The Small Field Parabolic Flow for Bosonic Many–body Models:

Part 3 — Nonperturbatively Small Errors

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the "parabolic flow" which exhibits the formation of a "Mexican hat" potential well. Here we provide arguments that suggest, but do not completey prove, that the difference between the "small field" approximation, analyzed in [5, 6], and full model is nonperturbatively small.

As part of the program to see symmetry breaking in an interacting many Boson system on a three dimensional lattice in the thermodynamic limit, we analyze in [5, 6] the "small field" approximation to the "parabolic flow" which exhibits the formation of a potential well. In this paper, we argue that the errors made with this approximation are nonperturbatively small, that is they are smaller than any power of the coupling constant. This note does not provide a proof of this fact; however we feel that the arguments given here can provide the core of such a proof.

The *first simplification* leading to the "small field" approximation is a simplification of the starting point. The outcome of the previous flow [1] (which treats the temporal ultraviolet problem in imaginary time) represents the partition function as a sum over "large–field/ small–field" decompositions of space. All but one term in this sum are nonperturbatively small, and the first simplification is to continue the flow with only this one term. It is of the form

$$\mathcal{Z}_{\text{in}}^{|\mathcal{X}_0|} \int \Big[\prod_{x \in \mathcal{X}_0} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi \imath} \Big] \, e^{\mathcal{A}_0(\psi^*, \psi)} \chi_0(\psi) \tag{1}$$

where $\mathcal{Z}_{\text{in}}$ is a normalization factor, $\mathcal{X}_0$ is a unit lattice, the action $\mathcal{A}_0$ has a very specific form and χ_0 is a function with compact support that implements the small field cutoff. See [5, (1.3) and (1.4)].

The output of the nth renormalization group step is an approximation to the partition function that is a constant times a functional integral over a space of complex valued fields ψ on a unit lattice $\mathcal{X}_0^{(n)}$. One can write the action in this functional integral as function $\mathfrak{A}_n(\psi^*, \psi)$ of the field ψ and its complex conjugate ψ^*, where $\mathfrak{A}_n(\psi_*, \psi)$ is an analytic function[1] of two independent complex fields ψ_*, ψ. The domain of integration at this point is a bounded subset $\text{In}(n)$ of the space of complex valued fields on $\mathcal{X}_0^{(n)}$. See (16) below. A block spin transformation amounts to rewriting the functional integral as

$$\frac{1}{N^{(n)}} \int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \int_{\text{In}(n)} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \, e^{-aL^{-2}\|\theta - Q\psi\|_{-1}^2 - \mathfrak{A}_n(\psi^*, \psi)} \tag{2}$$

where $N^{(n)}$ is the normalization constant for the Gaussian integral over θ, $\mathcal{X}_{-1}^{(n+1)}$ is a sublattice of $\mathcal{X}_0^{(n)}$ and Q is an averaging operator defined in [5, Definition 1.1]. The goal is to perform, for any fixed θ, the ψ integral in (2) to obtain a functional integral representation of the partition function in the θ variables. We view this ψ integral

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[1]The exact form of this function is stated in the main theorem [5, Theorem 1.17]

as an integral of a holomorphic differential form in the $2\big|\mathcal{X}_0^{(n)}\big|$ complex variables $\psi_*(x), \psi(x),\ x \in \mathcal{X}_0^{(n)}$ over the set $D = \big\{ (\psi_*, \psi) \in \mathrm{In}(n) \times \mathrm{In}(n) \ \big|\ \psi_* = \psi^* \big\}$:

$$\int_D \bigg[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)_* \wedge d\psi(x)}{2\pi i} \bigg]\ e^{-aL^{-2}\langle \theta^* - Q\psi_*\, ,\, \theta - Q\psi \rangle_{-1} - \mathfrak{A}_n(\psi_*, \psi)} \tag{3}$$

See [2], Step 3. Observe that D has $2\big|\mathcal{X}_0^{(n)}\big|$ real dimensions.

We will use stationary phase to evaluate the integral (3). To do so, we want to determine, for each fixed value of θ, an approximate critical point $\big(\psi_{*n}(\theta^*, \theta),\ \psi_n(\theta^*, \theta)\big)$ for the map

$$(\psi_*, \psi) \mapsto -aL^{-2}\, \langle \theta^* - Q\psi_*\, ,\, \theta - Q\psi \rangle_{-1} - \mathfrak{A}_n(\psi_*, \psi) \tag{4}$$

This approximate critical point lies $\mathrm{In}(n) \times \mathrm{In}(n)$ only if θ is not too big. Below, we argue that for large θ, the ψ integral in (3) gives nonperturbatively small contributions. Therefore we make an *approximation* by restricting the variable θ in (2) to a bounded subset $\check{\mathrm{In}}(n)$.

As pointed out in [2, Step 3], for general $\theta \in \check{\mathrm{In}}(n)$, the critical point of (4) does not fulfil the reality condition $\psi_{*n}(\theta^*, \theta) = \psi_n(\theta^*, \theta)^*$. In particular it does not lie in the domain of integration D. We choose a bounded subset[2] S of

$$\big\{ (\psi_*, \psi) \ \big|\ \psi_* - \psi_{*n}(\theta^*, \theta) = \big(\psi - \psi_n(\theta^*, \theta)\big)^* \big\}$$

containing $\big(\psi_{*n}(\theta^*, \theta),\ \psi_n(\theta^*, \theta)\big)$, and a $2\big|\mathcal{X}_0^{(n)}\big| + 1$ dimensional set $\mathcal{Y}$ whose boundary consists of D, S and some other component. Below we argue that the integral of

$$\bigg[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)_* \wedge d\psi(x)}{2\pi i} \bigg]\ e^{-aL^{-2}\langle \theta^* - Q\psi_*\, ,\, \theta - Q\psi \rangle_{-1} - \mathfrak{A}_n(\psi_*, \psi)}$$

over $\partial \mathcal{Y} \setminus (S \cup D)$ is nonperturbatively small. This, combined with Stokes' theorem, would justify our *last approximation*, which is the replacement of (3) with

$$\int_S \bigg[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)_* \wedge d\psi(x)}{2\pi i} \bigg]\ e^{-aL^{-2}\langle \theta^* - Q\psi_*\, ,\, \theta - Q\psi \rangle_{-1} - \mathfrak{A}_n(\psi_*, \psi)}$$

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[2]We wish to integrate over a neighborhood of the critical point. So we make a change of variables to "fluctuation fields $\delta\psi_* = \psi^* - \psi_{*n}(\theta^*, \theta)$, $\delta\psi = \psi - \psi_n(\theta^*, \theta)$". The condition in the set below is a reality condition on the fluctuation fields.

An important ingredient in the argument that the above approximations are justified is that, at the points considered, the effective action

$$aL^{-2} \langle \theta^* - Q\psi_* \, , \, \theta - Q\psi \rangle_{-1} + \mathfrak{A}_n(\psi_*, \psi)$$

has a large, positive, real part. Though positivity is suggested by the quadratic and quartic terms in the explicit form of the action (see [5, Definition 1.1]), we have to pay close attention since the fields ψ_*, ψ are complex valued.

