.

Remark B.2. Let $\tilde{\mathcal{F}}(\tilde\alpha_*,\tilde\alpha)$ be given and set

$$
\mathcal{F}\big(\alpha_*,\alpha\big) = \tilde{\mathcal{F}}\big((\alpha_*,\{\partial_\nu\alpha_*\}),\,(\alpha,\{\partial_\nu\alpha\})\big)
$$

then

$$
(g\mathcal{F})\big(\alpha_*,\alpha\big) = (g\tilde{\mathcal{F}})\big((\alpha_*,\{\partial_\nu\alpha_*\}),\,(\alpha,\{\partial_\nu\alpha\})\big)
$$

for all $g\in\mathfrak{S}$.

Proof. It suffices to consider g a generator of $\mathfrak{S}$. If g is a translation operator, the conclusion is obvious. If g is a reflection, and, for example, $1\le\nu\le 3$, observe that

$$
\begin{aligned}
\varepsilon_j\partial_\nu\big(R_\nu\alpha\big)(u) &= \big[\big(R_\nu\alpha\big)(u+\varepsilon_j e_\nu) - \big(R_\nu\alpha\big)(u) \qquad = \alpha(R_\nu u-\varepsilon_j e_\nu) - \alpha(R_\nu u)\\
&= -\big[\alpha(R_\nu u-\varepsilon_j e_\nu+\varepsilon_j e_\nu) - \alpha(R_\nu u-\varepsilon_j e_\nu)\big] = -\varepsilon_j\partial_\nu\alpha(R_\nu u-\varepsilon_j e_\nu)
\end{aligned}
$$

□

Example B.3. If

$$\mathcal{F}(\alpha_*, \alpha) = \int_{\mathcal{X}_j^{(n-j)}} du\, du'\ \alpha_*(u)\, K(u, u')\, \alpha(u')$$

is invariant under $\mathfrak{S}$, then

$$K(u + x, u' + x) = K(u, u') \quad \overline{K(R_0 u', R_0 u)} = K(u, u') \quad K(R_\nu u, R_\nu u') = K(u, u')$$

for all $u, u' \in \mathcal{X}_j^{(n-j)}$, $x \in \mathcal{X}_0^{(n)}$ and $1 \le \nu \le 3$.

Lemma B.4. *Let $1 \le \nu \le 3$ and assume that*

$$\begin{aligned}
\mathcal{F}_2 &= \int_{\mathcal{X}_j^{(n-j)}} du_1\, du_2\ \alpha_*(u_1)\, K_2(u_1, u_2)\, \alpha_\nu(u_2) \\
\mathcal{F}_4 &= \int_{\mathcal{X}_j^{(n-j)}} du_1 \cdots du_4\ K_4(u_1, u_2, u_3, u_4)\ \alpha_*(u_1)\alpha(u_2)\alpha_*(u_3)\alpha_\nu(u_4)
\end{aligned}$$

are invariant under $\mathfrak{S}_{\rm spatial}$. Then

$$\begin{aligned}
&K_2(u_1 + x, u_2 + x) = K_2(u_1, u_2) \qquad\qquad K_2(R_{\nu'} u_1, R_{\nu'} u_2) = K_2(u_1, u_2) \\
&K_2(u_1, u_2) = -K_2(R_\nu u_1, R_\nu u_2 - \varepsilon_j e_\nu)
\end{aligned}$$

and

$$\begin{aligned}
K_4(u_1 + x, u_2 + x, u_3 + x, u_4 + x) &= K_4(u_1, u_2, u_3, u_4) \\
K_4(R_{\nu'} u_1, R_{\nu'} u_2, R_{\nu'} u_3, R_{\nu'} u_4) &= K_4(u_1, u_2, u_3, u_4) \\
K_4(u_1, u_2, u_3, u_4) &= -K_4(R_\nu u_1, R_\nu u_2, R_\nu u_3, R_\nu u_4 - \varepsilon_j e_\nu)
\end{aligned}$$

for all $u_1, \cdots, u_4 \in \mathcal{X}_j^{(n-j)}$, $x \in \mathcal{X}_0^{(n)}$ and $1 \le \nu' \le 3$ with $\nu' \ne \nu$.

Proof. We prove the last K_2 identity. The other cases are similar.

$$\begin{aligned}
&\int du_1\, du_2\ \alpha_*(u_1)\, K_2(u_1, u_2)\, \alpha_\nu(u_2) \\
&\qquad = -\int du_1\, du_2\ \alpha_*(R_\nu u_1)\, K_2(u_1, u_2)\, \alpha_\nu(R_\nu u_2 - \varepsilon_j e_\nu) \\
&\qquad = -\int du_1\, du_2\ \alpha_*(u_1)\, K_2(R_\nu u_1, R_\nu u_2)\, \alpha_\nu(u_2 - \varepsilon_j e_\nu) \\
&\qquad = -\int du_1\, du_2\ \alpha_*(u_1)\, K_2(R_\nu u_1, R_\nu u_2 - \varepsilon_j e_\nu)\, \alpha_\nu(u_2)
\end{aligned}$$

□

There are obvious Lemma B.4 analogs for monomials of type $\alpha_{*\nu}\,\alpha$ and $\alpha_{*\nu}\alpha\,\alpha_*\,\alpha$.

Remark B.5. Let $\mathfrak{m} \geq 0$, $g \in \mathfrak{S}$, $0 \leq \nu, \nu' \leq 3$ and

$$\mathcal{F}\big(\alpha_{*\nu'}, \alpha_\nu\big) = \int du\, du'\ \alpha_{*\nu'}(u)\, K(u,u')\, \alpha_\nu(u')$$

Then, with the notation of Definition 1.11,

$$\|g\mathcal{F}\|_{\mathfrak{m}} \leq e^{2\varepsilon_j \mathfrak{m}} \|\mathcal{F}\|_{\mathfrak{m}}$$

Proof. If g is a translation, or a reflection $R_{\nu''}$ with $\nu'' \neq \nu, \nu'$, or if $g = R_\nu$ and $\nu' = \nu$, then $\|g\mathcal{F}\|_{\mathfrak{m}} = \|\mathcal{F}\|_{\mathfrak{m}}$. If $\nu' \neq \nu$ and $g = R_\nu$, with $1 \leq \nu \leq 3$, then

$$\begin{aligned} \max_u \sum_{u'} \big| -K(R_\nu u, R_\nu u' - \varepsilon_j e_\nu)\big| e^{\mathfrak{m}|u-u'|} &= \max_u \sum_{u'} \big|K(u,u')\big| e^{\mathfrak{m}|R_\nu u - R_\nu u' + \varepsilon_j e_\nu|} \\ &\leq e^{\varepsilon_j \mathfrak{m}} \|\mathcal{F}\|_{\mathfrak{m}} \end{aligned}$$

Similarly, $\|g\mathcal{F}\|_{\mathfrak{m}} \leq e^{\varepsilon_j^2 \mathfrak{m}} \|\mathcal{F}\|_{\mathfrak{m}}$ when $\nu' \neq \nu$ and $g = R_0$. By the relations of Definition B.1.d, every $g \in \mathfrak{S}$ may be written as a product of a translation and reflections, with each $R_{\nu''}$, $0 \leq \nu'' \leq 3$ appearing at most once. The claim follows. ☐

C Inequalities for Parameters

Lemma C.1. *Assume that ϵ is sufficiently small.*

(a) We have

$$L^{n_p} \leq \Big(\tfrac{\mathfrak{v}_0^{5\epsilon}}{\mu_0 - \mu_*}\Big)^{\frac{1}{2+5\epsilon}}$$

and, for $0 \leq n \leq n_p$,

$$\tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)} \log L^n \leq -\tfrac{1}{2}(1 - 6\epsilon) \log \mathfrak{v}_0 + O(\epsilon^2 |\log \mathfrak{v}_0|)$$

(b) For $0 \leq n \leq n_p$

$$\mathfrak{e}_{\mathfrak{l}}(n) \leq \mathfrak{v}_0^{\frac{3}{2}\epsilon} \tag{C.1.a}$$

$$\tfrac{\mathfrak{v}_0}{L^n} \kappa(n)^2 \kappa_{\mathfrak{l}}(n)^2 \leq \mathfrak{v}_0^{\epsilon/6} \mathfrak{e}_{\mathfrak{l}}(n) \tag{C.1.b}$$

(c) The quantity $\frac{\kappa_{\mathfrak{l}}(n)}{\kappa'(n)\mathfrak{e}_{\mathfrak{l}}(n)}$ *is monotonically decreasing with* n *and bounded above by* $\mathfrak{v}_0^{\epsilon/2}$ *.*

(d) Let $\mathfrak{v}_0$ *be sufficiently small, depending on* ϵ*. Then, for all* $C > 0$*, the infinite product* $\Pi_0^\infty(C) = \prod_{j=1}^\infty \big(1 + C\frac{\mathfrak{e}_{\mathfrak{l}}(j-1)}{\kappa(j)^2}\big)$ *is finite and, for* $n \geq 1$ *and all* $\vec{p} \in \mathfrak{D}$

$$\mathfrak{r}_{\vec{p}}(n, C) \leq (1 + C)\,\Pi_0^\infty(C)\ \mathfrak{v}_0^{1-5\epsilon} \begin{cases} 1 & \textit{if } \vec{p} = (1,1,0) \\ L^{-(2\eta'-\eta_{\mathfrak{l}})n} & \textit{if } \vec{p} = (0,1,1) \\ 1 & \textit{if } \vec{p} = (0,0,2) \\ \mathfrak{v}_0\, L^{-4n} & \textit{if } \vec{p} = (6,0,0) \end{cases}$$

Proof. (a) follows from Remark 1.12.

