The Small Field Parabolic Flow for Bosonic Many–body Models:
Part 4 — Background and Critical Field Estimates

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the "small field" approximation to the "parabolic flow" which exhibits the formation of a "Mexican hat" potential well. Here we prove the existence of and bounds on the background and critical fields that arise from the steepest descent attack that is at the core of the renormalization group step anaylsis of [5, 6].

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Contents

1 Introduction

In [5, 6], we use the block spin renormalization group formalism to exhibit the formation[1] of a potential well, signalling the onset of symmetry breaking in a many particle system of weakly interacting Bosons in three space dimensions. For an overview, see [1]. For a brief discussion of the algebraic aspects of the block spin method see [4].

In [1, 5, 6] the model is initially formulated as a functional integral with integration variables indexed by the lattice[2,3]

$$\mathcal{X}_0 = \big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$$

$\mathcal{X}_0$ is a unit lattice in the sense that the distance between nearest neighbours in the lattice is 1. During each renormalization group step this lattice is scaled down. In each of the first n_p steps, which are the steps considered in [1, 5, 6], we use (anisotropic) "parabolic scaling" which decreases the lattice spacing in the temporal direction by a factor of L^2 and in the spatial directions by a factor of L. Here $L \geq 3$ is a fixed odd natural number. So after n renormalization group steps the lattice spacing in the spatial directions is $\varepsilon_n = \frac{1}{L^n}$ and in the temporal direction is $\varepsilon_n^2 = \frac{1}{L^{2n}}$ and the lattice $\mathcal{X}_0$ has been scaled down to

$$\mathcal{X}_n = \big(\tfrac{1}{L^{2n}}\mathbb{Z}/\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\tfrac{1}{L^n}\mathbb{Z}^3/\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

We call $\mathcal{X}_n$ the "ε_n–lattice".

The dominant "pure small field" part of the original functional integral representation of this model is, after n renormalization group steps, reexpressed as a functional integral $\int \prod_{x\in\mathcal{X}_0^{(n)}} \frac{d\psi^*(x)d\psi(x)}{2\pi i}\, e^{\mathrm{Action}_n}$ with integration variables indexed by the unit sublattice

$$\mathcal{X}_0^{(n)} = \big(\mathbb{Z}/\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

of $\mathcal{X}_n$. More generally, we have to deal with the decreasing sequence of sublattices

$$\mathcal{X}_j^{(n-j)} = \big(\tfrac{1}{L^{2j}}\mathbb{Z}/\tfrac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\tfrac{1}{L^j}\mathbb{Z}^3/\tfrac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$$

[1] in the small field regime

[2] Of course $\mathcal{X}_0$ is a finite set and so is perhaps more accurately described as a discrete torus, rather than a lattice.

[3] In this introduction, we are only going to give "impressionistic" definitions. The detailed, technically complete, definitions are given in [5, Appendix A]. Specifically, for the lattices, see [5, §A.1].

of $\mathcal{X}_n$. The lower index gives the "scale" of the lattice. That is, the distance between nearest neighbour points of the lattice. The upper index determines the number of points in the sublattice $\big($namely $\big(\frac{L_{\rm tp}}{L^{2(n-j)}}\big)\big(\frac{L_{\rm sp}}{L^{n-j}}\big)^3$ $\big)$. The sum of the upper and lower indices gives the number of the renormalization group step. For fields ϕ, ψ on $\mathcal{X}_j^{(n-j)}$, we use the "real" inner product $\langle \phi, \psi \rangle_j = \frac{1}{L^{5j}} \sum_{u \in \mathcal{X}_j^{(n-j)}} \phi(u)\psi(u)$. The vector space $\mathbb{C}^{\mathcal{X}_j^{(n-j)}}$, equipped with the inner product $\langle \phi^*, \psi \rangle$, is a Hilbert space, which we denote $\mathcal{H}_j^{(n-j)}$.

Roughly speaking, in each block spin RG step one

- paves $\mathcal{X}_0^{(n)}$ by rectangles centered at the points of the sublattice $\mathcal{X}_{-1}^{(n+1)} \subset \mathcal{X}_0^{(n)}$ and then,
- for each $y \in \mathcal{X}_{-1}^{(n+1)}$, integrates out all values of ψ whose "average value" over the rectangle centered at y is equal to the value of a given field $\theta(y)$ on $\mathcal{X}_{-1}^{(n+1)}$. The precise "average value" used is determined by an averaging profile[4]. One uses this profile to define[5] an averaging operator Q from the space of fields on $\mathcal{X}_0^{(n)}$ to the space of fields on $\mathcal{X}_{-1}^{(n+1)}$. One then implements the "integrating out" by first inserting into the integrand 1, expressed as a constant times the Gaussian integral

$$\int \prod_{y \in \mathcal{X}_{-1}^{(n+1)}} \frac{d\theta^*(y) d\theta(y)}{2\pi i} e^{-a\langle \theta^* - Q\,\psi_*\,,\, \theta - Q\,\psi \rangle_{-1}} \tag{1.1}$$

  with some constant $a > 0$, and then interchanging the order of the θ and ψ integrals.

We use stationary phase/steepest descent to control these integrals. This naturally leads one to express the action not solely in terms of the integration variables ψ, but also in terms of "background fields", which are concatinations of "steepest descent" critical field maps for all previous steps. See [4, Remark 1 and Proposition 4.c]. The dominant part of the action is then of the form

$$A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V})\Big|_{\substack{\phi_* = \phi_{*n}(\psi^*, \psi) \\ \phi = \phi_n(\psi^*, \psi)}}$$

where

$$\begin{aligned} A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V}) = - \big\langle \psi_* - Q_n \phi_*, \mathfrak{Q}_n \big(\psi - Q_n \phi\big) \big\rangle_0 - \langle \phi_*,\, D_n \phi \rangle_n \\ - \mathcal{V}(\phi_*, \phi) + \mu \, \langle \phi_*,\, \phi \rangle_n \end{aligned} \tag{1.2}$$

and

[4]In [5, 6], the averaging profile is an iterated convolution of the characteristic function of the rectangle with itself. See [5, §A.3]

[5]For the detailed definition of the averaging operator Q, see [5, §A.3].