This note can be considered as a complement to [5, 6] and uses the notation introduced there. This notation is summarized in [5, Appendix A].

We emphasise again that this note is intended to provide motivation rather than a proof. Some of the bounds are not uniform in the volume $\mathcal{X}_0$. Furthermore some of the statements we make are handwavy. We concentrate on showing where the nonperturbatively small factors come from. A rigorous construction, with bounds uniform in the volume, would entail expressing the errors as sums over "large field subsets" $\mathfrak{L} \subset \mathcal{X}_0^{(n)}$ and exhibiting bounds which include a nonperturbatively small factor for each point of each $\mathfrak{L}$, as was done in [1].

As said above, we start with the approximation (1) for the partition function $\mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)}$ of the many Boson system (see [5, (1.1), (1.2) and (1.3)]). The approximations $\mathbb{T}_n^{(SF)}$ to the block spin transformations sketched above lead, for each $0 \le n < n_p$, to the approximation of

$$
\begin{aligned}
J_n &= \int \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \tfrac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \Big((\mathbb{ST}_{n-1}^{(SF)}) \circ (\mathbb{ST}_{n-2}^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big)(\psi^*, \psi) \chi_n(\psi) \\
&= \tfrac{\tilde{\mathcal{Z}}_n}{\tilde{\mathcal{Z}}_{n+1}} \int \Big[\prod_{x \in \mathcal{X}_0^{(n+1)}} \tfrac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] (\mathbb{ST}_n) \Big(\big[(\mathbb{ST}_{n-1}^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \, e^{\mathcal{A}_0} \big] \chi_n \Big)
\end{aligned}
$$

(5)

(see [5, Remarks 1.2.i and 1.4.i] and (17), below) by a constant times

$$
J_{n+1} = \int \Big[\prod_{x \in \mathcal{X}_0^{(n+1)}} \tfrac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \Big((\mathbb{ST}_n^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big)(\psi^*, \psi) \chi_{n+1}(\psi)
$$

In [5, 6] we did not say very much either about the "small field" cutoff functions $\chi_n(\psi)$ or about the errors introduced by these approximations. In this note we make a possible choice of $\chi_n(\psi)$, $n \ge 1$ (one of many possible choices) and argue that it is reasonable to expect that, for all $n \ge 0$, the error $\frac{1}{\tilde{\mathcal{Z}}_{n+1}} J_{n+1} - \frac{1}{\tilde{\mathcal{Z}}_n} J_n$ is

nonperturbatively small. By this we mean smaller than the dominant contribution by a factor of order $O(e^{-1/\mathfrak{v}_n^\varepsilon})$ for some $\varepsilon > 0$. We concentrate on the case $n \geq 1$. The case $n = 0$ is similar but simpler.

We use two mechanisms for "generating nonperturbatively small factors". The first consists in exhibiting large negative contributions to the leading part $-A_n$ of the representation

$$\Big((\mathbb{ST}_{n-1}^{(SF)}) \circ (\mathbb{ST}_{n-2}^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \Big)\Big(e^{\mathcal{A}_0} \Big)$$

$$= \tfrac{1}{\mathcal{Z}_n} \exp\Big\{ -A_n(\psi^*, \psi, \phi_*, \phi, \ \mu_n, \mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n \Big\}\Big|_{\phi_{(*)} = \phi_{(*)n}(\psi^*, \psi, \mu_n, \mathcal{V}_n)}$$

$$(6)$$

of [5, Theorem 1.17]. These large negative contributions arise whenever $|\psi(x)|$ or $|\partial_\nu \psi(x)|$ are sufficiently large for some $x \in \mathcal{X}_0^{(n)}$, $0 \leq \nu \leq 3$. See Proposition 1, below. The second mechanism appears in the course of the stationary phase approximation of [5, §1.2] when (ψ_*, ψ) is too far from the critical point $\big(\psi_{*n}(\theta^*, \theta), \psi_n(\theta^*, \theta)\big)$. See Step 3, below.

The background fields $\phi_{(*)n}$ and the actions $-A_n + \mathcal{R}_n + \mathcal{E}_n$ are well–defined on the "domain of analyticity"

$$\mathrm{An}(n) = \big\{ \ \psi \in \mathcal{H}_0^{(n)} \ \big| \ |\psi(x)| < \kappa(n), \ |\partial_\nu \psi(x)| < \kappa'(n) \text{ for all } x \in \mathcal{X}_0^{(n)}, \ 0 \leq \nu \leq 3 \ \big\}$$

On these domains we have the following lower and upper bounds on the real part of the dominant contribution, A_n, to the action.

Proposition 1. *Let $\delta > 0$. There are constants $\gamma, \tilde{\gamma} > 0$, independent of δ, such that if $\mathfrak{v}_0$ is sufficiently small, depending on δ,*

$$\gamma \sum_{\nu=0}^3 \int_{\mathcal{X}_0^{(n)}} dx \ |\partial_\nu \psi(x)|^2 - (1+\delta)\mu_n \int_{\mathcal{X}_0^{(n)}} dx \ |\psi(x)|^2 + \tfrac{1}{2}(1-\delta)\mathrm{v}_n \int_{\mathcal{X}_0^{(n)}} dx \ |\psi(x)|^4$$

$$\leq \mathrm{Re}\, A_n(\psi^*, \psi, \phi_*, \phi, \ \mu_n, \mathcal{V}_n)\Big|_{\phi_{(*)} = \phi_{(*)n}(\psi^*, \psi, \mu_n, \mathcal{V}_n)}$$

$$\leq \tilde{\gamma} \sum_{\nu=0}^3 \int_{\mathcal{X}_0^{(n)}} dx \ |\partial_\nu \psi(x)|^2 - (1-\delta)\mu_n \int_{\mathcal{X}_0^{(n)}} dx \ |\psi(x)|^2 + \tfrac{1}{2}(1+\delta)\mathrm{v}_n \int_{\mathcal{X}_0^{(n)}} dx \ |\psi(x)|^4$$

for all $1 \leq n \leq n_p$ and $\psi \in \mathrm{An}(n)$. Here

$$\mathrm{v}_n = \int_{\mathcal{X}_n^3} du_2 \, du_3 \, du_4 \ V_n(0, u_2, u_3, u_4)$$

is the "coupling constant at scale n".