(b) By part (a),

$$\begin{aligned} \log \mathfrak{e}_{\mathfrak{l}}(n) &= \big(\tfrac{1}{3} - 2\epsilon\big) \log \mathfrak{v}_0 + \big(\tfrac{2}{3} - 4\epsilon\big) \tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)} \log L^n \\ &\leq \big(\tfrac{1}{3} - 2\epsilon\big)\big(1 - (1 - 6\epsilon)\big) \log \mathfrak{v}_0 + O(\epsilon^2 |\log \mathfrak{v}_0|) \\ &= 2\epsilon \log \mathfrak{v}_0 + O(\epsilon^2 |\log \mathfrak{v}_0|) \end{aligned}$$

and

$$\begin{aligned} \log\big(\tfrac{\mathfrak{v}_0}{L^n}\kappa(n)^2\kappa_{\mathfrak{l}}(n)^2\big) - \log \mathfrak{e}_{\mathfrak{l}}(n) &= \log\big(\mathfrak{v}_0^{1/3+\epsilon} L^{(2\eta-1+\epsilon)n}\big) - \log\big(\mathfrak{v}_0^{\frac{1}{3}-2\epsilon} L^{\eta_{\mathfrak{l}} n}\ \big) \\ &= 3\epsilon \log \mathfrak{v}_0 + (2\eta - 1 - \eta_{\mathfrak{l}} + \epsilon) \log L^n \\ &\leq 3\epsilon \log \mathfrak{v}_0 + \big(4\epsilon \tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)} + \epsilon\big) \log L^n \\ &\leq 3\epsilon \log \mathfrak{v}_0 + \tfrac{16}{3}\epsilon \tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)} \log L^n \\ &\leq \tfrac{1}{3}\epsilon \log \mathfrak{v}_0 + O(\epsilon^2 |\log \mathfrak{v}_0|) \end{aligned}$$

(c) By Remark 1.12,

$$\begin{aligned} \log \kappa_{\mathfrak{l}}(n) - \log \kappa'(n) - \log \mathfrak{e}_{\mathfrak{l}}(n) &= \big(\tfrac{\epsilon}{2} - \eta' - \eta_{\mathfrak{l}}\big) \log L^n - \big(\tfrac{\epsilon}{2} - \tfrac{1}{3} + \epsilon + \tfrac{1}{3} - 2\epsilon\big) \log \mathfrak{v}_0 \\ &= -\big[\tfrac{3}{2}(1 - \epsilon) - (\tfrac{1}{3} + 4\epsilon)\tfrac{\log \mathfrak{v}_0}{\log(\mu_0 - \mu_*)}\big] \log L^n + \tfrac{\epsilon}{2} \log \mathfrak{v}_0 \end{aligned}$$

is monotonically decreasing with n.

(d) The fact that the infinite product is finite is immediate. Clearly,

$$\mathfrak{r}_{\vec{p}}(n,C) \le (1+C)\,\Pi_0^\infty(C) \begin{cases} \mathfrak{v}_0^{1-4\epsilon} + \mathfrak{v}_0^{1-3\epsilon} \sum_{\ell=1}^{n} \frac{L^{\eta_\mathfrak{l}\ell}}{L^{(\eta+\eta')\ell}} & \text{if } \vec{p}=(1,1,0) \\ \frac{\mathfrak{v}_0^{1-4\epsilon}}{L^n} + \frac{\mathfrak{v}_0^{1-3\epsilon}}{L^n} \sum_{\ell=1}^{n} L^\ell \frac{L^{\eta_\mathfrak{l}\ell}}{L^{2\eta'\ell}} & \text{if } \vec{p}=(0,1,1) \\ \mathfrak{v}_0^{1-4\epsilon} + \mathfrak{v}_0^{1-3\epsilon} \sum_{\ell=1}^{n} \frac{L^{\eta_\mathfrak{l}\ell}}{L^{2\eta'\ell}} & \text{if } \vec{p}=(0,0,2) \\ \frac{\mathfrak{v}_0^{2-\epsilon}}{L^{4n}} + \frac{\mathfrak{v}_0^{7/3-7\epsilon}}{L^{4n}} \sum_{\ell=1}^{n} L^{4\ell} \frac{L^{\eta_\mathfrak{l}\ell}}{L^{6\eta\ell}} & \text{if } \vec{p}=(6,0,0) \end{cases}$$

In the case that $\vec{p}=(0,1,1)$, the successive terms in the sum increase by a factor of at most $L^{1-\mathrm{const}\epsilon}$, while in the other cases they decrease by a factor of at least $L^{\mathrm{const}\epsilon}$. □

As in [13, Definition 5.2 and Lemma 5.5], we define, for $\vec{p}=(p_u,p_0,p_{\rm sp})$,

$$\Delta(\vec{p}) = \tfrac{3}{2}p_u + \tfrac{7}{2}p_0 + \tfrac{5}{2}p_{\rm sp} \qquad \kappa^{\vec{p}}(n) = \kappa(n)^{p_u}\kappa'(n)^{p_0+p_{\rm sp}}$$

With this notation

$$\mathfrak{r}_{\vec{p}}(n,C) = \mathfrak{r}_{\vec{p}}(0) L^{(5-\Delta(\vec{p}))n}\,\Pi_0^n(C) + C\sum_{\ell=1}^{n} L^{(5-\Delta(\vec{p}))(n-\ell)} \tfrac{\mathfrak{e}_\mathfrak{l}(\ell-1)}{\kappa^{\vec{p}}(\ell)}\,\Pi_\ell^n(C) \tag{C.2}$$

Lemma C.2. *Assume that ϵ is sufficiently small. Let $C>0$ and assume that $\mathfrak{v}_0$ is so small that $\epsilon|\log\mathfrak{v}_0| \ge 2\log(1+C)\,\Pi_0^\infty(C)$. Let $\vec{p}\in\mathfrak{D}$ and $1\le n\le n_p$. Then*

(a) $\frac{\kappa^{\vec{p}}(n)}{\kappa(n)^2}\mathfrak{r}_{\vec{p}}(n,C) \le \mathfrak{v}_0^{\frac{2}{3}-\frac{3}{2}\epsilon}$ *and*

$$\begin{aligned} &\tfrac{\kappa_\mathfrak{l}(n)}{\kappa'(n)}\kappa^{\vec{p}}(n)\,\mathfrak{r}_{\vec{p}}(n,C) \le \mathfrak{v}_0^{1/8}\mathfrak{e}_\mathfrak{l}(n) && \text{if } \vec{p}\ne(6,0,0) \\ &\tfrac{\kappa_\mathfrak{l}(n)}{\kappa(n)}\kappa^{\vec{p}}(n)\,\mathfrak{r}_{\vec{p}}(n,C) \le \mathfrak{v}_0^{\epsilon}\mathfrak{e}_\mathfrak{l}(n) && \text{if } \vec{p}=(6,0,0) \end{aligned}$$

(b) We have

$$\mathfrak{r}_{\vec{p}}(n,C) \le \mathfrak{v}_0^\epsilon \min\left\{\mathfrak{v}_0^{\frac{4}{3}-7\epsilon},\ \tfrac{\mathfrak{v}_0}{L^n}\right\} \begin{cases} \kappa(n)\kappa_\mathfrak{l}(n) & \text{if } \vec{p}=(1,1,0),\ (0,1,1),\ (0,0,2) \\ \frac{1}{\kappa_\mathfrak{l}(n)^2} & \text{if } \vec{p}=(6,0,0) \end{cases}$$

Proof. (a) By Lemma C.1.d

$$\log\big(\kappa^{\vec{p}}(n)\,\mathfrak{r}_{\vec{p}}(n,C)\,\big) \le \tfrac{\epsilon}{2}|\log\mathfrak{v}_0| + \begin{cases} (\frac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + (\eta+\eta')\log L^n & \text{if } \vec{p}=(1,1,0)\\ (\frac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + \eta_{\mathfrak{l}}\log L^n & \text{if } \vec{p}=(0,1,1)\\ (\frac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + 2\eta'\log L^n & \text{if } \vec{p}=(0,0,2)\\ \epsilon\log\mathfrak{v}_0 + (6\eta-4)\log L^n & \text{if } \vec{p}=(6,0,0) \end{cases} \tag{C.3}$$

Consequently

$$\log\big(\tfrac{\kappa^{\vec{p}}(n)}{\kappa(n)^2}\mathfrak{r}_p(n,C)\big) \le \begin{cases} (1-\frac{11}{2}\epsilon)\log\mathfrak{v}_0 - (\eta-\eta')\log L^n & \text{if } \vec{p}=(1,1,0)\\ (1-\frac{11}{2}\epsilon)\log\mathfrak{v}_0 - (2\eta-\eta_{\mathfrak{l}})\log L^n & \text{if } \vec{p}=(0,1,1)\\ (1-\frac{11}{2}\epsilon)\log\mathfrak{v}_0 - 2(\eta-\eta')\log L^n & \text{if } \vec{p}=(0,0,2)\\ (\frac{2}{3}-\frac{3}{2}\epsilon)\log\mathfrak{v}_0 - 4(1-\eta)\log L^n & \text{if } \vec{p}=(6,0,0) \end{cases}$$

The first inequality of the Lemma is immediate.

