# The Algebra of Block Spin Renormalization Group Transformations

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**Abstract**

Block spin renormalization group is the main tool used in our program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. In this paper, we discuss some of its purely algebraic aspects in an abstract setting. For example, we derive some "well known" identities like the composition rule and the relation between critical fields and background fields.

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One standard implementation of the renormalization group philosophy [9] uses block spin transformations. See [8, 1, 7, 2, 6]. Concretely, suppose we are to control a functional integral on a finite[1] lattice $\mathcal{X}_-$ of the form

$$\int \prod_{x\in\mathcal{X}_-} \tfrac{d\phi^*(x)d\phi(x)}{2\pi i}\, e^{A(\alpha_1,\cdots,\alpha_s;\phi^*,\phi)} \tag{1}$$

with an action $A(\alpha_1, \cdots, \alpha_s; \phi_*, \phi)$ that is a function of external complex valued fields $\alpha_1, \cdots, \alpha_s$, and the two[2] complex fields $\phi_*, \phi$ on $\mathcal{X}_-$. This scenario occurs in [4, 5], where we use block spin renormalization group maps to exhibit the formation of a potential well, signalling the onset of symmetry breaking in a many particle system of weakly interacting Bosons in three space dimensions. (For an overview, see [3].) For simplicity, we suppress the external fields in this paper.

Under the renormalization group approach to controlling integrals like (1) one successively "integrates out" lower and lower energy degrees of freedom. In the block spin formalism this is implemented by considering a decreasing sequence of sublattices of $\mathcal{X}_-$. The formalism produces, for each such sublattice, a representation of the integral (1) that is a functional integral whose integration variables are indexed by that sublattice. To pass from the representation associated with one sublattice $\mathcal{X} \subset \mathcal{X}_-$, with integration variables $\psi(x)$, $x \in \mathcal{X}$, to the representation associated to the next coarser sublattice $\mathcal{X}_+ \subset \mathcal{X}$, with integration variables $\theta(y)$, $y \in \mathcal{X}_+$, one

- paves $\mathcal{X}$ by rectangles centered at the points of $\mathcal{X}_+$ (this is illustrated in the figure below — the dots, both small and large, are the points of $\mathcal{X}$ and the large dots are the points of $\mathcal{X}_+$) and then,
- for each $y \in \mathcal{X}_+$ integrates out all values of $\psi$ whose "average value" over the rectangle centered at $y$ is equal to $\theta(y)$. The precise "average value" used is determined by an averaging profile. One uses this profile to define an averaging operator $Q$ from the space $\mathcal{H}$ of fields on $\mathcal{X}$ to the space $\mathcal{H}_+$ of fields on $\mathcal{X}_+$. One then implements the "integrating out" by first, inserting, into the integrand, 1 expressed as a constant times the Gaussian integral

$$\int \prod_{y\in\mathcal{X}_+} \tfrac{d\theta^*(y)d\theta(y)}{2\pi i} e^{-b\langle \theta^* - Q\,\psi_*\,,\,\theta - Q\,\psi\rangle} \tag{2}$$

with some constant $b > 0$, and then interchanging the order of the $\theta$ and $\psi$ integrals.

For example, in [3, 4, 5] the model is initially formulated as a functional integral with integration variables indexed by a lattice[3] $\big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$. After $n$

[1] Usually, the finite lattice is a "volume cutoff" infinite lattice and one wants to get bounds that are uniform in the size of the volume cutoff.

[2] In the actions, we treat $\phi$ and its complex conjugate $\phi^*$ as independent variables.

[3] The volume cutoff is determined by $L_{\rm tp}$ and $L_{\rm sp}$.

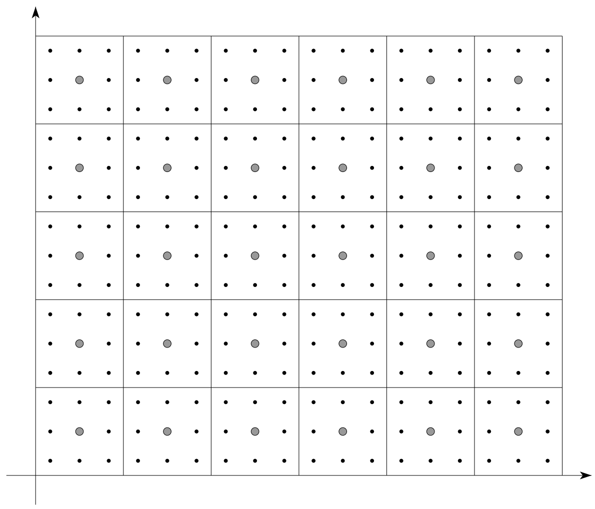

Figure 1: The lattices $\mathcal{X}$ and $\mathcal{X}_+$

renormalization group steps this lattice is scaled down to $\mathcal{X}_n = \big(\frac{1}{L^{2n}}\mathbb{Z}\big/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\frac{1}{L^n}\mathbb{Z}^3\big/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$. The decreasing family of sublattices is $\mathcal{X}_j^{(n-j)} = \big(\frac{1}{L^{2j}}\mathbb{Z}\big/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\frac{1}{L^j}\mathbb{Z}^3\big/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$, $j = n,\ n-1,\ \cdots$. The abstract lattices $\mathcal{X}_-$, $\mathcal{X}$, $\mathcal{X}_+$ in the above framework correspond to $\mathcal{X}_n$, $\mathcal{X}_0^{(n)}$ and $\mathcal{X}_{-1}^{(n+1)}$, respectively.

Return to the abstract setting. The integral is often controlled using stationary phase/steepest descent. The contributions to the integral that come from integration variables close to their critical values are called "small field" contributions. At the end of every step, the small field contribution to the original integral (1) is, up to a multiplicative normalization constant[4], of the form

$$\int \prod_{x\in\mathcal{X}} \frac{d\psi^*(x)d\psi(x)}{2\pi i}\, e^{-\langle \psi^*-Q_-\,\phi_*\,,\,\mathfrak{Q}(\psi-Q_-\,\phi)\rangle - \mathfrak{A}(\phi_*,\phi)+\mathcal{E}(\psi^*,\psi)}\Big|_{\substack{\phi_*=\phi_{*\rm bg}(\psi^*,\psi)\\ \phi=\phi_{\rm bg}(\psi^*,\psi)}} \tag{3}$$

where

- $Q_-$ is an averaging operator that maps the space $\mathcal{H}_-$ of fields on $\mathcal{X}_-$ to the space $\mathcal{H}$ of fields on $\mathcal{X}$. It is the composition of the averaging operations for all previous steps.
- the exponent $\langle \psi^*-Q_-\,\phi_*\,,\,\mathfrak{Q}(\psi-Q_-\,\phi)\rangle$ is a residue of the exponents in the Gaussian integrals (2) inserted in the previous steps. The operator[5] $\mathfrak{Q}$ is bounded and boundedly invertible on $L^2(\mathcal{X})$.

[4] See Remark 1 for the core of the recursion responsible for this form.
[5] See Remark 1 for the recursion relation that builds $\mathfrak{Q}$.

- the "background fields"

$$(\psi_*, \psi) \mapsto \phi_{*\mathrm{bg}}(\psi_*, \psi) \qquad (\psi_*, \psi) \mapsto \phi_{\mathrm{bg}}(\psi_*, \psi)$$

  map sufficiently small fields $\psi_*$, $\psi$ on $\mathcal{X}$ to fields on $\mathcal{X}_-$. They are the concatination[6] of "steepest descent" critical field maps for all previous steps.
- $\mathfrak{A}(\phi_*, \phi)$, the "dominant part" of the action, is an explicit function of $\phi_*, \phi \in \mathcal{H}_-$
- $\mathcal{E}(\psi_*, \psi)$ is the contribution to the action that consists of "perturbative corrections". It is an analytic function of $\psi_*, \psi \in \mathcal{H}$.