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Proof. We start by recalling, from [5, (3.2) and Theorem 1.13], that the background fields $\phi_{(*)n}(\psi^*, \psi, \mu_n, \mathcal{V}_n)$ are the solutions of

$$D_n^* \phi_* = Q_n^* \mathfrak{Q}_n \psi^* - Q_n^* \mathfrak{Q}_n Q_n \phi_* + \mu_n \phi_* - \mathcal{V}'_{n*}(\phi_*, \phi, \phi_*)$$
$$D_n \phi = Q_n^* \mathfrak{Q}_n \psi - Q_n^* \mathfrak{Q}_n Q_n \phi + \mu_n \phi - \mathcal{V}'_n(\phi, \phi_*, \phi) \tag{7}$$

For the rest of this proof, we'll write $\phi_{(*)}$ instead of $\phi_{(*)n}(\psi^*, \psi, \mu_n, \mathcal{V}_n)$ and A_n instead of $A_n(\psi^*, \psi, \phi_*, \phi, \mu_n, \mathcal{V}_n)$. Substituting (7) into the definition [5, (1.7)] of A_n gives

$$A_n = \langle \psi^*, \mathfrak{Q}_n(\psi - Q_n \phi) \rangle_0 - \langle \phi_*, \mathcal{V}'_n(\phi, \phi_*, \phi) \rangle_n + \mathcal{V}_n(\phi_*, \phi)$$
$$= \langle \psi^*, \mathfrak{Q}_n(\psi - Q_n \phi) \rangle_0 - \mathcal{V}_n(\phi_*, \phi) \tag{8}$$

since $\langle \phi_*, \mathcal{V}'_n(\phi, \phi_*, \phi) \rangle_n = 2\mathcal{V}_n(\phi_*, \phi)$.

By [7, Proposition 2.1.a and Remark 2.3],

$$\phi = \Phi + \phi_n^{(\geq 3)}(\psi^*, \psi) = \Phi - S_n(\mu_n) \mathcal{V}'_n(\Phi, \Phi_*, \Phi) + \phi_n^{(\geq 5)}(\psi^*, \psi) \tag{9}$$

with

$$\Phi_* = \Phi_*(\mu_n) = S_n(\mu_n)^* Q_n^* \mathfrak{Q}_n \psi^* \qquad \Phi = \Phi(\mu_n) = S_n(\mu_n) Q_n^* \mathfrak{Q}_n \psi$$

being the parts of $\phi_{(*)}$ that are of degree precisely one in $\psi^{(*)}$ and $\phi_n^{(\geq d)}(\psi^*, \psi)$ being the part that is of degree at least d in $\psi^{(*)}$. So

$$\psi - Q_n \phi = B_n^\Delta \psi + Q_n S_n(\mu_n) \mathcal{V}'_n(\Phi, \Phi_*, \Phi) - Q_n \phi_n^{(\geq 5)}(\psi^*, \psi) \tag{10}$$

where $B_n^\Delta = \mathbb{1} - Q_n S_n(\mu_n) Q_n^* \mathfrak{Q}_n$. For general ($O(1)$ small enough) μ

$$S_n(\mu) = (D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu)^{-1} = S_n(\mathbb{1} - \mu S_n)^{-1} = S_n + \mu S_n S_n(\mu)$$

so that, by [5, Proposition 1.15],

$$\langle \psi^*, \mathfrak{Q}_n B_n^\Delta \psi \rangle_0 = \langle \psi^*, \mathfrak{Q}_n(\mathbb{1} - Q_n S_n Q_n^* \mathfrak{Q}_n) \psi \rangle_0 - \mu_n \langle \psi^*, \mathfrak{Q}_n Q_n S_n(\mu_n) S_n Q_n^* \mathfrak{Q}_n) \psi \rangle_0$$
$$= \langle \psi^*, \Delta^{(n)} \psi \rangle_0 - \mu_n \langle \Phi_*(\mu_n), \Phi(0) \rangle_n \tag{11}$$

with $\Delta^{(n)} = \Delta^{(n)}(\mu = 0)$. Inserting (10) and (11) into the representation (8) of A_n gives

$$A_n = \langle \psi^*, \Delta^{(n)} \psi \rangle_0 - \mu_n \langle \Phi_*(\mu_n), \Phi(0) \rangle_n + \langle S_n(\mu_n)^* Q_n^* \mathfrak{Q}_n \psi^*, \mathcal{V}'_n(\Phi, \Phi_*, \Phi) \rangle_n$$
$$- \mathcal{V}_n(\phi_*, \phi) - \langle Q_n^* \mathfrak{Q}_n \psi^*, \phi_n^{(\geq 5)}(\psi^*, \psi) \rangle_n$$
$$= \langle \psi^*, \Delta^{(n)} \psi \rangle_0 - \mu_n \langle \Phi_*(\mu_n), \Phi(0) \rangle_n + 2\mathcal{V}_n(\Phi_*, \Phi) - \mathcal{V}_n(\Phi_* + \phi_{*n}^{(\geq 3)}, \Phi + \phi_n^{(\geq 3)})$$
$$- \langle Q_n^* \mathfrak{Q}_n \psi^*, \phi_n^{(\geq 5)}(\psi^*, \psi) \rangle_n$$
$$= \langle \psi^*, \Delta^{(n)} \psi \rangle_0 - \mu_n \langle \Phi_*(\mu_n), \Phi(0) \rangle_n + \mathcal{V}_n(\Phi_*, \Phi)$$
$$- \left[\mathcal{V}_n(\Phi_* + \phi_{*n}^{(\geq 3)}, \Phi + \phi_n^{(\geq 3)}) - \mathcal{V}_n(\Phi_*, \Phi) \right] - \langle Q_n^* \mathfrak{Q}_n \psi^*, \phi_n^{(\geq 5)}(\psi^*, \psi) \rangle_n$$

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In our bounds, we fix $\psi \in \mathcal{H}_0^{(n)}$ and denote

$$\mathfrak{k} = \|\psi\|_{L^\infty} \quad \mathfrak{k}_2 = \|\psi\|_{L^2} \quad \mathfrak{k}_4 = \|\psi\|_{L^4} \quad \mathfrak{k}' = \max_{0 \leq \nu \leq 3} \|\partial_\nu \psi\|_{L^\infty} \quad \mathfrak{k}'_2 = \sum_{\nu=0}^3 \|\partial_\nu \psi\|_{L^2}$$

Since $\psi \in \mathrm{An}(n)$, we have $\mathfrak{k} < \kappa(n)$ and $\mathfrak{k}' < \kappa'(n)$. Also, $\mathfrak{k}' \leq 2\mathfrak{k}$ since ∂_ν is a difference operator on a unit lattice. By [7, Remark 2.3] and [4, Lemma 2.5.b], we have $\left\| \phi_n^{(\geq 5)} \right\|_{L^{4/3}} = O\big(\mathfrak{v}_n^2 \mathfrak{k}^2 \mathfrak{k}_4^3\big)$ and consequently, by [4, Lemma A.1],

$$\big\langle Q_n^* \mathfrak{Q}_n \psi^*, \, \phi_n^{(\geq 5)}(\psi^*, \psi) \big\rangle_n = O\big(\mathfrak{v}_n^2 \mathfrak{k}^2 \mathfrak{k}_4^4\big)$$