As

$$\begin{aligned} \log\tfrac{\kappa_{\mathfrak{l}}(n)}{\kappa'(n)} - \log\mathfrak{e}_{\mathfrak{l}}(n) &= \tfrac{\epsilon}{2}\log\mathfrak{v}_0 - (\eta'+\eta_{\mathfrak{l}}-\tfrac{\epsilon}{2})\log L^n\\ \log\tfrac{\kappa_{\mathfrak{l}}(n)}{\kappa(n)} - \log\mathfrak{e}_{\mathfrak{l}}(n) &= \tfrac{\epsilon}{2}\log\mathfrak{v}_0 - (\eta+\eta_{\mathfrak{l}}-\tfrac{\epsilon}{2})\log L^n \end{aligned}$$

the inequalities (C.3) give for the case $\vec{p}=(1,1,0)$

$$\begin{aligned} \log\big(\tfrac{\kappa_{\mathfrak{l}}(n)}{\kappa'(n)}\kappa^{\vec{p}}(n)\,\mathfrak{r}_{\vec{p}}(n,C)\,\big) - \log\mathfrak{e}_{\mathfrak{l}}(n) &\le (\tfrac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + (\eta-\eta_{\mathfrak{l}}+\tfrac{\epsilon}{2})\log L^n\\ &\le (\tfrac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + \big(\tfrac{1+\epsilon}{2} - (\tfrac{1}{3}-4\epsilon)\tfrac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)}\big)\log L^n\\ &\le (\tfrac{1}{3}-3\epsilon)\log\mathfrak{v}_0 + (\tfrac{1}{3}+5\epsilon)\tfrac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)}\log L^n\\ &\le \tfrac{1}{8}\log\mathfrak{v}_0 \end{aligned}$$

again by Remark 1.12 and Lemma C.1.a. Since $\eta' < \eta$, the same bound applies in the case $\vec{p}=(0,0,2)$. In the case $\vec{p}=(0,1,1)$, the desired inequality is easy. Finally, in the case $\vec{p}=(6,0,0)$

$$\begin{aligned} \log\big(\tfrac{\kappa_{\mathfrak{l}}(n)}{\kappa(n)}\kappa^{\vec{p}}(n)\,\mathfrak{r}_{\vec{p}}(n,C)\,\big) - \log\mathfrak{e}_{\mathfrak{l}}(n) &\le \epsilon\log\mathfrak{v}_0 + (5\eta-4-\eta_{\mathfrak{l}}+\tfrac{\epsilon}{2})\log L^n\\ &\le \epsilon\log\mathfrak{v}_0 + \Big(-\tfrac{3}{2} + \big(1+4\varepsilon\big)\tfrac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)} + \tfrac{\epsilon}{2}\Big)\log L^n\\ &\le \epsilon\log\mathfrak{v}_0 \end{aligned}$$

(b) By Lemma C.1.d,

$$\mathfrak{r}_{\vec{p}}(n,C) \le \mathfrak{v}_0^{\epsilon} \begin{cases} \mathfrak{v}_0^{1-\frac{13}{2}\epsilon} & \text{if } \vec{p} = (1,1,0),\ (0,1,1),\ (0,0,2) \\ L^{-4n}\mathfrak{v}_0^{2-\frac{13}{2}\epsilon} & \text{if } \vec{p} = (6,0,0) \end{cases} \tag{C.4}$$

The case $(6,0,0)$ is obvious. The remaining cases follow from

$$\frac{\mathfrak{v}_0^{1-\frac{13}{2}\epsilon}}{\kappa(n)\kappa_{\mathfrak{l}}(n)} = L^{(-\eta-\frac{\epsilon}{2})n}\mathfrak{v}_0^{\frac{4}{3}-7\epsilon} \le \min\left\{\mathfrak{v}_0^{\frac{4}{3}-7\epsilon},\ \tfrac{\mathfrak{v}_0}{L^n}\, L^{(1-\eta)n_p}\mathfrak{v}_0^{\frac{1}{3}-7\epsilon}\right\} \le \min\left\{\mathfrak{v}_0^{\frac{4}{3}-7\epsilon},\ \tfrac{\mathfrak{v}_0}{L^n}\right\}$$

since, by Remark 1.12, $\quad L^{(1-\eta)n_p}\mathfrak{v}_0^{\frac{1}{3}-7\epsilon} \le L^{\frac{1}{4}n_p}\mathfrak{v}_0^{\frac{1}{3}-7\epsilon} \le \mathfrak{v}_0^{-\frac{1}{4}(\frac{2}{3}-3\epsilon)}\mathfrak{v}_0^{\frac{1}{3}-7\epsilon} \le 1.$ □

Lemma C.3.

(a) If $1 \le n \le n_p$,

$$\mathfrak{v}_0^{1-8\epsilon}\sum_{\ell=1}^{n}\tfrac{1}{L^{(2-3\epsilon)\ell}}\left[\mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + L^{2\ell}(\mu_0-\mu_*)\right] \le \tfrac{1}{2}\mathfrak{v}_0^{\frac{4}{3}-15\epsilon}$$

(b) $\quad \displaystyle\sum_{\ell=1}^{n}\tfrac{L^{\ell}}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1) \le \mathfrak{v}_0^{\frac{5}{3}-6\epsilon}.$

Proof. (a) The claim follows from

$$\begin{aligned}\sum_{\ell=1}^{n}\tfrac{1}{L^{(2-3\epsilon)\ell}}\left[\mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + L^{2\ell}(\mu_0-\mu_*)\right] &\le \mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + \mathfrak{v}_0^{-\epsilon}L^{4\epsilon n_p}(\mu_0-\mu_*) \\ &\le \mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + \mathfrak{v}_0^{-\epsilon}(\mu_0-\mu_*)^{1-2\epsilon} && \text{by Definition 1.11.b} \\ &\le \mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + \mathfrak{v}_0^{-\epsilon}\mathfrak{v}_0^{\frac{8}{9}(1-2\epsilon)} && \text{by §1.5}\end{aligned}$$

(b) By Definition 1.11.a,

$$\sum_{\ell=1}^{n}\tfrac{L^{\ell}}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1) \le \sum_{\ell=1}^{\infty}L^{(1-4\eta+\eta_{\mathfrak{l}})\ell}\,\mathfrak{v}_0^{\frac{5}{3}-6\epsilon} \le \mathfrak{v}_0^{\frac{5}{3}-6\epsilon}$$

□

Corollary C.4. *Let* $1 \le n \le n_p$.

(a) If the real number μ *obeys* $|\mu-\mu_n^*| \le L^{2n}\,\mathfrak{v}_0^{1-8\epsilon} \sum_{\ell=1}^n \frac{1}{L^{(2-3\epsilon)\ell}}\big[\mathfrak{v}_0^{\frac{1}{3}-6\epsilon}+L^{2\ell}(\mu_0-\mu_*)\big]$ *then*

$$\big|\mu - L^{2n}(\mu_0-\mu_*)\big| \le \mathfrak{v}_0^{1-\epsilon} + L^{2n}\mathfrak{v}_0^{\frac{4}{3}-15\epsilon}$$

and

$$|\mu| \le 2L^{2n}(\mu_0-\mu_*) + \mathfrak{v}_0^{1-\epsilon} \le 4\min\big\{\mathfrak{v}_0^{5\epsilon}\,,\ L^{2n}\mathfrak{v}_0^{\frac{8}{9}+\epsilon}\big\}$$

(b) If the quartic monomial $\mathcal{V}$ *obeys* $\big\|\mathcal{V}-\mathcal{V}_n^{(u)}\big\|_{2m} \le \frac{1}{L^n\mathfrak{v}_0^\epsilon}\sum_{\ell=1}^n \frac{L^\ell}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1)$ *then*

$$\|\mathcal{V}-\mathcal{V}_n^{(u)}\|_{2m} \le \tfrac{1}{L^n}\mathfrak{v}_0^{\frac{5}{3}-7\epsilon} \qquad \textit{and} \qquad \big\|\mathcal{V}\big\|_{2m} \le \tfrac{\mathfrak{v}_0}{L^n}$$

Proof. (a) By Lemma C.3.a and [13, Lemma A.1],

$$\begin{aligned}\big|\mu - L^{2n}(\mu_0-\mu_*)\big| &\le \big|\mu_n^* - L^{2n}(\mu_0-\mu_*)\big| \\ &\qquad + L^{2n}\,\mathfrak{v}_0^{1-8\epsilon}\sum_{\ell=1}^n \tfrac{1}{L^{(2-3\epsilon)\ell}}\big[\mathfrak{v}_0^{\frac{1}{3}-6\epsilon} + L^{2\ell}(\mu_0-\mu_*)\big] \\ &\le \mathfrak{v}_0^{1-\epsilon} + \tfrac{L^{2n}}{2}\mathfrak{v}_0^{\frac{4}{3}-15\epsilon}\end{aligned}$$

The second inequality now follows from $\mathfrak{v}_0^{\frac{4}{3}-16\epsilon} \le \mu_0-\mu_*$ and Definition 1.11.b.

(b) is trivial. □

Lemma C.5. *Let* $0 \le n \le n_p$.

(a) If $0 \le \alpha \le 1$, *then* $\quad L^{2n}(\mu_0-\mu_*) \le L^{2\alpha n}\,\mathfrak{v}_0^{5\epsilon(1-\alpha)}\,(\mu_0-\mu_*)^\alpha$.

(b) $\qquad L^{2n}(\mu_0-\mu_*) \le \min\big\{\mathfrak{v}_0^{5\epsilon}\,,\ L^{2n}\mathfrak{v}_0^{\frac{8}{9}+\epsilon}\big\}$

(c) $\qquad L^{2n}(\mu_0-\mu_*) \le \frac{\mathfrak{v}_0^{1+\epsilon}}{L^n}\kappa(n)^2$

(d) $\qquad L^{2n}(\mu_0-\mu_*)\,\kappa_{\mathfrak{l}}(n)^2 \le \mathfrak{v}_0^\epsilon\,\mathfrak{e}_{\mathfrak{l}}(n)$

Proof. (a) By Definition 1.11.b,

$$L^{2n}(\mu_0-\mu_*) \le L^{2\alpha n}\big[L^{2n_p}(\mu_0-\mu_*)\big]^{1-\alpha}(\mu_0-\mu_*)^\alpha \le L^{2\alpha n}\mathfrak{v}_0^{5\epsilon(1-\alpha)}(\mu_0-\mu_*)^\alpha$$

(b) By part (a) with $\alpha=0$, $L^{2n}(\mu_0-\mu_*) \le \mathfrak{v}_0^{5\epsilon}$. By §1.5, $L^{2n}(\mu_0-\mu_*) \le L^{2n}\mathfrak{v}_0^{\frac{8}{9}+\epsilon}$.