The next block spin renormalization group step then consists of

- rewriting (3), by inserting 1 expressed as a constant times (2), as

$$\int \prod_{y \in \mathcal{X}_+} \tfrac{d\theta^*(y) d\theta(y)}{2\pi i} \int \prod_{x \in \mathcal{X}} \tfrac{d\psi^*(x) d\psi(x)}{2\pi i}\, e^{-b\langle \theta^* - Q\,\psi_*\,,\, \theta - Q\,\psi\rangle} \; e^{-\langle \psi^* - Q_-\,\phi_*\,,\, \mathfrak{Q}(\psi - Q_-\,\phi)\rangle - \mathfrak{A}(\phi_*,\phi) + \mathcal{E}(\psi^*,\psi)}\Big|_{\substack{\phi_* = \phi_{*\mathrm{bg}}(\psi^*,\psi) \\ \phi = \phi_{\mathrm{bg}}(\psi^*,\psi)}} \tag{4}$$

  up to a multiplicative normalization constant,
- and performing a stationary phase argument, for the $\psi$ integral, around appropriate critical fields[7] $\psi_{*\mathrm{cr}}(\theta_*, \theta)$, $\psi_{\mathrm{cr}}(\theta_*, \theta)$ that map sufficiently small fields $\theta_*$, $\theta$ on $\mathcal{X}_+$ to fields on $\mathcal{X}$.

In this paper, we discuss some purely algebraic aspects of the block spin renormalization group in an abstract setting. We derive some "well known" identities like, in Proposition 4.c, the composition rule, and, in Proposition 4.a, the relation between critical fields and background fields, and, in Lemma 12, a formula for the dominant part of the action in the fluctuation integral. They are used in Proposition 3.4.b, Proposition 3.4.a, and Lemma 4.1.a of [4], respectively.

We use the following abstract environment:

- Let $H_-$, $H$, $H_+$ be finite dimensional, real vector spaces with positive definite symmetric bilinear forms $\langle\,\cdot\,,\,\cdot\,\rangle_-$, $\langle\,\cdot\,,\,\cdot\,\rangle$, $\langle\,\cdot\,,\,\cdot\,\rangle_+$. These bilinear forms extend to nondegenerate bilinear forms on their complexifications $\mathcal{H}_-$, $\mathcal{H}$, $\mathcal{H}_+$. Think of $H_-$, $H$ and $H_+$ as being the vector spaces of real valued functions on the finite lattices $\mathcal{X}_-$, $\mathcal{X}$ and $\mathcal{X}_+$, respectively, and think of the complexifications $\mathcal{H}_-$, $\mathcal{H}$, $\mathcal{H}_+$ as being $L^2(\mathcal{X}_-)$, $L^2(\mathcal{X})$ and $L^2(\mathcal{X}_+)$ respectively.
- Let $d\mu_{\mathcal{H}}(\phi^*, \phi)$ be the volume form on $\mathcal{H}$ determined by its bilinear form. If $\mathcal{H} = L^2(\mathcal{X})$, then $d\mu_{\mathcal{H}}(\phi^*, \phi) = \prod_{x \in \mathcal{X}} \frac{d\phi(x)^* \wedge d\phi(x)}{2\pi\imath}$.

[6] See Proposition 4.c for the recursion relation that builds $\phi_{(*)\mathrm{bg}}$.

[7] $\psi_{*\mathrm{cr}}(\theta_*, \theta)$ and $\psi_{\mathrm{cr}}(\theta_*, \theta)$ need not be complex conjugates of each other

- Let
$$Q_- : H_- \to H \qquad Q : H \to H_+$$
be linear maps. They induce $\mathbb{C}$ linear maps between $\mathcal{H}_-$, $\mathcal{H}$, $\mathcal{H}_+$ which are denoted by the same letter. We set
$$\check{Q}_- = Q \circ Q_-$$
- Fix $b > 0$ and a strictly positive definite (real) symmetric linear operator, $\mathfrak{Q}$, on $H$.
- Let $\mathfrak{A}$ be a polynomial on $\mathcal{H}_- \times \mathcal{H}_-$.

Set, for $\phi_*, \phi \in \mathcal{H}_-$, $\psi_*, \psi \in \mathcal{H}$ and $\theta_*, \theta \in \mathcal{H}_+$

$$\begin{aligned}
\mathcal{A}(\psi_*, \psi; \phi_*, \phi) &= \langle \psi_* - Q_-\, \phi_*\,,\, \mathfrak{Q}(\psi - Q_-\, \phi)\rangle + \mathfrak{A}(\phi_*, \phi) \\
\mathcal{A}_{\rm eff}(\theta_*, \theta; \psi_*, \psi; \phi_*, \phi) &= b\,\langle \theta_* - Q\psi_*\,,\, \theta - Q\psi\rangle_+ + \mathcal{A}(\psi_*, \psi; \phi_*, \phi) \\
\check{\mathcal{A}}(\theta_*, \theta; \phi_*, \phi) &= \big\langle \theta_* - \check{Q}_-\, \phi_*\,,\, \check{\mathfrak{Q}}\big(\theta - \check{Q}_-\, \phi\big)\big\rangle_+ + \mathfrak{A}(\phi_*, \phi)
\end{aligned}$$

where
$$\check{\mathfrak{Q}} = \big(\tfrac{1}{b}\mathbb{1}_{\mathcal{H}_+} + Q\,\mathfrak{Q}^{-1}Q^*\big)^{-1} \tag{5}$$

**Remark 1.** In this setting, the action of the functional integral (3) that appears at the beginning of the renormalization group step is

$$-\,\langle \psi^* - Q_-\, \phi_*\,,\, \mathfrak{Q}(\psi - Q_-\, \phi)\rangle - \mathfrak{A}(\phi_*, \phi) + \mathcal{E}(\psi^*, \psi) = -\mathcal{A}(\psi^*, \psi; \phi_*, \phi) + \mathcal{E}(\psi^*, \psi)$$

and the action of the functional integral (4) that appears in the middle of the renormalization group step is

$$\begin{aligned}
-\,b\,\langle \theta^* - Q\,\psi_*\,,\, \theta - Q\,\psi\rangle_+ - \langle \psi^* - Q_-\, \phi_*\,,\, \mathfrak{Q}(\psi - Q_-\, \phi)\rangle - \mathfrak{A}(\phi_*, \phi) + \mathcal{E}(\psi^*, \psi) \\
= -\mathcal{A}_{\rm eff}(\theta^*, \theta; \psi^*, \psi; \phi_*, \phi) + \mathcal{E}(\psi^*, \psi)
\end{aligned}$$

We show in Proposition 4.b, below, that when one substitutes the critical $\psi$ into $\mathcal{A}_{\rm eff}$ one gets $\check{\mathcal{A}}$. Upon scaling (and renormalizing) $\check{\mathcal{A}}$ becomes the $\mathcal{A}$ for the beginning of the next renormalization group step. Equation (5) is the recursion relation that builds the operator $\mathfrak{Q}$ in $\mathcal{A}(\psi_*, \psi; \phi_*, \phi)$.

**Remark 2.** $\quad \check{\mathfrak{Q}} = b\big[\mathbb{1}_{\mathcal{H}_+} - bQ\big(bQ^*Q + \mathfrak{Q}\big)^{-1}Q^*\big]$

*Proof.* Apply Lemma 13 with $V = \mathcal{H}$, $W = \mathcal{H}_+$, $q = Q$, $q_* = Q^*$, $f = \mathfrak{Q}$ and $g = b\mathbb{1}_W$. ☐

**Definition 3.**

(a) Let $\mathcal{N}$ be a domain in $\mathcal{H}$ which is invariant under complex conjugation. "Background fields on $\mathcal{N}$" are maps $\phi_{*\rm bg}, \phi_{\rm bg} : \mathcal{N} \times \mathcal{N} \to \mathcal{H}_-$ such that, for each $(\psi_*, \psi) \in \mathcal{N} \times \mathcal{N}$, the point $\big(\phi_{*\rm bg}(\psi_*, \psi),\, \phi_{\rm bg}(\psi_*, \psi)\big)$ is a critical point of the map

$$(\phi_*, \phi) \mapsto \mathcal{A}(\psi_*, \psi;\, \phi_*, \phi)$$

That is, it solves

$$\begin{aligned} Q_-^* \mathfrak{Q}\, Q_- \phi_* + \nabla_\phi \mathfrak{A}(\phi_*, \phi) &= Q_-^* \mathfrak{Q} \psi_* \\ Q_-^* \mathfrak{Q}\, Q_- \phi + \nabla_{\phi_*} \mathfrak{A}(\phi, \phi) &= Q_-^* \mathfrak{Q} \psi \end{aligned} \tag{6}$$

"Formal background fields" are formal power series $\phi_{*\rm bg}(\psi_*, \psi)$, $\phi_{\rm bg}(\psi_*, \psi)$, in $(\psi_*, \psi)$ with vanishing constant terms, that solve (6).