Also, by [7, Proposition 2.1.a], [4, Lemma 3.7, Remark 3.5.a and Lemma 2.5.a],

$$\left| \mathcal{V}_n\big(\Phi_* + \phi_{*n}^{(\geq 3)}, \Phi + \phi_n^{(\geq 3)}\big) - \mathcal{V}_n\big(\Phi_*, \Phi\big) \right| = O\big(\mathfrak{v}_n^2 \mathfrak{k}^2 \mathfrak{k}_4^4\big)$$

Thus

$$A_n = \big\langle \psi^*, \Delta^{(n)} \psi \big\rangle_0 - \mu_n \big\langle \Phi_*(\mu_n), \, \Phi(0) \big\rangle_n + \mathcal{V}_n\big(\Phi_*, \Phi\big) + O\big(\mathfrak{v}_n^2 \mathfrak{k}^2 \mathfrak{k}_4^4\big) \qquad (12)$$

By [7, Lemma 2.4]

$$\Phi_{(*)}(\mu)(u) = \big(S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n \psi^{(*)}\big)(u) = \frac{a_n}{a_n - \mu} \Psi^{(*)}(u) + F_{\mathrm{lb}(*)}(\mu)(\{\partial_\nu \psi_{(*)}\})(u)$$

with $\Psi^{(*)}(u) = \psi^{(*)}\big(X(u)\big)$ and with the maps $F_{\mathrm{lb}(*)}(\mu)$ being of degree precisely one. Hence, recalling that $\mathfrak{k}' \leq 2\mathfrak{k}$,

$$\left| \mu_n \big\langle \Phi_*(\mu_n), \, \Phi(0) \big\rangle_n - \frac{a_n \mu_n}{a_n - \mu_n} \big\langle \psi^*, \psi \big\rangle_0 \right| = |\mu_n| \, O\big(\mathfrak{k}_2 \mathfrak{k}'_2 + \mathfrak{k}'^2_2\big) = \sqrt{|\mu_n|} \, O\big(|\mu_n| \mathfrak{k}_2^2 + \mathfrak{k}'^2_2\big)$$

$$\left| \mathcal{V}_n\big(\Phi_*, \Phi\big) - \Big(\frac{a_n}{a_n - \mu_n}\Big)^4 \mathcal{V}_n\big(\Psi^*, \Psi\big) \right| = \mathfrak{v}_n \, O\big(\mathfrak{k}'_2 \mathfrak{k} \mathfrak{k}_4^2 + \mathfrak{k}'^2_2 \mathfrak{k}^2 + \mathfrak{k}' \mathfrak{k} \mathfrak{k}'^2_2 + \mathfrak{k}'^2 \mathfrak{k}'^2_2\big)$$

$$= \sqrt{\mathfrak{v}_n} \, \mathfrak{k} \, O\big((1 + \sqrt{\mathfrak{v}_n} \, \mathfrak{k}) \mathfrak{k}'^2_2 + \mathfrak{v}_n \mathfrak{k}_4^4\big) \qquad (13)$$

Using [5, Theorem 1.17] and localizing as in [6, Corollary B.2],

$$\mathcal{V}_n\big(\Psi^*, \Psi\big) = \tfrac{1}{2} \mathrm{v}_n \mathfrak{k}_4^4 + O\big(\sqrt{\mathfrak{v}_n \kappa(n)^2}\big) \mathfrak{k}'^2_2 + O\big(\mathfrak{v}_0^{\frac{2}{3} - 7\epsilon} + \sqrt{\mathfrak{v}_n \kappa(n)^2}\big) \mathfrak{v}_n \mathfrak{k}_4^4 \qquad (14)$$

Inserting (13) and (14) into (12) we get

$$A_n = \big\langle \psi^*, \Delta^{(n)} \psi \big\rangle_0 - \frac{a_n \mu_n}{a_n - \mu_n} \big\langle \psi^*, \psi \big\rangle_0 + \tfrac{1}{2} \Big(\frac{a_n}{a_n - \mu_n}\Big)^4 \mathrm{v}_n \mathfrak{k}_4^4$$

$$+ O\big(\sqrt{\mu_n} + \sqrt{\mathfrak{v}_n \kappa(n)^2}\big) \mathfrak{k}'^2_2 + O\big(\sqrt{\mu_n}\big) \mu_n \mathfrak{k}_2^2 + O\big(\sqrt{\mathfrak{v}_0} + \sqrt{\mathfrak{v}_n \kappa(n)^2}\big) \mathfrak{v}_n \mathfrak{k}_4^4 \qquad (15)$$

By [3, Lemma 4.2.b,d], the Fourier transform of $\Delta^{(n)}$ is

$$\widehat{\Delta^{(n)}}_k = -ik_0 + \left(\tfrac{1}{a_n} + \tfrac{\varepsilon_n^2}{2}\right)k_0^2 + \tfrac{1}{2}\sum_{\nu,\nu'=1}^{3} H_{\nu,\nu'}\mathbf{k}_\nu\mathbf{k}_{\nu'} + O\big(|k|^3\big)$$

and obeys $\operatorname{Re}\widehat{\Delta^{(n)}(0)}_k \geq \rho(c)$ when $|k| \geq c$. In particular, there are constants $\gamma, \tilde{\gamma}$, (independent of n and L) such that

$$8\gamma(k_0^2 + \mathbf{k}^2) \leq \operatorname{Re}\widehat{\Delta^{(n)}}_k \leq \tfrac{1}{2}\tilde{\gamma}(k_0^2 + \mathbf{k}^2) \implies 2\gamma\mathfrak{k}_2'^2 \leq \operatorname{Re}\big\langle \psi^*,\, \Delta^{(n)}\psi\big\rangle_0 \leq \tfrac{1}{2}\tilde{\gamma}\mathfrak{k}_2'^2$$

It now suffices to combine (15)–(14) and use that, by [5, (C.1.a,b) and Corollary C.4.a],

$$\mathfrak{v}_n\kappa(n)^2 < \mathfrak{v}_0^{\frac{3}{2}\epsilon} \qquad |\mu_n| < 4\mathfrak{v}_0^{5\epsilon} \qquad \tfrac{1}{2} \leq a_n \leq 2$$

$\square$

We choose the "small field" cutoff function $\chi_n(\psi)$ of (5) to be the characteristic function of

$$\operatorname{In}(n,c) = \big\{ \psi \in \mathcal{H}_0^{(n)} \mid |\psi(x)| < c\kappa(n),\ |\partial_\nu\psi(x)| < c\kappa'(n) \tag{16}$$
$$\text{for all } x \in \mathcal{X}_0^{(n)},\ 0 \leq \nu \leq 3 \big\}$$

with an appropriate value of c.