(c) By part (a) with $\alpha = \eta - \frac{1}{2}$ and Definition 1.11.b,

$$\begin{aligned}
&\log\big[L^{2n}(\mu_0 - \mu_*)\big] - \log\big[\tfrac{\mathfrak{v}_0^{1+\epsilon}}{L^n}\kappa(n)^2\big] \\
&\qquad \le 5\epsilon\big(\tfrac{3}{2} - \eta\big)\log\mathfrak{v}_0 + (\eta - \tfrac{1}{2})\log(\mu_0 - \mu_*) - \big(\tfrac{1}{3} + 3\epsilon\big)\log\mathfrak{v}_0 \\
&\qquad \le \tfrac{25}{8}\epsilon\log\mathfrak{v}_0 + \tfrac{1}{3}\log\mathfrak{v}_0 - \big(\tfrac{1}{3} + 3\epsilon\big)\log\mathfrak{v}_0 \\
&\qquad \le 0
\end{aligned}$$

(d) By part (a) with $\alpha = \frac{\eta_\mathfrak{l} - \epsilon}{2}$, Definition 1.11.b and Remark 1.12,

$$\begin{aligned}
&\log\big[L^{2n}(\mu_0 - \mu_*)\big] - \log\big[\mathfrak{v}_0^\epsilon \tfrac{\mathfrak{e}_\mathfrak{l}(n)}{\kappa_\mathfrak{l}(n)^2}\big] \\
&\qquad \le 5\epsilon(1 - \tfrac{\eta_\mathfrak{l} - \epsilon}{2})\log\mathfrak{v}_0 + \tfrac{\eta_\mathfrak{l} - \epsilon}{2}\log(\mu_0 - \mu_*) - \tfrac{1}{3}\log\mathfrak{v}_0 \\
&\qquad \le 5\epsilon(1 - \tfrac{\eta_\mathfrak{l}}{2})\log\mathfrak{v}_0 - 2\epsilon\log\mathfrak{v}_0 - \tfrac{\epsilon}{2}\log(\mu_0 - \mu_*) \\
&\qquad \le 5\epsilon(1 - \tfrac{1}{2} \times \tfrac{2}{3} \times \tfrac{9}{8})\log\mathfrak{v}_0 - 2\epsilon\log\mathfrak{v}_0 - \tfrac{2\epsilon}{3}\log\mathfrak{v}_0 \\
&\qquad \le 0
\end{aligned}$$

□

D Rewriting the Output of the Ultraviolet Flow

D.1 The Model

We now give a technically complete description of the output of [7] and, in Proposition D.1, give the mathematically precise description of the starting point (1.3) of our analysis. The models under consideration are characterized by

- a kinetic energy operator
$$\mathbf{h} = \nabla^* \mathcal{H} \nabla$$
where $\mathcal{H}$ is a real, translation invariant, reflection invariant, strictly positive definite operator on the space, $L^2\big((\mathbb{Z}^3)^*\big)$, of functions on the set, $(\mathbb{Z}^3)^*$, of nearest neighbor bonds of the lattice $\mathbb{Z}^3$.
- a real, symmetric, translation invariant, reflection invariant, strictly positive definite two–body interaction $\mathbf{v}$ on $\mathbb{Z}^3$ and
- a real chemical potential μ.

We denote by h and v the periodizations of $\mathbf{h}$ and $\mathbf{v}$ to the finite lattice $X = \mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3$.

The results of [7] apply under the following conditions on the above data. Pick any mass $\mathrm{m} > 0$ and constants $c_v, D_\mathcal{H}, K_\mu > 0$, $\frac{1}{2} < e_\mu \le 1$ and $0 < c_\mathcal{H} < C_\mathcal{H}$. There is a number $1 \geq \bar{\mathfrak{v}} > 0$, depending on these constants, such that for all $0 < \mathfrak{v} \leq \bar{\mathfrak{v}}$, the results of [7] hold for all μ's, $\mathbf{v}$'s and $\mathcal{H}$'s that satisfy

- $\sum_{\substack{\mathbf{x}\in\mathbb{Z}^3 \\ 1\le i,j\le 3}} e^{6\,\mathrm{m}d(\mathbf{x},0)}\big|\mathcal{H}\big(\langle 0,e_i\rangle\ ,\ \langle \mathbf{x},\mathbf{x}+e_j\rangle\big)\big| \leq D_{\mathcal{H}}$, where e_i is the unit vector in the $i^{\rm th}$ direction and $d(\mathbf{x},\mathbf{y})$ is the Euclidean distance from $\mathbf{x}$ to $\mathbf{y}$, and
- the eigenvalues of the periodization of $\mathcal{H}$ lie between $c_{\mathcal{H}}$ and $C_{\mathcal{H}}$ and
- the norm
$$|||\mathbf{v}||| = \sup_{\mathbf{x}\in\mathbb{Z}^3}\sum_{\mathbf{y}\in\mathbb{Z}^3} e^{5\mathrm{m}\,d(\mathbf{x},\mathbf{y})}\,|\mathbf{v}(\mathbf{x},\mathbf{y})|$$
obeys $\frac{1}{4}\mathfrak{v} \leq |||\mathbf{v}||| \leq \frac{1}{2}\mathfrak{v}$ and
- the smallest eigenvalue of v is at least $c_v|||\mathbf{v}|||$ and
- $|\mu| \leq K_\mu \mathfrak{v}^{e_\mu}$.

D.2 The Output of [7]

Let H be the second quantized Hamiltonian with kinetic energy operator h and two–body interaction v, and let N be the number operator. Fix, as in [7, Hypothesis 2.14], strictly positive exponents $e_{\rm r}$, $e_{\rm R}$, and $e_{\rm R'}$ that obey

$$3e_{\rm R}+4e_{\rm r}<1 \qquad 1\ \leq 4e_{\rm R}+2e_{\rm r} \qquad 2(e_{\rm R}+e_{\rm r})<e_\mu \qquad e_{\rm R'}+e_{\rm r}\ <1 \qquad \tfrac{1}{2}\leq e_{\rm R'}$$

Think of $e_{\rm r}$ as being just slightly larger than 0, $e_{\rm R}$ as being slightly smaller than $\frac{1}{3}$, e_μ as being slightly smaller than $\frac{2}{3}$, and $e'_{\rm R}$ as being between one half and one. In [7, Theorem 2.16] (a self–contained treatment of the pure small field part of the argument is also given in [6]) we prove that there exist constants $K,\theta>0$ (we may assume that $\theta\leq 1$) and a function $I_\theta(\alpha_*,\beta)$ of two complex valued fields α_* and β on X such that

$$\mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)} = \int \prod_{\tau\in\theta\mathbb{Z}\cap(0,\frac{1}{kT}]} \Big[\prod_{\mathbf{x}\in X}\tfrac{d\alpha_\tau(\mathbf{x})^*\wedge d\alpha_\tau(\mathbf{x})}{2\pi\imath}\, e^{-\alpha_\tau(\mathbf{x})^*\alpha_\tau(\mathbf{x})}\Big] I_\theta(\alpha^*_{\tau-\theta},\alpha_\tau) \tag{D.1}$$

for all temperatures $T>0$. Here $\alpha_0=\alpha_{\frac{1}{kT}}$. We also proved that it is possible to write I_θ as the sum of a dominant part $I_\theta^{(SF)}$, called the *pure small field contribution*[16], and terms, indexed by proper subsets of X, which are nonperturbatively

[16] $I_\theta^{(SF)}$ is the $\Omega = X$ term in the formula given for $I_\theta(\alpha^*,\beta)$ in [7, Theorem 2.16]

small, exponentially in the size of the subsets. The dominant part has a logarithm. More precisely

$$I_\theta^{(\mathrm{SF})}(\alpha^*,\beta) = \mathcal{Z}_\theta^{|X|} e^{\langle \alpha^*,\, j(\theta)\beta\rangle_X + V_\theta(\alpha^*,\beta) + \mathcal{D}_\theta(\alpha^*,\beta)} \chi_\theta(\alpha,\beta) \tag{D.2}$$

where

- $\mathcal{Z}_\theta$ is a normalization constant,
- $\langle f, g\rangle_X = \sum_{\mathbf{x}\in X} f(\mathbf{x})g(\mathbf{x})$ is the "real" inner product of $f, g \in L^2(X)$,
- $j(t) = e^{-t(h-\mu)}$
- $V_\theta(\alpha^*,\beta) = -\int_0^\theta \big\langle \big[j(t)\alpha^*\big]\big[j(\theta-t)\beta\big]\,,\, v\big[j(t)\alpha^*\big]\big[j(\theta-t)\beta\big]\big\rangle_X \; dt$
- the function $\mathcal{D}_\theta(\alpha_*,\beta)$ is analytic in the fields α_* and β and is invariant under the $U(1)$ symmetry $\alpha_* \to e^{-it}\alpha_*$, $\beta \to e^{it}\beta$. Furthermore, it can be decomposed in the form
$$\mathcal{D}_\theta(\alpha_*,\beta) = \mathcal{R}_\theta(\alpha_*,\beta) + \mathcal{E}_\theta(\alpha_*,\beta)$$
with
  - a function $\mathcal{R}_\theta(\alpha_*,\beta)$ that is bilinear in α_* and β whose norm, as in Definition 1.10, with mass 2m and weight $\kappa = 2\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_\mathrm{R}+e_\mathrm{r}}$, for both α_* and β, is bounded by $K\,\theta\,\mathfrak{v}^{\mathrm{m}\log\frac{1}{\mathfrak{v}}}$ and
  - a function $\mathcal{E}_\theta(\alpha_*,\beta)$ that has degree at least two[17] both in α_* and in β, whose norm with mass 2m and weight $\kappa = 2\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_\mathrm{R}+e_\mathrm{r}}$, is bounded by $K\,(\mathfrak{v}\theta)^{2-6e_\mathrm{R}-8e_\mathrm{r}}$.
- The "small field cut off function" $\chi_\theta(\alpha,\beta)$ is one if
  - $|\alpha(\mathbf{x})|, |\beta(\mathbf{x})| \le \big(\frac{1}{\theta\mathfrak{v}}\big)^{e_\mathrm{R}+e_\mathrm{r}}$ for all $\mathbf{x}\in X$ and
  - $|\nabla\alpha(b)|, |\nabla\beta(b)| \le \big(\frac{1}{\theta}\big)^{e_{\mathrm{R}'}}\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_\mathrm{r}}$ for all bonds b on X and
  - $|\alpha(\mathbf{x})-\beta(\mathbf{x})| \le \big(\frac{1}{\theta\mathfrak{v}}\big)^{e_\mathrm{r}}$ for all $\mathbf{x}\in X$

  and is zero otherwise.