(b) Let $\mathcal{N}_+$ and $\mathcal{N}$ be domains in $\mathcal{H}_+$ and $\mathcal{H}$, respectively, which are invariant under complex conjugation. Let $\phi_{*\rm bg}, \phi_{\rm bg}$ be background fields on $\mathcal{N}$. "Critical fields on $\mathcal{N}_+$ with respect to $\phi_{*\rm bg}, \phi_{\rm bg}$" are maps $\psi_{*\rm cr}, \psi_{\rm cr} : \mathcal{N}_+ \times \mathcal{N}_+ \to \mathcal{N}$ such that, for each $(\theta_*, \theta) \in \mathcal{N}_+ \times \mathcal{N}_+$, the point $\big(\psi_{*\rm cr}(\theta_*, \theta), \psi_{\rm cr}(\theta_*, \theta)\big)$ is a critical point for the map

$$(\psi_*, \psi) \mapsto \mathcal{A}_{\rm eff}(\theta_*, \theta; \psi_*, \psi; \phi_{*\rm bg}(\psi_*, \psi), \phi_{\rm bg}(\psi_*, \psi))$$

That is, it solves

$$\begin{aligned} (bQ^*Q + \mathfrak{Q})\psi_* &= bQ^*\theta_* + \mathfrak{Q}\, Q_-\, \phi_{*\rm bg}(\psi_*, \psi) \\ (bQ^*Q + \mathfrak{Q})\psi &= bQ^*\theta + \mathfrak{Q}\, Q_-\, \phi_{\rm bg}(\psi_*, \psi) \end{aligned} \tag{7}$$

If $\phi_{*\rm bg}, \phi_{\rm bg}$ are formal background fields, then "formal critical fields with respect to $\phi_{*\rm bg}, \phi_{\rm bg}$" are formal power series $\psi_{*\rm cr}(\theta_*, \theta), \psi_{\rm cr}(\theta_*, \theta)$, in $(\theta_*, \theta)$ with vanishing constant terms, that solve (7).

(c) Let $\mathcal{N}_+$ be a domain in $\mathcal{H}_+$ which is invariant under complex conjugation. "Next scale background fields on $\mathcal{N}_+$" are maps $\check\phi_{*\rm bg}, \check\phi_{\rm bg} : \mathcal{N}_+ \times \mathcal{N}_+ \to \mathcal{H}_-$ such that, for each $(\theta_*, \theta) \in \mathcal{N}_+ \times \mathcal{N}_+$, the point $\big(\check\phi_{*\rm bg}(\theta_*, \theta),\, \check\phi_{\rm bg}(\theta_*, \theta)\big)$ is a critical point of the map

$$(\phi_*, \phi) \mapsto \check{\mathcal{A}}(\theta_*, \theta;\, \phi_*, \phi)$$

That is, it solves

$$\begin{aligned} \check Q_-^* \check{\mathfrak{Q}}\, \check Q_- \check\phi_* + \nabla_{\check\phi} \mathfrak{A}(\check\phi_*, \check\phi) &= \check Q_-^* \check{\mathfrak{Q}}\, \theta_* \\ \check Q_-^* \check{\mathfrak{Q}}\, \check Q_- \check\phi + \nabla_{\check\phi_*} \mathfrak{A}(\check\phi_*, \check\phi) &= \check Q_-^* \check{\mathfrak{Q}}\, \theta \end{aligned} \tag{8}$$

Formal power series $\check\phi_{*\rm bg}(\theta_*,\theta)$, $\check\phi_{\rm bg}(\theta_*,\theta)$, in $(\theta_*,\theta)$ with vanishing constant terms, that solve (8) are called "formal next scale background fields".

**Proposition 4.** *Let $\mathcal{N}_+$ and $\mathcal{N}$ be domains in $\mathcal{H}_+$ and $\mathcal{H}$, respectively, which are invariant under complex conjugation. Let $\phi_{*\rm bg},\phi_{\rm bg}$ be background fields on $\mathcal{N}$ and $\psi_{*\rm cr},\psi_{\rm cr}$ be critical fields on $\mathcal{N}_+$ with respect to $\phi_{*\rm bg},\phi_{\rm bg}$. Define the composition*

$$\begin{aligned}\check\phi_{*\rm cp}(\theta_*,\theta)&=\phi_{*\rm bg}\big(\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta)\big)\\ \check\phi_{\rm cp}(\theta_*,\theta)&=\phi_{\rm bg}\big(\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta)\big)\end{aligned}\tag{9}$$

*Then, for all $(\theta_*,\theta)\in\mathcal{N}_+\times\mathcal{N}_+$,*

*(a) $\big(\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta)\big)$ fulfils the equations*

$$\begin{aligned}\psi_{*\rm cr}(\theta_*,\theta)&=(bQ^*Q+\mathfrak{Q})^{-1}\big(bQ^*\theta_*+\mathfrak{Q}\,Q_-\,\check\phi_{*\rm cp}(\theta_*,\theta)\big)\\ \psi_{\rm cr}(\theta_*,\theta)&=(bQ^*Q+\mathfrak{Q})^{-1}\big(bQ^*\theta+\mathfrak{Q}\,Q_-\,\check\phi_{\rm cp}(\theta_*,\theta)\big)\end{aligned}$$

*(b) The effective action*

$$\begin{aligned}&\mathcal{A}_{\rm eff}\big(\theta_*,\theta;\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta);\check\phi_{*\rm cp}(\theta_*,\theta),\check\phi_{\rm cp}(\theta_*,\theta)\big)\\ &\qquad\qquad=\check{\mathcal{A}}(\theta_*,\theta;\check\phi_{*\rm cp}(\theta_*,\theta),\check\phi_{\rm cp}(\theta_*,\theta))\end{aligned}$$

*(c) $\check\phi_{*\rm cp}(\theta_*,\theta)$ , $\check\phi_{\rm cp}(\theta_*,\theta)$ are next scale background fields on $\mathcal{N}_+$.*

*(d) For any continuous function $\mathcal{E}(\psi_*,\psi)$ on $\mathcal{N}\times\mathcal{N}$*

$$\begin{aligned}&\int_{\mathcal{N}\times\mathcal{N}}d\mu_{\mathcal{H}}(\psi^*,\psi)\;e^{-\mathcal{A}(\psi^*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi))+\mathcal{E}(\psi^*,\psi)}\\ &\quad=b^{\dim\mathcal{H}_+}\bigg\{\int_{\mathcal{N}_+\times\mathcal{N}_+}d\mu_{\mathcal{H}_+}(\theta^*,\theta)\;e^{-\check{\mathcal{A}}(\theta^*,\theta;\check\phi_{*\rm cp}(\theta^*,\theta),\check\phi_{\rm cp}(\theta^*,\theta))}\;e^{\mathcal{E}(\psi_{*\rm cr}(\theta^*,\theta),\psi_{\rm cr}(\theta^*,\theta))}\mathcal{F}(\theta^*,\theta)\\ &\qquad+\int_{(\mathcal{H}_+\times\mathcal{H}_+)\setminus(\mathcal{N}_+\times\mathcal{N}_+)}d\mu_{\mathcal{H}_+}(\theta^*,\theta)\int_{\mathcal{N}\times\mathcal{N}}d\mu_{\mathcal{H}}(\psi^*,\psi)\,e^{-\mathcal{A}_{\rm eff}(\theta^*,\theta;\psi^*,\psi;\phi_{*\rm bg}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}\bigg\}\end{aligned}$$

*where the fluctuation integral*

$$\mathcal{F}(\theta_*,\theta)=\int_{\mathcal{D}(\theta_*,\theta)}d\mu_{\mathcal{H}}(\delta\psi_*,\delta\psi)\,e^{-\delta\mathcal{A}(\theta_*,\theta;\delta\psi_*,\delta\psi)}e^{\delta\mathcal{E}(\theta_*,\theta;\delta\psi_*,\delta\psi)}$$

*Here the functions $\delta\mathcal{A}$ and $\delta\mathcal{E}$ are given by*

$$\delta\mathcal{A}(\theta_*,\theta;\delta\psi_*,\delta\psi) = \mathcal{A}_{\rm eff}\big(\theta_*,\theta;\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)\Big|_{\psi_*=\psi_{*\rm cr},\ \psi=\psi_{\rm cr}}^{\psi_*=\psi_{*\rm cr}+\delta\psi_*,\ \psi=\psi_{\rm cr}+\delta\psi}$$

$$\delta\mathcal{E}(\theta_*,\theta;\delta\psi_*,\delta\psi) = \mathcal{E}\big(\psi_*,\psi\big)\Big|_{\psi_*=\psi_{*\rm cr},\ \psi=\psi_{\rm cr}}^{\psi_*=\psi_{*\rm cr}+\delta\psi_*,\ \psi=\psi_{\rm cr}+\delta\psi}$$