Using the procedure starting at [5, (1.6)] and leading up to [5, Definition 1.6], and then applying [5, Theorem 1.17], we would expect the answer to the integral

$$\int_{\operatorname{In}(n,c)} \left[\prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\psi(x)^*\wedge d\psi(x)}{2\pi i} \right] e^{-A_n(\psi^*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n)+\mathcal{R}_n+\mathcal{E}_n}\bigg|_{\phi_{(*)}=\phi_{(*)n}(\psi^*,\psi,\mu_n,\mathcal{V}_n)}$$

to have the main contribution a normalization constant times

$$\int_{\operatorname{In}(n+1,c)} \left[\prod_{x\in\mathcal{X}_0^{(n+1)}} \frac{d\psi(x)^*\wedge d\psi(x)}{2\pi i} \right] e^{-A_{n+1}(\psi^*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_{n+1})}\bigg|_{\phi_{(*)}=\phi_{(*)n+1}(\psi^*,\psi,\mu_{n+1},\mathcal{V}_{n+1})}$$

The logarithm of the normalization constant is bounded in magnitude by a constant, which depends only on L and Γ_{op}, times $|\mathcal{X}_0^{(n)}|$. For constant ψ close to the bottom of the potential well, the integrand has magnitude greater than one, by the upper bound of Proposition 1. Observe that if $\psi \in \operatorname{An}(n) \setminus \operatorname{In}(n,c)$ then there is some $x \in \mathcal{X}_0^{(n)}$ and possibly some $0 \leq \nu \leq 3$ such that either $|\psi(x)| \geq c\kappa(n)$ or $|\partial_\nu\psi(x)| \geq c\kappa'(n)$. So the lower bound of Proposition 1, suggests the following "corollary". The significance of the quotation marks is that this is a "moral" rather than a "mathematical" statement.

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"Corollary" 2. Let $c > 0$ and let $\mathfrak{v}_0$ be small enough, depending on c. Then, for any $S \subset \mathrm{An}(n) \setminus \mathrm{In}(n, c)$

$$\int_S \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \Big| e^{-A_n(\psi^*, \psi, \phi_*, \phi, \mu_n, \mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n} \Big|$$

is nonperturbatively small.

We shall later choose a small, possibly L–dependent constant, $c_0 > 0$. Then our cutoff functions $\chi_n(\psi)$ are chosen to be $\mathrm{In}(n) = \mathrm{In}(n, c_0)$. With these cutoff functions, we now sketch the argument that $\frac{1}{\mathcal{Z}_{n+1}} J_{n+1} - \frac{1}{\mathcal{Z}_n} J_n$ is nonperturbatively small in the case that $n \geq 1$. It goes in three steps. First we just state what the steps are. We'll discuss them in more detail shortly.

Step 1: Substituting

$$1 = \frac{1}{N_{\mathbb{T}}^{(n)}} \int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] e^{-aL^{-2} \|\theta - Q\psi\|_{-1}^2}$$

from [5, Remark 1.2.a], into (5) and (6), we have

$$J_n = \frac{1}{N_{\mathbb{T}}^{(n)} \mathcal{Z}_n} \int \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \int_{\mathrm{In}(n)} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \\ e^{-aL^{-2} \|\theta - Q\psi\|_{-1}^2 - A_n(\psi^*, \psi, \phi_*, \phi, \mu_n, \mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n} \tag{17}$$
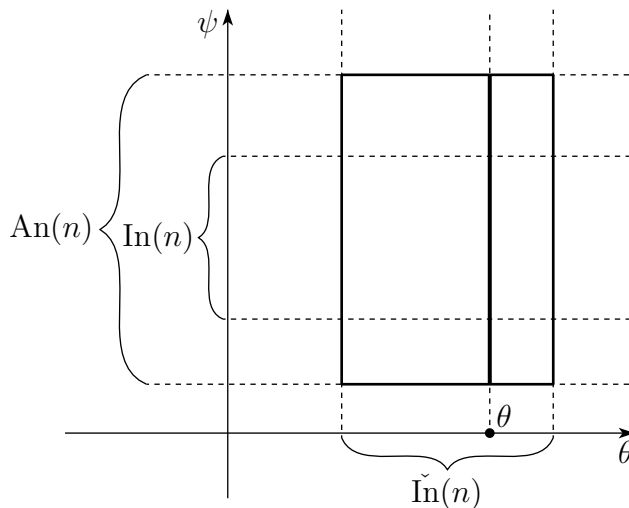
with $\phi_{(*)} = \phi_{(*)n}(\psi^*, \psi, \mu_n, \mathcal{V}_n)$. The domain of integration for the double integral in (17) is $(\theta, \psi) \in \mathcal{H}_{-1}^{(n+1)} \times \mathrm{In}(n)$. The first step consists in restricting the domain to $(\theta, \psi) \in \check{\mathrm{In}}(n) \times \mathrm{In}(n)$ where

$$\check{\mathrm{In}}(n) = \big\{ \mathbb{S}^{-1} \psi \in \mathcal{H}_{-1}^{(n+1)} \mid \psi \in \mathrm{In}(n+1) \big\}$$
$$= \big\{ \theta \in \mathcal{H}_{-1}^{(n+1)} \mid |\theta(y)| < c_0 \frac{\kappa(n+1)}{L^{\frac{3}{2}}}, \ |\partial_\nu \theta(y)| < c_0 \frac{\kappa'(n+1)}{L^{\frac{3}{2}} L_\nu} \ \forall y \in \mathcal{X}_{-1}^{(n+1)}, \ 0 \leq \nu \leq 3 \big\}$$

and showing that the difference between $N_{\mathbb{T}}^{(n)} \mathcal{Z}_n J_n$ and

$$\int_{\check{\mathrm{In}}(n)} \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \int_{\mathrm{In}(n)} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \\ e^{-aL^{-2} \|\theta - Q\psi\|_{-1}^2 - A_n(\psi^*, \psi, \phi_*, \phi, \mu_n, \mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n} \tag{18}$$

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is nonperturbatively small.

Step 2: This step consists in enlarging the integration domain $\check{\mathrm{In}}(n) \times \mathrm{In}(n)$ of (18) to $\check{\mathrm{In}}(n) \times \mathrm{An}(n)$ and showing that the difference between (18) and

$$\int_{\check{\mathrm{In}}(n)} \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \int_{\mathrm{An}(n)} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \tag{19}$$
$$e^{-aL^{-2}\|\theta - Q\psi\|_{-1}^2 - A_n(\psi^*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n}$$

is nonperturbatively small.

Step 3: The third step consists in showing that, for each fixed $\theta \in \check{\mathrm{In}}(n)$ the inner integral

$$\int_{\mathrm{An}(n)} \Big[\prod_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \, e^{-aL^{-2}\|\theta - Q\psi\|_{-1}^2 - A_n(\psi^*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n) + \mathcal{R}_n + \mathcal{E}_n} \tag{20}$$

of (19) is nonperturbatively close to $\det C^{(n)} \, e^{\check{\mathcal{C}}_n(\theta_*,\theta)} \, \check{\mathcal{F}}_n(\theta_*,\theta)$ with the $\check{\mathcal{C}}_n(\theta_*,\theta)$ and $\check{\mathcal{F}}_n(\theta_*,\theta)$ of [5, Proposition 4.2.a]. (They are defined at the beginning of of [5, §4].)