[17] By this we mean that every monomial appearing in the power series expansion of these functions contains a factor of the form $\alpha_*(\mathbf{x}_1)\,\alpha_*(\mathbf{x}_2)\,\beta(\mathbf{x}_3)\,\beta(\mathbf{x}_4)$.

D.3 The Rewriting

Proposition D.1. *Make the hypotheses of §D.1 and §D.2. Then*

$$\int \prod_{\tau\in\theta\mathbb{Z}\cap(0,\frac{1}{kT}]} \Big[\prod_{\mathbf{x}\in X} \tfrac{d\alpha_\tau(\mathbf{x})^*\wedge d\alpha_\tau(\mathbf{x})}{2\pi\imath}\, e^{-\alpha_\tau(\mathbf{x})^*\alpha_\tau(\mathbf{x})}\Big] I_\theta^{(SF)}(\alpha^*_{\tau-\theta},\alpha_\tau)$$
$$= \mathcal{Z}_{\rm in}^{|\mathcal{X}_0|}\int \Big[\prod_{x\in\mathcal{X}_0} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi\imath}\Big]\, e^{-\langle\psi_*,\,D_0\psi\rangle_0-\mathcal{V}_0(\psi_*,\psi)+\mu_0\langle\psi_*,\,\psi\rangle_0+\mathcal{R}_0(\psi^*,\psi)+\mathcal{E}_0(\psi^*,\psi)}\chi_0(\psi)$$

where

- $\mathcal{Z}_{\rm in}=\mathcal{Z}_\theta e^{-\theta\mu}$
- $\mathcal{X}_0=\big(\mathbb{Z}\times\mathbb{Z}^3\big)/\big(\frac{1}{\theta kT}\mathbb{Z}\times L_{\rm sp}\mathbb{Z}^3\big)$
- $D_0=\mathbb{1}-e^{-h_0}-e^{-h_0}\partial_0$ *with* $h_0=\theta h$ *and* ∂_0 *the forward time derivative.*
- *There is a real–valued kernel* $\mathbf{V}_0(x_1,x_2,x_3,x_4)$ *on* $\big((\mathbb{Z}/\frac{1}{\theta kT}\mathbb{Z})\times\mathbb{Z}^3\big)^4$ *that is invariant under* $x_1\leftrightarrow x_3$ *and under* $x_2\leftrightarrow x_4$ *and under the symmetry group* $\mathfrak{S}$*, and there is a constant* K_v*, depending only on* θ, m, c_v, K_μ *and* $\mathcal{H}$*, such that*
  - $\mathcal{V}_0(\psi_*,\psi)=\frac{1}{2}\int_{\mathcal{X}_0^4}dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\,\psi_*(x_1)\psi(x_2)\psi_*(x_3)\psi(x_4)$ *where* V_0 *is the spatial periodization of* $\mathbf{V}_0$*,*
  - $\|\mathbf{V}_0\|_{\frac{2}{3}{\rm m}}\le K_v\mathfrak{v}$ *and* $\|\mathbf{V}_0\|_0\ge\frac{1}{K_v}\mathfrak{v}$
  - $\Big|\frac{1}{2}\int dx_2\,dx_3\,dx_4\ V_0(x_1,\cdots,x_4)-\theta\int_X d\mathbf{x}\ v(\mathbf{0},\mathbf{x})\Big|\le K_v\mathfrak{v}^{2-2e_{\rm R}-4e_{\rm r}}$
  - $\big\|\mathbf{V}_0-\delta_{x_{1,0},x_{3,0}}\,\delta_{x_{2,0},x_{4,0}}\,\delta_{x_{1,0},x_{2,0}-1}\,\mathbf{v}_\theta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\big\|_{\frac{2}{3}{\rm m}}\le K_v\mathfrak{v}^{2-2e_{\rm R}-4e_{\rm r}}$ *with*

$$\begin{aligned}\mathbf{v}_\theta(\mathbf{x}_1,\cdots,\mathbf{x}_4)&\\ =\sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^3}&\int_0^\theta dt\ e^{-t\mathbf{h}}(\mathbf{x},\mathbf{x}_1)\,e^{-(\theta-t)\mathbf{h}}(\mathbf{x},\mathbf{x}_2)\ \mathbf{v}(\mathbf{x},\mathbf{y})\ e^{-t\mathbf{h}}(\mathbf{y},\mathbf{x}_3)\,e^{-(\theta-t)\mathbf{h}}(\mathbf{y},\mathbf{x}_4)\\ +\sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^3}&\int_0^\theta dt\ e^{-t\mathbf{h}}(\mathbf{x},\mathbf{x}_3)\,e^{-(\theta-t)\mathbf{h}}(\mathbf{x},\mathbf{x}_2)\ \mathbf{v}(\mathbf{x},\mathbf{y})\ e^{-t\mathbf{h}}(\mathbf{y},\mathbf{x}_1)\,e^{-(\theta-t)\mathbf{h}}(\mathbf{y},\mathbf{x}_4)\end{aligned}$$

- $\big|\mu_0-\big(1-e^{-\theta\mu}\big)\big|\le K\theta\mathfrak{v}^{{\rm m}\log\frac{1}{\mathfrak{v}}}$
- $\mathcal{R}_0(\psi_*,\psi)=\tilde{\mathcal{R}}_{\rm in}\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big)+\mathcal{R}_0^{(6)}(\psi_*,\psi)$
  where $\tilde{\mathcal{R}}_{\rm in}((\psi_*,\{\psi_{*\nu}\}),(\psi,\{\psi_\nu\})$ *is an* $\mathfrak{S}$ *invariant, particle–number preserving function with real valued kernels that*
  - *is of degree two in the fields, with either one* $\psi_{(*)}$ *field and one* $\psi_{(*)0}$ *field or two* $\psi_{(*)\nu}$ *fields, with both having* $1\le\nu\le3$ *and*
  - *obeys the bound* $\|\tilde{\mathcal{R}}_{\rm in}\|_{\rm m}\le C_r\mathfrak{v}^{{\rm m}\log\frac{1}{\mathfrak{v}}}$ *with a constant* C_r *that depends only on* m, K_μ *and* K.

and $\mathcal{R}_0^{(6)}(\psi_*, \psi)$ *is an* $\mathfrak{S}$ *invariant, particle–number preserving function with real valued kernel*

- ◦ *that thas degree three both in* ψ_* *and* ψ*, and*
- ◦ *fulfils the estimate* $\|\mathcal{R}_0^{(6)}\|_{2\mathrm{m}} \le e^{-3\theta\mu+6\mathrm{m}} K\,(\mathfrak{v}\theta)^{2-2e_{\mathrm{r}}}$

- $\mathcal{E}_0(\psi_*, \psi)$ *is an* $\mathfrak{S}$ *invariant, particle–number preserving function with real valued kernels that*
  - ◦ *is of degree at least four both in* ψ_* *and in* ψ*, and*
  - ◦ *has norm with mass* $2\mathrm{m}$ *and weight* $2e^{\theta\mu/2-\mathrm{m}}\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_{\mathrm{R}}+e_{\mathrm{r}}}$ *at most* $K\,(\mathfrak{v}\theta)^{2-6e_{\mathrm{R}}-8e_{\mathrm{r}}}$.
- *the "small field cut off function"* $\chi_0(\psi)$ *is one if*
  - ◦ $|\psi(x)| \le e^{\theta\mu/2}\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_{\mathrm{R}}+e_{\mathrm{r}}}$ *for all* $x \in \mathcal{X}_0$ *and*
  - ◦ $|\partial_\nu\psi(x)| \le e^{\theta\mu/2}\big(\frac{1}{\theta}\big)^{e_{\mathrm{R}'}}\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_{\mathrm{r}}}$ *for all* $1 \le \nu \le 3$ *and all* $x \in \mathcal{X}_0$ *and*
  - ◦ $|\partial_0\psi(x)| \le e^{\theta\mu/2}\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_{\mathrm{r}}}$ *for all* $x \in \mathcal{X}_0$

  and is zero otherwise.