*with $\psi_{*\rm cr} = \psi_{*\rm cr}(\theta_*,\theta)$, $\psi_{\rm cr} = \psi_{\rm cr}(\theta_*,\theta)$, and the domain*

$$\mathcal{D}(\theta_*,\theta) = \Big\{\ (\delta\psi_*,\delta\psi)\in\mathcal{H}\times\mathcal{H}\ \Big|\ \psi_{*\rm cr}(\theta_*,\theta)+\delta\psi_* = \big(\psi_{\rm cr}(\theta_*,\theta)+\delta\psi\big)^* \in\mathcal{N}\ \Big\}$$

The formal power series versions of parts (a), (b) and (c) of Proposition 4 are

**Proposition 4'.** *Let $\phi_{*\rm bg},\phi_{\rm bg}$ be formal background fields and $\psi_{*\rm cr},\psi_{\rm cr}$ be formal critical fields with respect to $\phi_{*\rm bg},\phi_{\rm bg}$. Set*[8]

$$\check\phi_{(*)\rm cp}(\theta_*,\theta) = \phi_{(*)\rm bg}\big(\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta)\big) \tag{9'}$$

*(a) $\big(\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta)\big)$ fulfils the equations*

$$\psi_{(*)\rm cr}(\theta_*,\theta) = (bQ^*Q+\mathfrak{Q})^{-1}\big(bQ^*\theta_{(*)}+\mathfrak{Q}\,Q_-\,\check\phi_{(*)\rm cp}(\theta_*,\theta)\big)$$

*(b) The effective action*

$$\begin{aligned}\mathcal{A}_{\rm eff}\big(\theta_*,\theta;\psi_{*\rm cr}(\theta_*,\theta),\psi_{\rm cr}(\theta_*,\theta);\check\phi_{*\rm cp}(\theta_*,\theta),\check\phi_{\rm cp}(\theta_*,\theta)\big)&\\ = \check{\mathcal{A}}(\theta_*,\theta;\check\phi_{*\rm cp}(\theta_*,\theta),\check\phi_{\rm cp}(\theta_*,\theta))&\end{aligned}$$

*(c) $\check\phi_{*\rm cp}(\theta_*,\theta)$ , $\check\phi_{\rm cp}(\theta_*,\theta)$ are formal next scale background fields.*

The proof of these Propositions will be given after Lemma 7.

**Remark 5.**

(a) Part (c) of the Proposition is often called the "composition rule".

(b) In applications, the domain $\mathcal{N}_+$ is chosen so that the second integral on the right hand side of the formula in part (d) is small. In that integral either $\theta$ or $\theta_*$ is bounded away from the origin ("large fields").

[8] We routinely use the "optional $*$" notation $\alpha_{(*)}$ to denote "$\alpha_*$ or $\alpha$". The equation "$\alpha_{(*)}=\beta_{(*)}$" means "$\alpha_*=\beta_*$ and $\alpha=\beta$".

(c) As in Proposition 4', let $\phi_{*\text{bg}}, \phi_{\text{bg}}$ be formal background fields and $\psi_{*\text{cr}}, \psi_{\text{cr}}$ be formal critical fields with respect to $\phi_{*\text{bg}}, \phi_{\text{bg}}$. Assume, in addition, that the equations (8), for the next scale background fields, have a unique formal power series solution, that we denote $\check\phi_{*\text{bg}}, \check\phi_{\text{bg}}$. Then by part (c) of Proposition 4', $\check\phi_{(*)\text{bg}}(\theta_*, \theta) = \check\phi_{(*)\text{cp}}(\theta_*, \theta)$ and, by part (a) of Proposition 4',

$$\psi_{(*)\text{cr}}(\theta_*, \theta) = (bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta_{(*)} + \mathfrak{Q}\,Q_-\,\check\phi_{(*)\text{bg}}(\theta_*, \theta)\big)$$

If, in addition, $\check\phi_{(*)\text{bg}}(\theta_*, \theta)$ are analytic functions on some domain, then so are $\psi_{(*)\text{cr}}(\theta_*, \theta)$. So to construct analytical critical fields, it suffices to have

- uniqueness of formal power series solutions to the next scale background field equations
- existence of analytic solutions to the next scale background field equations
- formal background fields
- formal critical fields with respect to the formal background fields

Lemma 6, below, provides existence and uniqueness for formal power series solutions of the critical field equations.

**Lemma 6.** *Let $\phi_{*\text{bg}}, \phi_{\text{bg}}$ be formal background fields of the form*

$$\phi_{(*)\text{bg}}(\psi_*, \psi) = L_{(*)}\psi_{(*)} + \phi^{(\geq 2)}_{(*)\text{bg}}(\psi_*, \psi)$$

*with $\phi^{(\geq 2)}_{(*)\text{bg}}(\psi_*, \psi)$ being of degree at least two[9] in $(\psi_*, \psi)$ and with the $L_{(*)}$'s being linear operators. If the linear operators $bQ^*Q + \mathfrak{Q} - \mathfrak{Q}Q_-L_{(*)}$ are invertible, then there exist unique formal critical fields with respect to $\phi_{*\text{bg}}, \phi_{\text{bg}}$.*

*Proof.* Rewrite the equations (7) in the form

$$\begin{aligned}(bQ^*Q + \mathfrak{Q} - \mathfrak{Q}Q_-L_*)\psi_* &= bQ^*\theta_* + \mathfrak{Q}\,Q_-\,\phi^{(\geq 2)}_{*\text{bg}}(\psi_*, \psi)\\ (bQ^*Q + \mathfrak{Q} - \mathfrak{Q}Q_-L)\psi &= bQ^*\theta + \mathfrak{Q}\,Q_-\,\phi^{(\geq 2)}_{\text{bg}}(\psi_*, \psi)\end{aligned}$$

As $\psi_*$ and $\psi$ are to have vanishing constant terms, this provides a "lower triangular" recursion relation for the coefficients of $(\psi_*, \psi)$. As $\mathcal{H}$ and $\mathcal{H}_+$ are finite dimensional, this recursion relation trivially generates a unique solution. □

The proof of Proposition 4 is based on

[9] By this we mean that each nonzero monomial in $\phi^{(\geq 2)}_{(*)\text{bg}}$ has degree at least two.

**Lemma 7.** *For $\phi_*, \phi \in \mathcal{H}_-$ and $\theta_*, \theta \in \mathcal{H}_+$ set*

$$\tilde\psi_{(*)}(\theta_{(*)}, \phi_{(*)}) = (bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta_{(*)} + \mathfrak{Q}\, Q_-\, \phi_{(*)}\big)$$

*Then $\check{\mathcal{A}}\big(\theta_*, \theta;\, \phi_*, \phi\big) = \mathcal{A}_{\rm eff}\big(\theta_*, \theta; \tilde\psi_*(\theta_*, \phi_*), \tilde\psi(\theta, \phi);\, \phi_*, \phi\big)$ and*

$$\begin{aligned}
&(\nabla_{\phi_{(*)}}\check{\mathcal{A}})(\theta_*, \theta; \phi_*, \phi) \\
&\qquad = (\nabla_{\phi_{(*)}}\mathcal{A})\big(\tilde\psi_*(\theta_*, \phi_*), \tilde\psi(\theta, \phi); \phi_*, \phi\big) \\
&\qquad\qquad + Q_-^*\mathfrak{Q}\,(bQ^*Q + \mathfrak{Q})^{-1}\big[(\nabla_{\psi_{(*)}}\mathcal{A}_{\rm eff})\big(\theta_*, \theta; \tilde\psi_*(\theta_*, \phi_*), \tilde\psi(\theta, \phi);\, \phi_*, \phi\big)\big]
\end{aligned} \tag{10}$$