Putting these three steps together, we see that $\frac{1}{\check{\mathcal{Z}}_n} J_n$ is nonperturbatively close

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to

$$\frac{\det C^{(n)}}{N_{\mathbb{T}}^{(n)} \mathcal{Z}_n \check{\mathcal{Z}}_n} \int_{\check{\mathrm{In}}(n)} \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] e^{\check{\mathcal{C}}_n(\theta_*, \theta)} \; \check{\mathcal{F}}_n(\theta_*, \theta)$$

$$= \frac{1}{\check{\mathcal{Z}}_n} \int_{\check{\mathrm{In}}(n)} \Big[\prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta(y)^* \wedge d\theta(y)}{2\pi i} \Big] \Big(\mathbb{T}_n^{(SF)} \circ (\mathbb{ST}_{n-1}^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big) (\theta^*, \theta)$$

$$= \frac{1}{\check{\mathcal{Z}}_n L^{3|\mathcal{X}_0^{(n+1)}|}} \int_{\mathrm{In}(n+1)} \Big[\prod_{x \in \mathcal{X}_0^{(n+1)}} \frac{d\psi(x)^* \wedge d\psi(x)}{2\pi i} \Big] \Big((\mathbb{ST}_n^{(SF)}) \circ \cdots \circ (\mathbb{ST}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big) (\psi^*, \psi)$$

$$= \frac{1}{\check{\mathcal{Z}}_{n+1}} J_{n+1}$$

by (6), [5, Definition 1.6, Proposition 4.2.a and (1.6)].

We now elaborate on these three steps.

Step 1: Fix any $\theta \notin \check{\mathrm{In}}(n)$ and decompose the domain of integration for the ψ integral

$$\mathrm{In}(n) = \mathrm{In}_s(n, \theta) \cup \mathrm{In}_b(n, \theta)$$

with

$$\mathrm{In}_s(n, \theta) = \big\{ \; \psi \in \mathrm{In}(n) \; \big| \; L^{-1} \|\theta - Q\psi\|_{-1} < \tfrac{1}{\mathrm{v}_n^\epsilon} \; \big\}$$
$$\mathrm{In}_b(n, \theta) = \big\{ \; \psi \in \mathrm{In}(n) \; \big| \; L^{-1} \|\theta - Q\psi\|_{-1} \geq \tfrac{1}{\mathrm{v}_n^\epsilon} \; \big\}$$

We would expect that the integral over $\psi \in \mathrm{In}_b(n, \theta)$ gives a nonperturbatively small contribution because of the $-aL^{-2}\|\theta - Q\psi\|_{-1}^2$ in the exponent. Furthermore we claim that $\mathrm{In}_s(n, \theta) \subset \mathrm{An}(n) \setminus \mathrm{In}(n, c)$ with

$$c = \min_{0 \leq \nu \leq 3} \Big\{ \frac{c_0}{2L^{\frac{3}{2}-\eta} \|Q\|_{m=0}} \, , \, \frac{c_0}{2L^{\frac{3}{2}-\eta'} L_\nu \|Q_{+,\nu}^{(-)}\|_{m=0}} \Big\}$$

(The operator $Q_{n,\nu}^{(-)}$ was defined in [3, (2.11)], $L_0 = L^2$ and $L_\nu = L$ for $\nu = 1, 2, 3$.) This will "imply", by "Corollary 2", that the integral over $\mathrm{In}_s(n, \theta)$ is also nonperturbatively small. So let $\psi \in \mathrm{In}_s(n, \theta)$.

- If there is a $y \in \mathcal{X}_{-1}^{(n+1)}$ with $|\theta(y)| \geq c_0 \frac{\kappa(n+1)}{L^{\frac{3}{2}}}$, then since

$$|\theta(y)| \leq |\theta(y) - (Q\psi)(y)| + |(Q\psi)(y)|$$
$$\leq \tfrac{1}{L^{5/2}} \|\theta - Q\psi\|_{-1} + \|Q\|_{m=0} \|\psi\|_{\ell^\infty}$$
$$\leq \tfrac{1}{L^{3/2} \mathrm{v}_n^\epsilon} + \|Q\|_{m=0} \|\psi\|_{\ell^\infty}$$

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we have

$$\|\psi\|_{\ell^\infty} \geq \tfrac{1}{L^{3/2}\|Q\|_{m=0}}\big(c_0\kappa(n+1) - \tfrac{1}{\mathsf{v}_n^\epsilon}\big) \geq \tfrac{c_0}{2L^{3/2}\|Q\|_{m=0}}\kappa(n+1) \geq c\kappa(n)$$

and $\psi \notin \mathrm{In}(n,c)$.

- If there is a $y \in \mathcal{X}_{-1}^{(n+1)}$ and a $0 \leq \nu \leq 3$ with $|\partial_\nu\theta(y)| \geq c_0\frac{\kappa'(n+1)}{L^{3/2}L_\nu}$, then since

$$
\begin{aligned}
|\partial_\nu\theta(y)| &\leq |\partial_\nu\theta(y) - (\partial_\nu Q\psi)(y)| + |(\partial_\nu Q\psi)(y)| \\
&= \big|\partial_\nu\big[\theta - Q\psi\big](y)\big| + |(Q_{+,\nu}^{(-)}\partial_\nu\psi)(y)| \\
&\leq \tfrac{2}{L^{5/2}L_\nu}\|\theta - Q\psi\|_{-1} + \|Q_{+,\nu}^{(-)}\|_{m=0}\|\partial_\nu\psi\|_{\ell^\infty} \\
&\leq \tfrac{2}{L^{3/2}L_\nu \mathsf{v}_n^\epsilon} + \|Q_{+,\nu}^{(-)}\|_{m=0}\|\partial_\nu\psi\|_{\ell^\infty}
\end{aligned}
$$

we have

$$\|\partial_\nu\psi\|_{\ell^\infty} \geq \tfrac{1}{L^{3/2}L_\nu\|Q_{+,\nu}^{(-)}\|_{m=0}}\big(c_0\kappa'(n+1) - \tfrac{2}{\mathsf{v}_n^\epsilon}\big) \geq \tfrac{c_0}{2L^{3/2}L_\nu\|Q_{+,\nu}^{(-)}\|_{m=0}}\kappa'(n+1) \geq c\kappa'(n)$$

and, again, $\psi \notin \mathrm{In}(n,c)$.

Step 2   "follows" directly from "Corollary 2" with $S = \mathrm{An}(n) \setminus \mathrm{In}(n)$.