Proof. We start by defining a field ψ on the lattice $\mathcal{X}_0$ by

$$\psi(x_0, \mathbf{x}) = e^{\theta\mu/2}\alpha_{\theta x_0}(\mathbf{x})$$

Making a change of variables from $\alpha_\tau(\mathbf{x})$ to $\psi(x)$ converts the integral on the left hand side to

$$\mathcal{Z}_\theta^{|\mathcal{X}_0|} e^{-\theta\mu|\mathcal{X}_0|}\int\Big[\prod_{x\in\mathcal{X}_0}\tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi\imath}\Big]\, e^{\mathcal{F}(\psi^*,\psi)}\chi_0(\psi)$$

where

$$\begin{aligned}
\mathcal{F}(\psi_*, \psi) &= \sum_{\tau\in\theta\mathbb{Z}\cap(0,\frac{1}{kT}]}\big\{-\langle\alpha_{*\tau},\,\alpha_\tau\rangle_X + \langle\alpha_{*\tau-\theta},\, j(\theta)\alpha_\tau\rangle_X + V_\theta(\alpha_{*\tau-\theta},\alpha_\tau) + \mathcal{D}_\theta(\alpha_{*\tau-\theta},\alpha_\tau)\big\}\\
&= -e^{-\theta\mu}\langle\psi_*,\psi\rangle_{\mathcal{X}_0} + \sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]}\Big\{\big\langle\psi_*(x_0-1,\,\cdot\,)\ ,\ \big(e^{-\theta h}\psi\big)(x_0,\,\cdot\,)\big\rangle_X\\
&\qquad\qquad + e^{-2\theta\mu}V_\theta\big(\psi_*(x_0-1,\,\cdot\,),\psi(x_0,\,\cdot\,)\big)\\
&\qquad\qquad + \mathcal{D}_\theta\big(e^{-\theta\mu/2}\psi_*(x_0-1,\,\cdot\,), e^{-\theta\mu/2}\psi(x_0,\,\cdot\,)\big)\Big\}\\
&= -\sum_{x\in\mathcal{X}_0}\psi_*(x)\,\big([\mathbb{1}-e^{-\theta h}-e^{-\theta h}\partial_0]\psi\big)(x) + (1-e^{-\theta\mu})\,\langle\psi_*,\psi\rangle_{\mathcal{X}_0}\\
&\qquad - \mathcal{V}_{\mathrm{in}}(\psi_*,\psi) + \mathcal{R}'_{\mathrm{in}}(\psi_*,\psi) + \mathcal{E}_{\mathrm{in}}(\psi_*,\psi)
\end{aligned}\tag{D.3}$$

with

$$\begin{aligned}
\mathcal{V}_{\rm in}(\psi_*,\psi) &= -\sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]} e^{-2\theta\mu}\, V_\theta\big(\psi_*(x_0-1,\,\cdot\,),\psi(x_0,\,\cdot\,)\big)\\
\mathcal{R}'_{\rm in}(\psi_*,\psi) &= \sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]} e^{-\theta\mu}\, \mathcal{R}_\theta\big(\psi_*(x_0-1,\,\cdot\,),\psi(x_0,\,\cdot\,)\big)\\
\mathcal{E}_{\rm in}(\psi_*,\psi) &= \sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]} \mathcal{E}_\theta\big(e^{-\theta\mu/2}\psi_*(x_0-1,\,\cdot\,),e^{-\theta\mu/2}\psi(x_0,\,\cdot\,)\big)
\end{aligned}$$

All of $\mathcal{V}_{\rm in}$, $\mathcal{R}'_{\rm in}$, $\mathcal{E}_{\rm in}$ are invariant under $\mathfrak{S}$ and have real–valued kernels.

Observe that

$$\begin{aligned}
\mathcal{V}_{\rm in}(\psi_*,\psi) &= e^{-2\theta\mu}\sum_{\substack{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]\\ \mathbf{x},\mathbf{y}\in X\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\in X}} \textstyle\int_0^\theta dt\ j(t)(\mathbf{x},\mathbf{x}_1)\psi_*(x_0-1,\mathbf{x}_1)\ j(\theta-t)(\mathbf{x},\mathbf{x}_2)\psi(x_0,\mathbf{x}_2)\\
&\qquad\qquad v(\mathbf{x},\mathbf{y})\ j(t)(\mathbf{y},\mathbf{x}_3)\psi_*(x_0-1,\mathbf{x}_3)\ j(\theta-t)(\mathbf{y},\mathbf{x}_4)\psi(x_0,\mathbf{x}_4)\\
&= \tfrac{1}{2}\int_{\mathcal{X}_0^4} dx_1\cdots dx_4\ V_{\rm in}(x_1,\cdots,x_4)\ \psi_*(x_1)\psi(x_2)\psi_*(x_3)\psi(x_4)
\end{aligned}$$

where $V_{\rm in}$ is the spatial periodization of

$$\mathbf{V}_{\rm in}(x_1,\cdots,x_4)=\delta_{x_{1,0},x_{3,0}}\,\delta_{x_{2,0},x_{4,0}}\,\delta_{x_{1,0},x_{2,0}-1}\,\mathbf{v}_\theta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$$

As in [7, Lemma 3.21],

$$\|\mathbf{V}_{\rm in}\|_{5\rm m}\le 2\theta\, e^{2K_j\theta}e^{5\rm m}|||\mathbf{v}||| \tag{D.4}$$

By translation invariance

$$\begin{aligned}
&\tfrac{1}{2}\int dx_2\,dx_3\,dx_4\ V_{\rm in}(x_1,\cdots,x_4)\\
&\qquad= \Big[\textstyle\int_X d\mathbf{x}\ v(\mathbf{0},\mathbf{x})\Big]\displaystyle\int_0^\theta dt\ \Big[\textstyle\int_X d\mathbf{x}\ e^{-th}(\mathbf{0},\mathbf{x})\Big]^2\Big[\textstyle\int_X d\mathbf{x}\ e^{-(\theta-t)h}(\mathbf{0},\mathbf{x})\Big]^2\\
&\qquad= \theta\int_X d\mathbf{x}\ v(\mathbf{0},\mathbf{x})
\end{aligned}$$

since, using $\hat{h}$ to denote the Fourier transform of h,

$$\int_X d\mathbf{x}\ e^{-\tau h}(\mathbf{0},\mathbf{x}) = e^{-\tau\hat{h}(\mathbf{0})} = 1$$

Similarly

$$\begin{aligned}\|\mathbf{V}_{\rm in}\|_0 &\ge \int dx_2\, dx_3\, dx_4\ \mathbf{V}_{\rm in}(x_1,\cdots,x_4) = 2\theta\int_{\mathbb{Z}^3} d\mathbf{x}\ \mathbf{v}(\mathbf{0},\mathbf{x}) = 2\theta\int_X d\mathbf{x}\ v(\mathbf{0},\mathbf{x})\\ &= 2\theta\tfrac{\langle 1,v1\rangle_{L^2(X)}}{\langle 1,1\rangle_{L^2(X)}}\\ &\ge \tfrac{\theta c_v}{2}\mathfrak{v}\end{aligned} \tag{D.5}$$

We now move onto a discussion of $\mathcal{R}'_{\rm in}$. By the bound on $\mathcal{R}_\theta$ following (D.2)

$$\|\mathcal{R}'_{\rm in}\|_{2\rm m} \le e^{2\rm m}e^{-\theta\mu}\|\mathcal{R}_\theta\|_{2\rm m} \le e^{2\rm m}e^{-\theta\mu}\tfrac{(\theta\mathfrak{v})^{2(e_{\rm R}+e_{\rm r})}}{4}\ K\theta\mathfrak{v}^{{\rm m}\log\frac{1}{\mathfrak{v}}}$$

By [13, Lemma B.3.c],

$$\begin{aligned}\mathcal{R}'_{\rm in}(\psi_*,\psi) &= \Big[\sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]} e^{-\theta\mu}\,\mathcal{R}_\theta\big(\psi_*(x_0,\,\cdot\,),\psi(x_0,\,\cdot\,)\big)\Big]\\ &\qquad + \Big[\sum_{x_0\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}]} e^{-\theta\mu}\,\mathcal{R}_\theta\big(\psi_*(x_0,\,\cdot\,),(\partial_0\psi)(x_0,\,\cdot\,)\big)\Big]\\ &= \Delta\mu\int_{\mathcal{X}_0} dx\ \psi_*(x)\psi(x)\ +\ \tilde{\mathcal{R}}_{\rm in}\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big)\end{aligned}$$

with a real number $\Delta\mu$ obeying $|\Delta\mu| \le K\theta\mathfrak{v}^{{\rm m}\log\frac{1}{\mathfrak{v}}}$ and a function $\tilde{\mathcal{R}}_{\rm in}$ that has the properties specified in the statement of the proposition. (The contribution with one time derivative and one space derivative that is allowed by [13, Lemma B.3.c] vanishes in this case since the time arguments of the two fields in $\mathcal{R}_\theta\big(\psi_*(x_0,\,\cdot\,),\psi(x_0,\,\cdot\,)\big)$ are always equal.)