*Proof.* With the abbreviation $\tilde\psi_{(*)} = \tilde\psi_{(*)}(\theta_{(*)}, \phi_{(*)})$

$$\begin{aligned}
\theta - Q\tilde\psi &= \theta - Q(bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta + \mathfrak{Q}\, Q_-\,\phi\big) \\
&= \big[\mathbb{1} - bQ(bQ^*Q + \mathfrak{Q})^{-1}Q^*\big]\theta - \check{Q}_-\,\phi + QQ_-\,\phi - Q(bQ^*Q + \mathfrak{Q})^{-1}\mathfrak{Q}\, Q_-\,\phi \\
&= \big[\mathbb{1} - bQ(bQ^*Q + \mathfrak{Q})^{-1}Q^*\big]\theta - \check{Q}_-\,\phi \\
&\qquad\qquad + Q(bQ^*Q + \mathfrak{Q})^{-1}\big[(bQ^*Q + \mathfrak{Q}) - \mathfrak{Q}\big]Q_-\,\phi \\
&= \big[\mathbb{1} - bQ(bQ^*Q + \mathfrak{Q})^{-1}Q^*\big]\big(\theta - \check{Q}_-\,\phi\big) \\
\tilde\psi - Q_-\,\phi &= (bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta + \mathfrak{Q}\, Q_-\,\phi\big) - Q_-\,\phi \\
&= (bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta + \mathfrak{Q}\, Q_-\,\phi - bQ^*QQ_-\,\phi - \mathfrak{Q}\, Q_-\,\phi\big) \\
&= b(bQ^*Q + \mathfrak{Q})^{-1}Q^*\big(\theta - \check{Q}_-\,\phi\big)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\check{\mathcal{A}}\big(\theta_*, \theta;\, \phi_*, \phi\big) - \mathcal{A}_{\rm eff}\big(\theta_*, \theta; \tilde\psi_*, \tilde\psi;\, \phi_*, \phi\big) \\
&\qquad = \big\langle \theta_* - \check{Q}_-\,\phi_*\,,\, \check{\mathfrak{Q}}\big(\theta - \check{Q}_-\,\phi\big)\big\rangle_+ - b\big\langle \theta_* - Q\tilde\psi_*\,,\, \theta - Q\tilde\psi\big\rangle_+ \\
&\qquad\qquad\qquad - \big\langle \tilde\psi_* - Q_-\,\phi_*\,,\, \mathfrak{Q}(\tilde\psi - Q_-\,\phi)\big\rangle \\
&\qquad = b\big\langle \theta_* - \check{Q}_-\,\phi_*\,,\, \mathcal{O}\big(\theta - \check{Q}_-\,\phi\big)\big\rangle_+
\end{aligned}$$

where, by Remark 2,

$$\begin{aligned}
\mathcal{O} &= \big[\mathbb{1} - bQ\big(bQ^*Q + \mathfrak{Q}\big)^{-1}Q^*\big] - \big[\mathbb{1} - bQ(bQ^*Q + \mathfrak{Q})^{-1}Q^*\big]^2 \\
&\qquad - bQ(bQ^*Q + \mathfrak{Q})^{-1}\mathfrak{Q}(bQ^*Q + \mathfrak{Q})^{-1}Q^* \\
&= b\big[\mathbb{1} - bQ\big(bQ^*Q + \mathfrak{Q}\big)^{-1}Q^*\big]Q(bQ^*Q + \mathfrak{Q})^{-1}Q^* \\
&\qquad - bQ(bQ^*Q + \mathfrak{Q})^{-1}\mathfrak{Q}(bQ^*Q + \mathfrak{Q})^{-1}Q^* \\
&= bQ\big[\mathbb{1} - \big(bQ^*Q + \mathfrak{Q}\big)^{-1}bQ^*Q - (bQ^*Q + \mathfrak{Q})^{-1}\mathfrak{Q}\big](bQ^*Q + \mathfrak{Q})^{-1}Q^* \\
&= 0
\end{aligned}$$

This proves the first statement. The second follows by the chain rule and the observation that $\nabla_{\phi_{(*)}}\mathcal{A}_{\rm eff} = \nabla_{\phi_{(*)}}\mathcal{A}\,.$ $\square$

*Proof of Propositions 4 and 4'.* The proof of Proposition 4' is virtually identical to that of Proposition 4.a,b,c, so we just give the proof of Proposition 4. Part (a) follows immediately from (7) and (9). Now evaluate the conclusions of Lemma 7 at $\phi_{(*)} = \check\phi_{(*){\rm cp}}(\theta_*,\theta)\big)$. The formula for $\check{\mathcal{A}}$ in Lemma 7 directly gives part (b). The right hand side of (10) vanishes upon this evaluation by parts (a) and (b) of Definition 3. This shows that $\big(\check\phi_{*{\rm cp}}(\theta_*,\theta)\,,\,\check\phi_{\rm cp}(\theta_*,\theta)\big)$ is critical for the map $(\phi_*,\phi)\mapsto\check{\mathcal{A}}\big(\theta_*,\theta;\,\phi_*,\phi\big)$, which proves part (c). Now

$$
\begin{aligned}
b^{-\dim\mathcal{H}_+}&\int_{\mathcal{N}\times\mathcal{N}} d\mu_{\mathcal{H}}(\psi^*,\psi)\, e^{-\mathcal{A}(\psi^*,\psi;\phi_{*{\rm bg}}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}\\
&=\int d\mu_{\mathcal{H}_+}(\theta^*,\theta)\int_{\mathcal{N}\times\mathcal{N}} d\mu_{\mathcal{H}}(\psi^*,\psi)\, e^{-b\langle\theta^*-Q\psi^*\,,\,\theta-Q\psi\rangle_+-\mathcal{A}(\psi^*,\psi;\phi_{*{\rm bg}}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}\\
&=\int d\mu_{\mathcal{H}_+}(\theta^*,\theta)\int_{\mathcal{N}\times\mathcal{N}} d\mu_{\mathcal{H}}(\psi^*,\psi)\, e^{-\mathcal{A}_{\rm eff}(\theta^*,\theta;\psi^*,\psi;\phi_{*{\rm bg}}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}\\
&=\int_{\mathcal{N}_+\times\mathcal{N}_+} d\mu_{\mathcal{H}_+}(\theta^*,\theta)\int_{\mathcal{N}\times\mathcal{N}} d\mu_{\mathcal{H}}(\psi^*,\psi)\, e^{-\mathcal{A}_{\rm eff}(\theta^*,\theta;\psi^*,\psi;\phi_{*{\rm bg}}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}\\
&\quad+\int_{\mathcal{H}_+\times\mathcal{H}_+\setminus\mathcal{N}_+\times\mathcal{N}_+} d\mu_{\mathcal{H}_+}(\theta^*,\theta)\int_{\mathcal{N}\times\mathcal{N}} d\mu_{\mathcal{H}}(\psi^*,\psi)\, e^{-\mathcal{A}_{\rm eff}(\theta^*,\theta;\psi^*,\psi;\phi_{*{\rm bg}}(\psi^*,\psi),\phi_{\rm bg}(\psi^*,\psi))+\mathcal{E}(\psi^*,\psi)}
\end{aligned}
$$

Making the change of variables $\psi^* = \psi_{*{\rm cr}}(\theta^*,\theta)+\delta\psi_*$, $\psi = \psi_{\rm cr}(\theta^*,\theta)+\delta\psi$ in the inner integral of the upper line and applying part (b) gives part (d). $\square$

From now on we assume that the function $\mathfrak{A}(\phi_*,\phi)$ in the definitions of $\mathcal{A}$ and $\check{\mathcal{A}}$ is of the form

$$
\mathfrak{A}(\phi_*,\phi) = \langle\phi_*, D\phi\rangle_- + P(\phi_*,\phi) \tag{11}
$$

where

- $P$ is a polynomial whose nonzero monomials are each of degree at least two and
- $D$ a linear operator on $\mathcal{H}_-$ such that both the operators $(D + Q_-^*\mathfrak{Q}\,Q_-)$ and $(D+\check Q_-^*\check{\mathfrak{Q}}\,\check Q_-)$ are invertible. We define the "Green's functions"

$$
S = (D + Q_-^*\mathfrak{Q}\,Q_-)^{-1} \qquad\qquad \check S = (D+\check Q_-^*\check{\mathfrak{Q}}\,\check Q_-)^{-1} \tag{12}
$$

We think of $D$ as a differential operator, possibly shifted by a chemical potential.