Step 3:   Fix any $\theta \in \check{\mathrm{In}}(n)$. Set

$$\rho_n(\theta) = \psi_{*n}(\theta^*,\theta)^* - \psi_n(\theta^*,\theta)$$

By Remark [7, 5.4],

$$\|\rho_n(\theta)\|_\infty \leq \sqrt{c_0}\,\kappa'(n)$$

if c_0 is small enough. Clearly (θ^*,θ) is in the domain of $\psi_{(*)n}$. Recall that

$$\psi_{(*)n}(\theta_*,\theta) = \tfrac{1}{L^{3/2}}\mathbb{L}_*\big[\hat{\psi}_{(*)n}(\mathbb{S}\theta_*,\mathbb{S}\theta)\big]$$

Since $\|\theta\|_\infty < \frac{c_0}{L^{3/2}}\kappa(n+1)$ and $\|\partial_\nu\theta\|_\infty < \frac{c_0}{L^{3/2}L_\nu}\kappa'(n+1)$ for all $0 \leq \nu \leq 3$, we have, by [7, Proposition 5.1], with $\mathfrak{k} = c_0\bar{\kappa}$, $\mathfrak{k}' = c_0\bar{\kappa}'$,

$$
\begin{aligned}
\|\psi_{(*)n}\|_\infty &= \tfrac{1}{L^{3/2}}\|\hat{\psi}_{(*)n}\|_\infty &\leq \tfrac{c_0}{L^{3/2}}K_{\mathrm{op}}\kappa(n+1) &= \tfrac{c_0}{L^{3/2-\eta}}K_{\mathrm{op}}\kappa(n) \\
\|\partial_\nu\psi_{(*)n}\|_\infty &= \tfrac{1}{L^{3/2}L_\nu}\|\partial_\nu\hat{\psi}_{(*)n}\|_\infty &\leq \tfrac{c_0}{L^{3/2}L_\nu}K_{\mathrm{op}}\kappa'(n+1) &= \tfrac{c_0}{L^{3/2-\eta'}L_\nu}K_{\mathrm{op}}\kappa'(n)
\end{aligned}
$$

with $\eta < \frac{7}{8}$ and $\eta' < \frac{3}{4}$. Hence, if we pick L large enough or c_0 small enough, depending only on K_{op},

$$\psi_{(*)n}(\theta^*,\theta) + \delta\psi_{(*)} \in \mathrm{An}(n)$$

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for all $\delta\psi$ obeying $\|\delta\psi_{(*)}\| < \frac{1}{2}\kappa(n)$, $\|\partial_\nu\delta\psi_{(*)}\| < \frac{1}{2}\kappa'(n)$. We may rewrite the integral (20) as

$$\int_{\mathrm{An}(n)} \left[\prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\psi(x)^*\wedge d\psi(x)}{2\pi i} \right] e^{-aL^{-2}\|\theta-Q\psi\|_{-1}^2 - A_n(\psi^*,\psi,\phi_*,\phi,\mu_n,\mathcal{V}_n)+\mathcal{R}_n+\mathcal{E}_n} = \int_{I_n(\theta^*,\theta)} \tilde{\omega}_n$$

where $\tilde{\omega}_n$ is the holomorphic differential form obtained from the integrand on the left hand side through the substitution

$$\psi^* = \psi_{*n}(\theta^*,\theta) + \delta\psi_* \qquad \psi = \psi_n(\theta^*,\theta) + \delta\psi \tag{21}$$

and the domain

$$I_n(\theta^*,\theta) = \left\{ (\delta\psi_*,\delta\psi) \in \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)} \,\middle|\, \begin{array}{l} \delta\psi = \delta\psi_*^* + \rho_n(\theta), \\[2mm] \psi_n(\theta^*,\theta) + \delta\psi \in \mathrm{An}(n) \end{array} \right\}$$

As in the proof of [5, Proposition 4.2],

$$\tilde{\omega}_n = e^{\check{C}_n(\theta_*,\theta)} e^{-<\delta\psi_*, C^{(n)-1}\delta\psi>_0} e^{-\delta\check{A}_n(\theta_*,\theta,\delta\psi_*,\delta\psi)+\delta\check{\mathcal{R}}_n+\delta\check{\mathcal{E}}_n} \prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\delta\psi_*(x)\wedge d\delta\psi(x)}{2\pi i}$$

We next make the change of variables $\delta\psi_* = D^{(n)*}\zeta_*$, $\delta\psi = D^{(n)}\zeta$, with $D^{(n)}$ being an operator square root of $C^{(n)}$ and $D^{(n)*}$ being the transpose (not adjoint) of $D^{(n)}$. Then

$$\int_{I_n(\theta^*,\theta)} \tilde{\omega}_n = \int_{I'_n(\theta)} \omega_n$$

with

$$I'_n(\theta) = \left\{ (\zeta_*,\zeta) \in \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)} \,\middle|\, \begin{array}{l} D^{(n)}\zeta = \overline{D^{(n)*}}\zeta_*^* + \rho_n(\theta), \\[2mm] \psi_n(\theta^*,\theta) + D^{(n)}\zeta \in \mathrm{An}(n) \end{array} \right\}$$

and

$$\omega_n = \det C^{(n)} \ e^{\check{C}_n(\theta_*,\theta)} e^{-<\zeta_*,\zeta>_0} e^{-\delta\check{A}_n+\delta\check{\mathcal{R}}_n+\delta\check{\mathcal{E}}_n} \prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\zeta_*(x)\wedge d\zeta(x)}{2\pi i}$$

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When $D^{(n)}\zeta = \overline{D^{(n)*}}\zeta_*^* + \rho_n$ we have $\zeta_* = D^{(n)*-1}\overline{D^{(n)}}\zeta^* - D^{(n)*-1}\rho_n^*$ so that

$$\langle \zeta_*, \zeta \rangle = \langle D^{(n)*-1}\overline{D^{(n)}}\zeta^*, \zeta \rangle - \langle D^{(n)*-1}\rho_n^*, \zeta \rangle \tag{22}$$

To convert the integral $\int_{I'_n(\theta)} \omega_n$ into an integral of ω_n over the "real" disk

$$S_{\mathrm{Bot}} = \left\{ (\zeta_*, \zeta) \,\middle|\, \zeta_* = \zeta^*, \ \|\zeta\| < r_n \right\}$$

we now choose a "Stokes' Cylinder" $\mathcal{Y}$ that contains S_{Bot} in its boundary. For each $0 \leq t \leq 1$, set $C(t) = \left[tC^{(n)^{-1}} + (1-t)\mathbb{1} \right]^{-1}$. Note that $C(t)^{-1} = \frac{at}{L^2}Q^*Q + t\Delta^{(n)} + (1-t)\mathbb{1}$ has strictly positive real part. (See [3, (4.3), Lemma 4.2.b,d and Lemma 2.3.c].) Denote by $D(t)$ the square root of $C(t)$ given by the contour integral as in [3, Corollary 4.5]. Set

$$I'_t = \left\{ (\zeta_*, \zeta) \in \mathcal{H}_0^{(n)} \times \mathcal{H}_0^{(n)} \,\middle|\, D(t)\zeta = D(t)^\dagger\zeta_*^* + t\rho_n(\theta) \right\}$$

$$\mathcal{Y} = \left\{ (\zeta_*, \zeta) \in \bigcup_{0 \leq t \leq 1} I'_t \,\middle|\, \|\zeta\| \leq \sqrt[4]{c_0}\kappa'(n) \right\}$$