Next, we discuss $\mathcal{E}_{\rm in}$. If the part of $\mathcal{E}_\theta$ that is homogeneous of degree $2n$ has kernel $E_{\theta,n}(\mathbf{x}_1,\cdots,\mathbf{x}_n;\mathbf{y}_1,\cdots,\mathbf{y}_n)$, then the part of $\mathcal{E}_{\rm in}$ that is homogeneous of degree $2n$ has kernel

$$\begin{aligned}E_{{\rm in},n}(x_1,\cdots,x_n;y_1,\cdots,y_n) &= E_{\theta,n}(\mathbf{x}_1,\cdots,\mathbf{x}_n;\mathbf{y}_1,\cdots,\mathbf{y}_n)e^{-n\theta\mu}\\ &\qquad\Big[\prod_{i=1}^n \delta x_{1,0}, x_{i,0}\Big]\Big[\prod_{i=1}^n \delta x_{i,0}, y_{i,0}-1\Big]\end{aligned}$$

Let $\mathcal{T}$ be a shortest tree on X having $\mathbf{x}_1,\ \cdots,\ \mathbf{x}_n,\ \mathbf{y}_1,\ \cdots,\ \mathbf{y}_n$ among its vertices. Then the tree on $\mathcal{X}_0$ whose edges are

- $\{(x_{1,0},\mathbf{z})\,,\,(x_{1,0},\mathbf{z}')\}$ if $\{\mathbf{z},\mathbf{z}'\}$ is an edge of $\mathcal{T}$

• and an edge from $(x_{1,0}, \mathbf{y}_i)$ to $y = (x_{1,0}+1, \mathbf{y}_i)$ for each $1 \le i \le n$

has $x_1, \cdots, x_n, y_1, \cdots, y_n$ among its vertices so that

$$\tau\big(x_1, \cdots, x_n, y_1, \cdots, y_n\big) \le \tau\big(\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_1, \cdots, \mathbf{y}_n\big) + n$$

Consequently,

$$\|E_{\mathrm{in},n}\|_{2\mathrm{m}} \le \|E_{\theta,n}\|_{2\mathrm{m}} e^{-n\theta\mu} e^{2\mathrm{m}n}$$

and the norm of $\mathcal{E}_{\mathrm{in}}$ with mass 2m and weight κ_{in} is bounded by the norm of $\mathcal{E}_\theta$ with mass 2m and weight κ provided $\kappa_{\mathrm{in}} e^{-\theta\mu/2+\mathrm{m}} \le \kappa$. So the norm, with mass 2m and weight κ_{in}, of $\mathcal{E}_{\mathrm{in}}$, obeys the bound quoted for $\mathcal{E}_\theta$ (but with mass 2m and weight κ) following (D.2), if we choose $\kappa_{\mathrm{in}} = e^{\theta\mu/2-\mathrm{m}}\kappa = e^{\theta\mu/2-\mathrm{m}} 2\big(\frac{1}{\theta\mathfrak{v}}\big)^{e_{\mathrm{R}}+e_{\mathrm{r}}}$.

Denote by $\mathcal{E}_4$ and $\mathcal{R}_0^{(6)}$ the two monomials in $\mathcal{E}_{\mathrm{in}}$ that are of degree four and six, respectively, and set

$$\mu_0 = \big(1 - e^{-\theta\mu}\big) + \Delta\mu \qquad \mathcal{V}_0 = \mathcal{V}_{\mathrm{in}} - \mathcal{E}_4 \qquad \mathcal{E}_0 = \mathcal{E}_{\mathrm{in}} - \mathcal{E}_4 - \mathcal{R}_0^{(6)}$$

and

$$\mathcal{R}_0(\psi_*, \psi) = \tilde{\mathcal{R}}_{\mathrm{in}}\big((\psi_*, \{\partial_\nu\psi_*\}), (\psi, \{\partial_\nu\psi\})\big) + \mathcal{R}_0^{(6)}(\psi_*, \psi)$$

Obviously

$$-\mathcal{V}_{\mathrm{in}} + \mathcal{R}_{\mathrm{in}} + \mathcal{E}_{\mathrm{in}} = -\mathcal{V}_0 + \mathcal{R}_0 + \mathcal{E}_0$$

and $\mathcal{R}_0$ and $\mathcal{E}_0$ have the desired properties. Except for the definition and properties of $\mathbf{V}_0$, the Proposition now follows from (D.3) and the discussion above.

Set

$$F = \Big\{\ x_1, x_2, x_3, x_4 \in (\mathbb{Z}/\tfrac{1}{\theta kT}\mathbb{Z}) \times \mathbb{Z}^3 \ \Big|\ \tfrac{L_{\mathrm{sp}}}{2} < x_{i,j} - x_{1,j} < \tfrac{L_{\mathrm{sp}}}{2} \\ \text{for all } i = 2,3,4 \text{ and } j = 1,2,3 \Big\}$$

Here $x_{i,j}$ is the j^{th} (spatial) coordinate of x_i. Set, for $x_1, x_2, x_3, x_4 \in (\mathbb{Z}/\frac{1}{\theta kT}\mathbb{Z}) \times \mathbb{Z}^3$

$$\mathbf{E}_4(x_1, x_2, x_3, x_4) = \begin{cases} E_4\big([x_1], \cdots, [x_4]\big) & \text{if } (x_1, x_2, x_3, x_4) \in F \\ 0 & \text{otherwise} \end{cases}$$

where E_4 is the kernel of $\mathcal{E}_4$. Then E_4 is the spatial periodization of $\mathbf{E}_4$. Define

$$\mathbf{V}_0 = \mathbf{V}_{\mathrm{in}} - \text{symmetrization of } \mathbf{E}_4$$

It remains only to prove that $\|\mathbf{E}_4\|_{\frac{2}{3}\mathrm{m}} \le \|E_4\|_{2\mathrm{m}}$. The desired bounds on $\mathbf{V}_0$ will then follow from (D.4) and (D.5).

Denote by $\tilde{\tau}$ the tree length in $(\mathbb{Z}/\frac{1}{\theta kT}\mathbb{Z}) \times \mathbb{Z}^3$. See Definition 1.9. If we have $(x_1, x_2, x_3, x_4) \in F$, then

$$\tilde{\tau}(x_1, x_2, x_3, x_4) \le |x_2 - x_1| + |x_3 - x_1| + |x_4 - x_1| \le 3\tau([x_1], [x_2], [x_3], [x_4])$$

It follows that $\|\mathbf{E}_4\|_{\frac{2}{3}\mathrm{m}} \le \|E_4\|_{2\mathrm{m}}$. ☐

In the setting of Proposition D.1, the μ_* of (1.19) has a particularly simple form. We thank Martin Lohmann for pointing this out.

Lemma D.2. *We have*

$$\mu_* = 2\theta \int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} \tfrac{\hat{\mathbf{v}}(\mathbf{0})+\hat{\mathbf{v}}(\mathbf{k})}{e^{\beta\hat{\mathbf{h}}(\mathbf{k})}-1} \; + \; O(\mathfrak{v}^{2-2e_{\mathrm{R}}-4e_{\mathrm{r}}})$$

where $\hat{\mathbf{h}}(\mathbf{k})$ *and* $\hat{\mathbf{v}}(\mathbf{k})$ *are the Fourier transforms of* $\mathbf{h}(\mathbf{x}, \mathbf{0})$ *and* $\mathbf{v}(\mathbf{x}, \mathbf{0})$ *and* $\beta = \frac{1}{kT}$.

Proof. As a preliminary calculation, we evaluate

$$\begin{aligned}\sum_{\mathbf{x}_2,\mathbf{x}_3\in\mathbb{Z}^3} & e^{-t\mathbf{h}}(\mathbf{x}, \mathbf{x}_2)\mathbf{D}_0^{-1}(x_3, x_2)e^{-(\theta-t)\mathbf{h}}(\mathbf{y}, \mathbf{x}_3) \\ &= \big(e^{-(\theta-t)\mathbf{h}}\mathbf{D}_0^{-1}e^{-t\mathbf{h}}\big)\big((x_{3,0}, \mathbf{y})\,,\,(x_{2,0}, \mathbf{x})\big) \\ &= \big(\mathbf{D}_0^{-1}e^{-\theta\mathbf{h}}\big)\big((x_{3,0}, \mathbf{y})\,,\,(x_{2,0}, \mathbf{x})\big) \\ &= \big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1}\big((x_{3,0}, \mathbf{y})\,,\,(x_{2,0}, \mathbf{x})\big)\end{aligned}$$

where

- we have used that $\mathbf{h}$ is a symmetric operator
- we are thinking of $e^{-\tau\mathbf{h}}(\mathbf{x}, \mathbf{y})$ as being tensored with an identity operator in the temporal arguments x_0, y_0
- we have used that $e^{-\tau\mathbf{h}}$ and $\mathbf{D}_0^{-1}$ are both translation invariant operators and hence commute with each other
- the operator $\big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1}$ is the inverse of $e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0 = e^{\theta\mathbf{h}}\mathbf{D}_0$ acting on the space $L^2\big((\mathbb{Z}/\frac{1}{\theta kT}\mathbb{Z}) \times \mathbb{Z}^3\big)$.

Since

$$\sum_{\mathbf{y}'\in\mathbb{Z}^3} e_X^{-\tau\mathbf{h}}(\mathbf{x}', \mathbf{y}') = e^{-\tau\hat{\mathbf{h}}(\mathbf{0})} = 1$$

we have (recalling the definition of $\mathbf{v}_\theta$ from Proposition D.1

$$\begin{aligned}
\sum_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\mathbb{Z}^3} & \mathbf{v}_\theta(\mathbf{0},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\ \mathbf{D}_0^{-1}(x_3,x_2) \\
&= \int_0^\theta dt \sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^3} v(\mathbf{x},\mathbf{y}) \Big\{ e^{-t\mathbf{h}}(\mathbf{x},\mathbf{0}) \Big[\big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((x_{3,0},\mathbf{y})\,,\,(x_{2,0},\mathbf{y})\big)\Big] \\
&\qquad\qquad + e^{-t\mathbf{h}}(\mathbf{y},\mathbf{0}) \Big[\big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((x_{3,0},\mathbf{y})\,,\,(x_{2,0},\mathbf{x})\big)\Big] \Big\} \\
&= \theta \sum_{\mathbf{x}\in\mathbb{Z}^3} v(\mathbf{0},\mathbf{x}) \Big\{ \big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((x_{3,0},\mathbf{0})\,,\,(x_{2,0},\mathbf{0})\big) \\
&\qquad\qquad + \big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((x_{3,0},\mathbf{x})\,,\,(x_{2,0},\mathbf{0})\big) \Big\}
\end{aligned}$$

Recall from Proposition D.1 that

$$\big\| \mathbf{V}_0 - \delta_{x_{1,0},x_{3,0}}\,\delta_{x_{2,0},x_{4,0}}\,\delta_{x_{1,0},x_{2,0}-1}\,\mathbf{v}_\theta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \big\|_{\frac{2}{3}\mathrm{m}} \le K_v \mathfrak{v}^{2-2e_{\mathrm{R}}-4e_{\mathrm{r}}}$$