**Remark 8.** In this setting, the background field equations (6) become

$$
\phi_{(*)} = S^{(*)}Q_-^*\mathfrak{Q}\psi_{(*)} - S^{(*)}P'_{(*)}(\phi_*,\phi) \tag{6'}
$$

where $P'_*(\phi_*, \phi) = \nabla_\phi P(\phi_*, \phi)$ and $P'(\phi_*, \phi) = \nabla_{\phi_*} P(\phi_*, \phi)$. Similarly, the next scale background field equations (8) become

$$\check{\phi}_{(*)} = \check{S}^{(*)} \check{Q}_-^* \check{\mathfrak{Q}}\, \theta_{(*)} - \check{S}^{(*)} P'_{(*)}(\check{\phi}_*, \check{\phi}) \tag{8'}$$

We now continue with our study of the critical field, following the plan of Remark 5.c. To describe the leading part of the critical field, we set

$$\Delta = \mathfrak{Q} - \mathfrak{Q}\, Q_- S Q_-^* \mathfrak{Q} : \ \mathcal{H} \longrightarrow \mathcal{H} \tag{13}$$

From now on we assume that $\Delta + bQ^*Q$ is invertible and define[10] the "covariance"

$$C = (\Delta + bQ^*Q)^{-1} : \ \mathcal{H} \longrightarrow \mathcal{H} \tag{14}$$

**Proposition 9.** *Assume that in the setting* (11)*, each nonzero monomial of $P$ is of degree at least three. Then there exist unique formal background fields $\phi_{(*)\mathrm{bg}}$ and unique formal next scale background fields $\check{\phi}_{(*)\mathrm{bg}}$. They are of the form*

$$\phi_{(*)\mathrm{bg}}(\psi_*, \psi) = S^{(*)} Q_-^* \mathfrak{Q} \psi_{(*)} + \phi^{(\geq 2)}_{(*)\mathrm{bg}}(\psi_*, \psi)$$
$$\check{\phi}_{(*)\mathrm{bg}}(\theta_*, \theta) = \check{S}^{(*)} \check{Q}_-^* \check{\mathfrak{Q}} \theta_{(*)} + \check{\phi}^{(\geq 2)}_{(*)\mathrm{bg}}(\theta_*, \theta)$$

*with $\phi^{(\geq 2)}_{(*)\mathrm{bg}}(\psi_*, \psi)$ and $\check{\phi}^{(\geq 2)}_{(*)\mathrm{bg}}(\theta_*, \theta)$ being of degree at least two. Furthermore, there are unique formal critical fields with respect to $\phi_{(*)\mathrm{bg}}$. They are of the form*

$$\begin{aligned}\psi_{(*)\mathrm{cr}}(\theta_*, \theta) &= (bQ^*Q + \mathfrak{Q})^{-1}\big(bQ^*\theta_{(*)} + \mathfrak{Q}\, Q_-\, \check{\phi}_{(*)\mathrm{bg}}(\theta_*, \theta)\big) \\ &= bC^{(*)} Q^*\, \theta_{(*)} + \psi^{(\geq 2)}_{(*)\mathrm{cr}}(\theta_*, \theta)\big)\end{aligned}$$

*with $\psi^{(\geq 2)}_{(*)\mathrm{cr}}$ being of degree at least two.*

*Proof.* The existence, uniqueness and forms of the formal background and next scale background fields are proven as Lemma 6 was proven. The existence and uniqueness of the formal critical field now follows from Lemma 6. The first representation of the critical fields follows from parts (a) and (c) of Proposition 4'. For the second representation, rewrite the equations (7) as

$$(bQ^*Q + \mathfrak{Q})\psi_{(*)} = bQ^*\theta_{(*)} + \mathfrak{Q}\, Q_-\, S^{(*)} Q_-^* \mathfrak{Q} \psi_{(*)} + \mathfrak{Q}\, Q_-\, \phi^{(\geq 2)}_{(*)\mathrm{bg}}(\psi_*, \psi)$$

or

$$\psi_{(*)} = bC^{(*)} Q^* \theta_{(*)} + C^{(*)} \mathfrak{Q}\, Q_-\, \phi^{(\geq 2)}_{(*)\mathrm{bg}}(\psi_*, \psi)$$

□

[10] We shall show, in Lemma 12, below, that $C$ is the covariance for the fluctuation integral.

The two representations of the critical field, $\psi_{\rm cr}$, given in Proposition 9, combined with the representation of $\check\phi_{\rm bg}$, suggest a formula for $bCQ^*$. In Remark 10, below, we give an algebraic proof of this formula, together with a number of representations for the Green's functions, $S$ and $\check S$, and covariance $C$. Then, in Lemma 12 below, we analyze the fluctuation integral of Proposition 4.d in more detail.

**Remark 10.** Assume that $D$ is invertible.

(a) $\Delta = \big(\mathbb{1}_{\mathcal H} + \mathfrak{Q}\, Q_- D^{-1} Q_-^*\big)^{-1}\mathfrak{Q} = \mathfrak{Q}\big(\mathbb{1}_{\mathcal H} + Q_- D^{-1} Q_-^*\,\mathfrak{Q}\big)^{-1}$

(b) Let $R : \mathcal{H}_- \to \mathcal{H}$ and $R_* : \mathcal{H} \to \mathcal{H}_-$ be linear maps such that $R\, D^{-1} R_* = Q_- D^{-1} Q_-^*$ and such that $D + R_*\mathfrak{Q}\, R$ is invertible. Then

$$[D + R_*\mathfrak{Q}\, R]^{-1} = D^{-1} - D^{-1} R_* \Delta\, R\, D^{-1}$$

In particular

$$S = D^{-1} - D^{-1} Q_-^* \Delta\, Q_- D^{-1}$$

(c) $\check S = \big[S^{-1} - Q_-^*\mathfrak{Q}(\mathfrak{Q} + bQ^*Q)^{-1}\mathfrak{Q}Q_-\big]^{-1} = S + SQ_-^*\,\mathfrak{Q}\, C\, \mathfrak{Q}\, Q_- S$

(d) $C = \big(bQ^*Q + \mathfrak{Q}\big)^{-1} + (bQ^*Q + \mathfrak{Q})^{-1}\,\mathfrak{Q}Q_-\check S Q_-^*\mathfrak{Q}\,(bQ^*Q + \mathfrak{Q})^{-1}$

(e) $bC^{(*)}Q^* = \big(bQ^*Q + \mathfrak{Q}\big)^{-1}\Big[bQ^* + \mathfrak{Q}Q_-\check S^{(*)}\check Q_-^*\check{\mathfrak{Q}}\Big]$

*Proof.* (a) By Lemma 13, with $V = \mathcal{H}_-$, $W = \mathcal{H}$, $q = Q_-$, $q_* = Q_-^*$, $f = D$ and $g = \mathfrak{Q}$

$$\begin{aligned}
\big\{\mathbb{1} + \mathfrak{Q}\, Q_- D^{-1} Q_-^*\big\}^{-1}\mathfrak{Q} &= \big\{\mathbb{1} - \mathfrak{Q}\, Q_-(D + Q_-^*\mathfrak{Q}Q_-)^{-1}Q_-^*\big\}\mathfrak{Q} = \Delta\\
\mathfrak{Q}\big\{\mathbb{1} + Q_- D^{-1} Q_-^*\,\mathfrak{Q}\big\}^{-1} &= \mathfrak{Q}\big\{\mathbb{1} - Q_-(D + Q_-^*\mathfrak{Q}Q_-)^{-1}Q_-^*\,\mathfrak{Q}\big\} = \Delta
\end{aligned}$$

(b) By part (a)

$$\begin{aligned}
\big[D + R_*\mathfrak{Q}\, R\big]\,\big[D^{-1} - D^{-1}R_*\Delta\, R\, D^{-1}\big] &= \mathbb{1} + R_*\big[\mathfrak{Q} - (\mathbb{1} + \mathfrak{Q}\, R\, D^{-1} R_*)\,\Delta\big]\, R\, D^{-1}\\
&= \mathbb{1} + R_*\big[\mathfrak{Q} - (\mathbb{1} + \mathfrak{Q}\, Q_- D^{-1} Q_-^*)\,\Delta\big]\, R\, D^{-1}\\
&= \mathbb{1}
\end{aligned}$$