If $(\zeta_*, \zeta) \in \mathcal{Y}$, then both $\psi_n(\theta^*, \theta) + D(t)\zeta$ and $\psi_{*n}(\theta^*, \theta) + D(t)^*\zeta_*$ are in $\mathrm{An}(n)$, by [7, Remark 5.4], provided we choose c_0 small enough. This is illustrated, for $t = 1$, in the figure
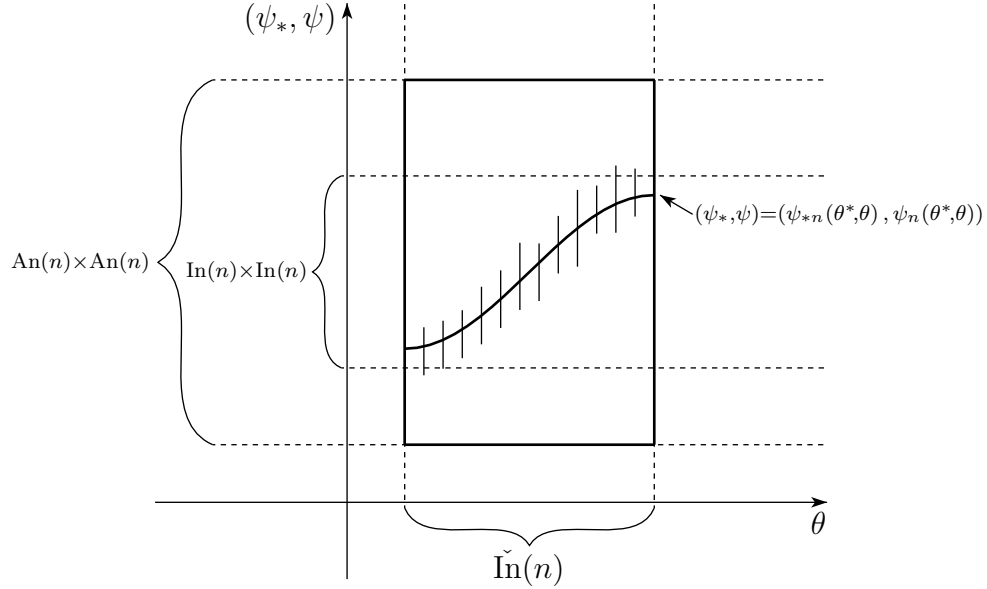
Each small vertical line in this figure is

$$\left\{ \left(\psi_{*n}(\theta^*, \theta) + D^{(n)}\zeta, \, \psi_{*n}(\theta^*, \theta) + D^{(n)*}\zeta \right) \,\middle|\, D^{(n)}\zeta = \overline{D^{(n)*}}\zeta_*^* + \rho_n(\theta) \ \|\zeta\|_\infty \leq \sqrt[4]{c_0}\,\kappa'(n) \right\}$$

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We now show that if $(\zeta_*, \zeta) \in I_t'$, for some $0 \leq t \leq 1$, and $\|\zeta\| \geq \sqrt[4]{c_0}\, \kappa'(n)$, then

$$\mathrm{Re}\,\langle \zeta_*, \zeta \rangle \geq \mathrm{const}\|\zeta\|^2 \geq \mathrm{const}\sqrt{c_0}\,\kappa'(n)^2 \gg r_n^2 \tag{23}$$

Since $(\zeta_*, \zeta) \in I_t'$

$$\langle \zeta_*, \zeta \rangle = \big\langle D(t)^{*-1}\overline{D(t)}\zeta^*, \zeta \big\rangle - t\big\langle D(t)^{*-1}\rho_n^*, \zeta \big\rangle$$
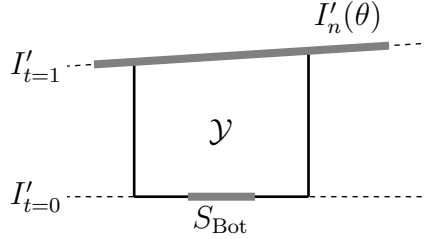
The real part of $\big\langle D(t)^{*-1}\overline{D(t)}\zeta^*, \zeta \big\rangle$ is

$$\tfrac{1}{2}\Big\{ \big\langle D(t)^{*-1}\overline{D(t)}\zeta^*, \zeta \big\rangle + \big\langle \overline{D(t)^{*-1}}D(t)\zeta, \zeta^* \big\rangle \Big\}$$

$$= \tfrac{1}{2}\big\langle \overline{D(t)}\zeta^*, D(t)^{\dagger -1}\big[D(t)^{\dagger}D(t)^{-1} + D(t)^{\dagger -1}D(t) \big] D(t)^{-1}D(t)\zeta \big\rangle$$

$$= \tfrac{1}{2}\big\langle \eta^*, \big[D(t)^{-2} + (D(t)^{-2})^{\dagger} \big]\eta \big\rangle \qquad \text{with } \eta = D(t)\zeta$$

$$= \big\langle \eta^*, \big[\tfrac{at}{L^2}Q^*Q + t\mathrm{Re}\,\Delta^{(n)} + (1-t)\mathbb{1} \big]\,\eta \big\rangle$$

$$\geq \mathrm{const}\|\eta\|^2$$

$$\geq \mathrm{const}\|\zeta\|^2 \tag{24}$$

since $D(t)^{-2} = C(t)^{-1} = \frac{at}{L^2}Q^*Q + t\Delta^{(n)} + (1-t)\mathbb{1}$ and since $D(t)^{-1}$ is a bounded operator.

Since $\int_{\partial \mathcal{Y}} \omega_n = 0$ by Stokes' theorem,

$$\int_{I_n'(\theta)} \omega_n = \int_{I_n'(\theta)\backslash \partial \mathcal{Y}} \omega_n + \int_{I_n'(\theta)\cap \partial \mathcal{Y}} \omega_n - \int_{\partial \mathcal{Y}} \omega_n$$

$$= \int_{I_n'(\theta)\backslash \partial \mathcal{Y}} \omega_n - \int_{\partial \mathcal{Y}\backslash (I_n'(\theta)\cap \partial \mathcal{Y})} \omega_n$$

$$= \int_{S_{\mathrm{Bot}}} \omega_n + \int_{I_n'(\theta)\backslash \partial \mathcal{Y}} \omega_n - \int_{\partial \mathcal{Y}\backslash [(I_n'(\theta)\cap \partial \mathcal{Y})\cup S_{\mathrm{Bot}}]} \omega_n$$

By (23) and the definition of S_{Bot}, the last two integrals are both nonperturbatively small. By the definition of $\check{\mathcal{F}}_n(\theta_*, \theta)$, the first integral

$$\int_{S_{\mathrm{Bot}}} \omega_n = \det C^{(n)}\ e^{\check{\mathcal{C}}_n(\theta_*, \theta)}\ \check{\mathcal{F}}_n(\theta_*, \theta)$$

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