As $\mathbf{D}_0^{-1}(x_3,x_2)$ is bounded, we have

$$\begin{aligned}
\mu_* = 2\theta \sum_{\mathbf{x}\in\mathbb{Z}^3} v(\mathbf{0},\mathbf{x}) \Big\{ \big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((1,\mathbf{0})\,,\,0\big) + \big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((1,\mathbf{x})\,,\,0\big) \Big\} \\
+ O\big(\mathfrak{v}^{2-2e_{\mathrm{R}}-4e_{\mathrm{r}}}\big)
\end{aligned} \tag{D.6}$$

On the other hand

$$\begin{aligned}
\big(e^{\theta\mathbf{h}} - \mathbb{1} - \partial_0\big)^{-1} \big((1,\mathbf{x}),0\big) &= \int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} e^{-\theta\mathbf{h}(\mathbf{k})}\ \tfrac{1}{L_{\mathrm{tp}}} \sum_{k_0\in\frac{2\pi}{L_{\mathrm{tp}}}\mathbb{Z}/2\pi\mathbb{Z}} \tfrac{e^{ik_0}}{1-e^{ik_0}e^{-\hat{\mathbf{h}}_0(\mathbf{k})}} \\
&= \int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} e^{-\theta\mathbf{h}(\mathbf{k})}\ \tfrac{e^{-(L_{\mathrm{tp}}-1)\hat{\mathbf{h}}_0(\mathbf{k})}}{1-e^{-L_{\mathrm{tp}}\hat{\mathbf{h}}_0(\mathbf{k})}} \\
&= \int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}}\ \tfrac{e^{-\beta\hat{\mathbf{h}}(\mathbf{k})}}{1-e^{-\beta\hat{\mathbf{h}}(\mathbf{k})}}
\end{aligned} \tag{D.7}$$

Here, we applied Lemma D.4, below, with $p = L_{\mathrm{tp}}$, $\zeta = e^{\frac{2\pi i}{L_{\mathrm{tp}}}}$ and $w = e^{-\hat{\mathbf{h}}_0(\mathbf{k})}$, to the k_0 sum. The Lemma now follows by combining (D.6) and (D.7).

□

Corollary D.3. *With* $\mathbf{H} = \theta\mathcal{H}$*, the data of Proposition D.1 fulfill the conditions of §1.5 provided* $\mathfrak{v}$ *is small enough (depending on* ϵ *and* θ*),* $\mathrm{m} \geq 3m$ *and*

$$2\int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} \tfrac{\hat{\mathbf{v}}(\mathbf{0})+\hat{\mathbf{v}}(\mathbf{k})}{e^{\beta\hat{\mathbf{h}}(\mathbf{k})}-1} + \mathfrak{v}^{\frac{4}{3}-2\epsilon} < \mu < \mathfrak{v}^{\frac{8}{9}+2\epsilon}$$

and $\frac{1}{3} - \epsilon < e_{\mathrm{R}} < \frac{1}{3} - \frac{5}{6}\epsilon$ *and* $e_{\mathrm{r}} < \frac{\epsilon}{2}$*. Observe that*

$$\lim_{\beta\to\infty} \int_{\mathbb{R}^3/2\pi\mathbb{Z}^3} \tfrac{d^3\mathbf{k}}{(2\pi)^3} \tfrac{\hat{\mathbf{v}}(\mathbf{0})+\hat{\mathbf{v}}(\mathbf{k})}{e^{\beta\hat{\mathbf{h}}(\mathbf{k})}-1} = 0$$

Proof. By the definition of $\mathfrak{v}_0$ in §1.5 and Proposition D.1,

$$\tfrac{2}{K_v}\mathfrak{v} \leq 2\|\mathbf{V}_0\|_0 \leq \mathfrak{v}_0 = 2\|\mathbf{V}_0\|_{2m} \leq 2\|\mathbf{V}_0\|_{\frac{2}{3}\mathrm{m}} \leq 2K_v\mathfrak{v}$$

The condition on μ_0 in §1.5 is satisfied since $\theta\mu = \mu_0 + O(\mu^2) + O(\mathfrak{v}^{\mathrm{m}\log\frac{1}{\mathfrak{v}}})$.

□

Lemma D.4. *Let* ζ *be a primitive* p^{th} *root of unity and* $w \in \mathbb{C}$ *not be a* p^{th} *root of unity. Then*

$$\tfrac{1}{p}\sum_{k=0}^{p-1} \tfrac{\zeta^k}{1-w\zeta^k} = \tfrac{w^{p-1}}{1-w^p}$$

Proof. First consider the case $0 < w < 1$. Expanding the geometric series and interchanging sums

$$\tfrac{1}{p}\sum_{k=0}^{p-1} \tfrac{\zeta^k}{1-w\zeta^k} = \sum_{n=0}^{\infty} \tfrac{1}{p}\sum_{k=0}^{p-1} w^n \zeta^{(n+1)k}$$

Now

$$\tfrac{1}{p}\sum_{k=0}^{p-1} \zeta^{(n+1)k} = \begin{cases} 0 & \text{if } n+1 \text{ is not an integer multiple of } p \\ 1 & \text{if } n = mp-1 \text{ for some integer } m \end{cases}$$

If $n \geq 0$, the integer m above has to be at least one. Therefore

$$\tfrac{1}{p}\sum_{k=0}^{p-1} \tfrac{\zeta^k}{1-w\zeta^k} = \sum_{m=1}^{\infty} w^{mp-1} = \tfrac{w^{p-1}}{1-w^p}$$

The claim now follows by analytic continuation in w.

□

References

[1] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski. *Methods of Quantum Field Theory in Statistical Physics*. Dover Publications, 1963.

[2] T. Balaban. The Ultraviolet Stability Bounds for Some Lattice σ–Models and Lattice Higgs–Kibble Models. In *Proc. of the International Conference on Mathematical Physics, Lausanne, 1979*, pages 237–240. Springer, 1980.

[3] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. A Functional Integral Representation for Many Boson Systems. I: The Partition Function. *Annales Henri Poincaré*, 9:1229–1273, 2008.

[4] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. A Functional Integral Representation for Many Boson Systems. II: Correlation Functions. *Annales Henri Poincaré*, 9:1275–1307, 2008.

[5] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Power Series Representations for Complex Bosonic Effective Actions. I. A Small Field Renormalization Group Step. *Journal of Mathematical Physics*, 51:053305, 2010.

[6] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Power Series Representations for Complex Bosonic Effective Actions. II. A Small Field Renormalization Group Flow. *Journal of Mathematical Physics*, 51:053306, 2010.

[7] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Temporal Ultraviolet Limit for Complex Bosonic Many-body Models. *Annales Henri Poincaré*, 11:151–350, 2010.

[8] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Temporal Ultraviolet Limit. In *Quantum Theory from Small to Large Scales, Ecole de Physique des Houches, 2010*, pages 99–170. Oxford University Press, 2012.

[9] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Complex Bosonic Many–body Models: Overview of the Small Field Parabolic Flow. Preprint, 2016.

[10] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Operators for Parabolic Block Spin Transformations. Preprint, 2016.

[11] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Power Series Representations for Complex Bosonic Effective Actions. III. Substitution and Fixed Point Equations. Preprint, 2016.

[12] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Algebra of Block Spin Renormalization Group Transformations. Preprint, 2016.

[13] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Small Field Parabolic Flow for Bosonic Many–body Models: Part 2 — Fluctuation Integral and Renormalization. Preprint, 2016.

[14] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Small Field Parabolic Flow for Bosonic Many–body Models: Part 3 — Nonperturbatively Small Errors. Preprint, 2016.

[15] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Small Field Parabolic Flow for Bosonic Many–body Models: Part 4 — Background and Critical Field Estimates. Preprint, 2016.

[16] G. Benfatto. Renormalization Group Approach to Zero Temperature Bose Condensation. In *Constructive Physics. Springer Lecture Notes in Physics 446*, pages 219–247. Springer, 1995.

[17] N. N. Bogoliubov. On the theory of superfluidity. *J. Phys. (USSR)*, 11:23–32, 1947.

[18] S. Cenatiempo and A. Giuliani. Renormalization theory of a two dimensional Bose gas: quantum critical point and quasi-condensed state. *Jour. Stat. Phys.*, 157:755–829, 2014.

[19] S. Coleman. Secret Symmetry. An Introduction to Spontaneous Symmetry Breakdown and Gauge Fields. In *Laws of Hadronic Matter*, pages 138–215. Academic Press, 1975.

[20] A.L. Fetter and J.D. Walecka. *Quantum Theory of Many-Particle Systems*. McGraw-Hill, 1971.

[21] K. Gawedzki and A. Kupiainen. A rigorous block spin approach to massless lattice theories. *Comm. Math. Phys.*, 77:31–64, 1980.

[22] K. Gawedzki and A. Kupiainen. Massless lattice φ_4^4 theory: rigorous control of a renormalizable asymptotically free model. *Comm. Math. Phys.*, 99:197–252, 1985.

[23] L.P. Kadanoff. Scaling laws for Ising models near T_c. *Physics*, 2:263, 1966.

[24] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason. *The Mathematics of the Bose Gas and its Condensation.* Birkhäuser, 2005.

[25] J. W. Negele and H. Orland. *Quantum Many–Particle Systems.* Addison–Wesley, 1988.

[26] L. Pitaevskii and S. Stringari. *Bose–Einstein Condensation.* Clarendon Press, Oxford, 2003.

[27] R. Seiringer. Cold quantum gases and Bose-Einstein condensation. In *Quantum Theory from Small to Large Scales, Ecole de Physique des Houches, 2010*, pages 429–466. Oxford University Press, 2012.

[28] S. Weinberg. *The Quantum Theory of Fields, Volume II. Modern Applications.* Cambridge Press, 1998.