(c) By Remark 2

$$\begin{aligned}
Q^*\check{\mathfrak{Q}}\, Q &= bQ^*Q\big[\mathbb{1} - (bQ^*Q + \mathfrak{Q})^{-1} bQ^*Q\big]\\
&= bQ^*Q\big[(bQ^*Q + \mathfrak{Q})^{-1}(bQ^*Q + \mathfrak{Q}) - (bQ^*Q + \mathfrak{Q})^{-1} bQ^*Q\big]\\
&= (\mathfrak{Q} + bQ^*Q - \mathfrak{Q})(bQ^*Q + \mathfrak{Q})^{-1}\mathfrak{Q}\\
&= \mathfrak{Q} - \mathfrak{Q}(\mathfrak{Q} + bQ^*Q)^{-1}\mathfrak{Q}
\end{aligned}$$

Therefore

$$S^{-1} - \check{S}^{-1} = Q_-^* \mathfrak{Q}\, Q_- - Q_-^* Q^* \check{\mathfrak{Q}}\, Q Q_- = Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \mathfrak{Q} Q_-$$

which gives the first representation of $\check{S}$. For the proof of the second representation, first observe that, by (13) and (14),

$$\begin{aligned} C^{-1}(\mathfrak{Q} + bQ^*Q)^{-1} &= (\mathfrak{Q} + bQ^*Q - \mathfrak{Q}Q_- S Q_-^* \mathfrak{Q})(\mathfrak{Q} + bQ^*Q)^{-1} \\ &= \mathbb{1} - \mathfrak{Q}Q_- S Q_-^* \mathfrak{Q}(\mathfrak{Q} + bQ^*Q)^{-1} \end{aligned}$$

so that

$$C = (\mathfrak{Q} + bQ^*Q)^{-1} \big\{ \mathbb{1} - \mathfrak{Q}\, Q_- S Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \big\}^{-1} \tag{15}$$

Hence, by the first representation of $\check{S}$,

$$\begin{aligned} &\big[ S + SQ_-^*\, \mathfrak{Q}\, C\, \mathfrak{Q}\, Q_- S \big] \check{S}^{-1} - \mathbb{1} \\ &\quad = \big[ \mathbb{1} + SQ_-^*\, \mathfrak{Q}\, C\, \mathfrak{Q}\, Q_- \big] \big[ \mathbb{1} - SQ_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \mathfrak{Q} Q_- \big] - \mathbb{1} \\ &\quad = SQ_-^*\, \mathfrak{Q} \Big[ C \big\{ \mathbb{1} - \mathfrak{Q} Q_- S Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \big\} - (\mathfrak{Q} + bQ^*Q)^{-1} \Big] \mathfrak{Q} Q_- \\ &\quad = 0 \end{aligned}$$

which implies the second representation of $\check{S}$.

(d) By Lemma 13 with $q = \mathfrak{Q} Q_-$, $q_* = Q_-^* \mathfrak{Q}$, $f = S^{-1}$ and $g = -(\mathfrak{Q} + bQ^*Q)^{-1}$

$$\begin{aligned} &\big\{ \mathbb{1} - \mathfrak{Q}\, Q_- S Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \big\}^{-1} \\ &\qquad = \mathbb{1} + \mathfrak{Q} Q_- \big[ S^{-1} - Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \mathfrak{Q} Q_- \big]^{-1} Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \\ &\qquad = \mathbb{1} + \mathfrak{Q} Q_- \check{S} Q_-^* \mathfrak{Q} (\mathfrak{Q} + bQ^*Q)^{-1} \end{aligned} \tag{16}$$

The second equality follows by the first representation of $\check{S}$ in part (c). Substituting (16) into (15) gives the desired representation of $C$.

(e) By Remark 2

$$\begin{aligned} \check{Q}_-^* \check{\mathfrak{Q}} &= bQ_-^* Q^* \big[ \mathbb{1} - bQ(bQ^*Q + \mathfrak{Q})^{-1} Q^* \big] \\ &= bQ_-^* \big[ \mathbb{1} - bQ^*Q(bQ^*Q + \mathfrak{Q})^{-1} \big] Q^* \\ &= bQ_-^* \mathfrak{Q} (bQ^*Q + \mathfrak{Q})^{-1} Q^* \end{aligned}$$

Therefore by part (d)

$$\begin{aligned} bC^{(*)}Q^* &= \big(bQ^*Q + \mathfrak{Q}\big)^{-1}\big[bQ^* + b\mathfrak{Q}Q_-\check{S}^{(*)}Q_-^*\mathfrak{Q}\,(bQ^*Q+\mathfrak{Q})^{-1}Q^*\big] \\ &= \big(bQ^*Q + \mathfrak{Q}\big)^{-1}\Big[bQ^* + \mathfrak{Q}Q_-\check{S}^{(*)}\check{Q}_-^*\check{\mathfrak{Q}}\Big] \end{aligned}$$

□

Define, in the setting of Proposition 4, $\delta\phi_{(*)\rm bg}\big(\psi_*,\psi,\delta\psi_*,\delta\psi\big)$ by

$$\phi_{(*)\rm bg}\big(\psi_*+\delta\psi_*,\psi+\delta\psi\big) = \phi_{(*)\rm bg}\big(\psi_*,\psi\big) + \delta\phi_{(*)\rm bg}\big(\psi_*,\psi,\delta\psi_*,\delta\psi\big) \tag{17.a}$$

and set

$$\delta\check{\phi}_{(*)\rm bg}\big(\theta_*,\theta,\delta\psi_*,\delta\psi\big) = \delta\phi_{(*)\rm bg}\big(\psi_{*\rm cr}(\theta_*,\theta)\,,\,\psi_{\rm cr}(\theta_*,\theta)\,,\,\delta\psi_*\,,\,\delta\psi\big) \tag{17.b}$$

With the $\check{\phi}_{(*)\rm bg}(\theta_*,\theta)$ of Proposition 4 and (9),

$$\phi_{(*)\rm bg}\big(\psi_{*\rm cr}(\theta_*,\theta)+\delta\psi_*,\psi_{\rm cr}(\theta_*,\theta)+\delta\psi\big) = \check{\phi}_{(*)\rm bg}(\theta_*,\theta) + \delta\check{\phi}_{(*)\rm bg}\big(\theta_*,\theta;\delta\psi_*,\delta\psi\big) \tag{18}$$

Also define $\delta\check{\phi}^{(+)}_{(*)}\big(\theta_*,\theta;\delta\psi_*,\delta\psi\big)$ by

$$\delta\check{\phi}_{(*)\rm bg}\big(\theta_*,\theta;\delta\psi_*,\delta\psi\big) = S^{(*)}Q_-^*\mathfrak{Q}\,\delta\psi_{(*)} + \delta\check{\phi}^{(+)}_{(*)}\big(\theta_*,\theta;\delta\psi_*,\delta\psi\big) \tag{19}$$

**Remark 11.** By Remark 8, the fields $\delta\check{\phi}_{(*)\rm bg}\big(\theta_*,\theta,\delta\psi_*,\delta\psi\big)$ introduced in (17) obey

$$\delta\check{\phi}_{(*)\rm bg} = S^{(*)}Q_-^*\mathfrak{Q}\,\delta\psi_{(*)} - S^{(*)}P'_{(*)}(\phi_*,\phi)\Big|^{\phi_{(*)}=\check{\phi}_{(*)\rm bg}(\theta_*,\theta)+\delta\check{\phi}_{(*)\rm bg}}_{\phi_{(*)}=\check{\phi}_{(*)\rm bg}(\theta_*,\theta)}$$

In particular, if $P=0$, then $\delta\check{\phi}_{(*)\rm bg} = S^{(*)}Q_-^*\mathfrak{Q}\,\delta\psi_{(*)}$. This is the motivation for the definition of $\delta\check{\phi}^{(+)}_{(*)}$ in (19).

**Lemma 12.** *The function $\delta\mathcal{A}$ appearing in the exponent of the fluctuation integral $\mathcal{F}(\theta_*,\theta)$ of Proposition 4.d is*

$$\begin{aligned} \delta\mathcal{A}(\theta_*,\theta;\delta\psi_*,\delta\psi) = \big\langle\delta\psi_*, C^{-1}\,\delta\psi\big\rangle &- \int_0^1 dt\ \big\langle\delta\psi_*\,,\,\mathfrak{Q}\,Q_-\,\delta\check{\phi}^{(+)}\big(\theta_*,\theta;t\,\delta\psi_*,t\,\delta\psi\big)\big\rangle \\ &- \int_0^1 dt\ \big\langle\mathfrak{Q}\,Q_-\,\delta\check{\phi}^{(+)}_*\big(\theta_*,\theta;t\,\delta\psi_*,t\,\delta\psi\big)\,,\,\delta\psi\big\rangle \end{aligned}$$

*Proof.* Set $\mathcal{B}(\psi_*,\psi)=\mathcal{A}\big(\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)$. As

$$\big(\nabla_{\phi_*}\mathcal{A}\big)\big(\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)=\big(\nabla_{\phi}\mathcal{A}\big)\big(\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)=0$$

we have

$$\begin{aligned}\big(\nabla_{\psi_*}\mathcal{B}\big)\big(\psi_*,\psi\big)&=\big(\nabla_{\psi_*}\mathcal{A}\big)\big(\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)=\mathfrak{Q}\big(\psi-Q_-\,\phi_{\rm bg}(\psi_*,\psi)\big)\\ \big(\nabla_{\psi}\mathcal{B}\big)\big(\psi_*,\psi\big)&=\big(\nabla_{\psi}\mathcal{A}\big)\big(\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)\ \ =\mathfrak{Q}\big(\psi_*-Q_-\,\phi_{*\rm bg}(\psi_*,\psi)\big)\end{aligned}$$

Therefore

$$\begin{aligned}&\mathcal{B}(\psi_*+\delta\psi_*,\psi+\delta\psi)-\mathcal{B}(\psi_*,\psi)\\ &\quad=\int_0^1 dt\,\Big[\big\langle\delta\psi_*\,,\,(\nabla_{\psi_*}\mathcal{B})(\psi_*+t\delta\psi_*,\psi+t\delta\psi)\big\rangle+\big\langle(\nabla_{\psi}\mathcal{B})(\psi_*+t\delta\psi_*,\psi+t\delta\psi)\,,\,\delta\psi\big\rangle\Big]\\ &\quad=\int_0^1 dt\ \big\langle\delta\psi_*\,,\,\mathfrak{Q}(\psi+t\delta\psi)-\mathfrak{Q}\,Q_-\,\phi_{\rm bg}(\psi_*+t\delta\psi_*,\psi+t\delta\psi)\big\rangle\\ &\qquad\qquad+\int_0^1 dt\ \big\langle\mathfrak{Q}(\psi_*+t\delta\psi_*)-\mathfrak{Q}\,Q_-\,\phi_{*\rm bg}(\psi_*+t\delta\psi_*,\psi+t\delta\psi)\,,\,\delta\psi\big\rangle\\ &\quad=\big\langle\delta\psi_*,\mathfrak{Q}\,\delta\psi\big\rangle+\big\langle\delta\psi_*,\mathfrak{Q}\,\psi\big\rangle+\big\langle\psi_*,\mathfrak{Q}\,\delta\psi\big\rangle-I\end{aligned}$$

where

$$\begin{aligned}I=&\int_0^1 dt\ \big\langle\delta\psi_*\,,\,\mathfrak{Q}\,Q_-\,\phi_{\rm bg}(\psi_{*\rm cr}+t\delta\psi_*,\psi_{\rm cr}+t\delta\psi)\big\rangle\\ &\quad+\int_0^1 dt\ \big\langle\mathfrak{Q}\,Q_-\,\phi_{*\rm bg}(\psi_{*\rm cr}+t\delta\psi_*,\psi_{\rm cr}+t\delta\psi)\,,\,\delta\psi\big\rangle\end{aligned}$$

Since

$$\mathcal{A}_{\rm eff}\big(\theta_*,\theta;\psi_*,\psi;\phi_{*\rm bg}(\psi_*,\psi),\phi_{\rm bg}(\psi_*,\psi)\big)=b\,\langle\theta_*-Q\psi_*,\theta-Q\psi\rangle_+ +\mathcal{B}(\psi_*,\psi)$$

we get, using Proposition 4,

$$\begin{aligned}\delta\mathcal{A}&=b\,\langle Q\,\delta\psi_*\,,\,Q\,\delta\psi\rangle_+-b\,\langle Q\,\delta\psi_*\,,\,\theta-Q\psi_{\rm cr}\rangle_+-b\,\langle\theta_*-Q\psi_{*\rm cr}\,,\,Q\,\delta\psi\rangle_+\\ &\qquad+\big\langle\delta\psi_*\,,\,\mathfrak{Q}\,\delta\psi\big\rangle+\big\langle\delta\psi_*\,,\,\mathfrak{Q}\,\psi_{\rm cr}\big\rangle+\big\langle\psi_{*\rm cr}\,,\,\mathfrak{Q}\,\delta\psi\big\rangle-I\\ &=\langle\delta\psi_*\,,\,(bQ^*Q+\mathfrak{Q})\,\delta\psi\rangle+\langle\delta\psi_*\,,\,(bQ^*Q+\mathfrak{Q})\psi_{\rm cr}-bQ^*\theta\rangle\\ &\qquad\qquad+\langle(bQ^*Q+\mathfrak{Q})\psi_{*\rm cr}-bQ^*\theta_*\,,\,\delta\psi\rangle-I\\ &=\langle\delta\psi_*\,,\,(bQ^*Q+\mathfrak{Q})\,\delta\psi\rangle+\big\langle\delta\psi_*\,,\,\mathfrak{Q}\,Q_-\check\phi_{\rm bg}\big\rangle+\big\langle\mathfrak{Q}\,Q_-\check\phi_{*\rm bg}\,,\,\delta\psi\big\rangle-I\end{aligned}$$

$$= \langle \delta\psi_* \, , \, (bQ^*Q + \mathfrak{Q})\, \delta\psi \rangle - \int_0^1 dt \, \big\langle \delta\psi_* \, , \, \mathfrak{Q}\, Q_- \big[ \phi_{\rm bg}(\psi_{*\rm cr} + t\delta\psi_*, \psi_{\rm cr} + t\delta\psi) - \check\phi_{\rm bg} \big] \big\rangle$$
$$- \int_0^1 dt \, \big\langle \mathfrak{Q}\, Q_- \big[ \phi_{*\rm bg}(\psi_{*\rm cr} + t\delta\psi_*, \psi_{\rm cr} + t\delta\psi) - \check\phi_{*\rm bg} \big] \, , \, \delta\psi \big\rangle$$
$$= \big\langle \delta\psi_*, (bQ^*Q + \mathfrak{Q} - \mathfrak{Q}Q_- S Q_-^* \mathfrak{Q})\, \delta\psi \big\rangle - \int_0^1 dt \, \big\langle \delta\psi_*, \mathfrak{Q}\, Q_- \, \delta\check\phi^{(+)}\big(\theta_*, \theta; t\delta\psi_*, t\delta\psi\big) \big\rangle$$
$$- \int_0^1 dt \, \big\langle \mathfrak{Q}\, Q_- \, \delta\check\phi_*^{(+)}\big(\theta_*, \theta; t\delta\psi_*, t\delta\psi\big), \, \delta\psi \big\rangle$$

By the definition of $C$ in (14), this is the desired representation. □

In the course of the arguments above the following simple algebraic observation was used several times.

**Lemma 13.** *Let $V$ and $W$ be vector spaces and let $q : V \to W$, $q_* : W \to V$, $f : V \to V$ and $g : W \to W$ be linear maps. Assume that $f$ and $f + q_* g\, q$ are invertible. Then $\mathbb{1}_W + gq\, f^{-1} q_*$ and $\mathbb{1}_W + q\, f^{-1} q_* g$ are also invertible and*

$$\big(\mathbb{1}_W + gq\, f^{-1} q_*\big)^{-1} = \mathbb{1}_W - gq(f + q_* gq)^{-1} q_*$$
$$\big(\mathbb{1}_W + q\, f^{-1} q_* g\big)^{-1} = \mathbb{1}_W - q(f + q_* gq)^{-1} q_* g$$

*Proof.* Replacing $q$ by $gq$ for the first line and $q_*$ by $q_* g$ for the second, we may assume that $g = \mathbb{1}_W$. Write $\mathbb{1}_W = \mathbb{1}$. Then

$$\big(\mathbb{1} - q(f + q_* q)^{-1} q_*\big)\big(\mathbb{1} + q f^{-1} q_*\big) = \mathbb{1} + q\big[\mathbb{1} - (f + q_* q)^{-1} f - (f + q_* q)^{-1} q_* q\big] f^{-1} q_*$$
$$= \mathbb{1}$$

and similarly $\big(\mathbb{1} + q f^{-1} q_*\big)\big(\mathbb{1} - q(f + q_* q)^{-1} q_*\big) = \mathbb{1}$. □

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