- $Q_n : \mathbb{C}^{\mathcal{X}_n} \to \mathbb{C}^{\mathcal{X}_0^{(n)}}$ is an averaging operator that is the composition of the averaging operations for all previous steps. For the precise definition of Q_n, see [5, §A.3]. For bounds on Q_n, see [2, Remark 2.1.a and Lemma 2.2].
- the term $\langle \psi^* - Q_n\,\phi_*\,,\,\mathfrak{Q}_n(\psi - Q_n\,\phi)\rangle_0$ is a residue of the exponents in the Gaussian integrals (1.1) inserted in the previous steps. The operator $\mathfrak{Q}_n$ is bounded and boundedly invertible. For the precise definition of $\mathfrak{Q}_n$, see [5, §A.3]. See [4, Remark 1] for the recursion relation that builds $\mathfrak{Q}_n$. For bounds on $\mathfrak{Q}_n$, see [2, Remark 2.1.c and Proposition 2.4].
- D_n is a discrete differential operator. It is simply a scaled version of the discrete differential operator that appeared in the initial action, which, in turn, was built from the single particle "kinetic energy" operator. Think of D_n as behaving like $-\partial_0 - \Delta$. For the detailed definition of D_n, see [5, §A.4]. Various properties of and bounds on D_n are provided in [2, §3].
- $\mathcal{V}$ is an interaction. It is a quartic monomial

$$\mathcal{V}(\phi_*,\phi) = \tfrac{1}{2}\int_{\mathcal{X}_n^4} du_1 \cdots du_4\ V(u_1,u_2,u_3,u_4)\,\phi_*(u_1)\phi(u_2)\phi_*(u_3)\phi(u_4)$$

  where $\int_{\mathcal{X}_n} du = \frac{1}{L^{5n}}\sum_{u\in\mathcal{X}_n}$ and the kernel $V(u_1,u_2,u_3,u_4)$ is translation invariant and exponentially decaying.
- μ is a chemical potential. In this paper, we are interested in $\mu > 0$ that are sufficiently small. For more details, see [5, Theorem 1.17].
- The background fields[6] $\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})$, in addition to being concatinations of "steepest descent" critical field maps for all previous steps, are critical points for the map

$$(\phi_*,\phi) \mapsto A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V})$$

In this paper we fix an integer $1 \le n \le n_p$, where n_p is the number of "parabolic scaling" renormalization group steps considered in [5, 6], and prove existence and properties of the background fields as above, in the concrete setting of [5, 6]. By definition, they are solutions of the "background field equations"

$$\tfrac{\partial}{\partial\phi_*}A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V}) = \tfrac{\partial}{\partial\phi}A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V}) = 0$$

or

$$\begin{aligned} S_n^*(\mu)^{-1}\phi_* + \mathcal{V}_*'(\phi_*,\phi,\phi_*) &= Q_n^*\mathfrak{Q}_n\psi_* \\ S_n(\mu)^{-1}\phi + \mathcal{V}'(\phi,\phi_*,\phi) &= Q_n^*\mathfrak{Q}_n\psi \end{aligned} \tag{1.3}$$

[6] We routinely use the "optional $*$" notation $\alpha_{(*)}$ to denote "α_* or α".

where[7]

$$S_n(\mu) = (D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu)^{-1}$$

and

$$\begin{aligned}
\mathcal{V}'_*(u;\zeta_{*1},\zeta,\zeta_{*2}) &= \int du_1 du_2 du_3 \ V(u_1,u_2,u_3,u)\,\zeta_{*1}(u_1)\zeta(u_2)\zeta_{*2}(u_3) \\
\mathcal{V}'(u;\zeta_1,\zeta_*,\zeta_2) &= \int du_2 du_3 du_4 \ V(u,u_2,u_3,u_4)\,\zeta_1(u_2)\zeta_*(u_3)\zeta_2(u_4)
\end{aligned}$$

We also write $S_n = S_n(0) = (D_n + Q_n^* \mathfrak{Q}_n Q_n)^{-1}$.

In §2 we write these equations as a fixed point equation and use the variant of the Banach fixed point theorem developed in [3], and summarized in Proposition A.1, to control them. We also show, in Proposition 2.1, that

$$\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) = S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n\, \psi_{(*)} + \phi_{(*)n}^{(\ge 3)}(\psi_*,\psi,\mu,\mathcal{V})$$

where $\phi_{(*)n}^{(\ge 3)}$ are analytic maps in (ψ_*,ψ) from a neighbourhood of the origin in $\mathbb{C}^{\mathcal{X}_0^{(n)}} \times \mathbb{C}^{\mathcal{X}_0^{(n)}}$ to $\mathbb{C}^{\mathcal{X}_n}$, and, in Corollary 2.5, that

$$\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})(u) = \tfrac{a_n}{a_n-\mu}\psi_{(*)}\big(X(u)\big) + \check{\phi}_{(*)n}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\,,\,\mu,\mathcal{V}\big)(u) \tag{1.4}$$

where, for each point u of the fine lattice $\mathcal{X}_n$, $X(u)$ denotes the point of the unit lattice $\mathcal{X}_0^{(n)}$ nearest to u, $a_n = a\big(1+\sum_{j=1}^{n-1}\frac{1}{L^{2j}}\big)^{-1}$ and $\check{\phi}_{(*)n}$ are analytic maps.

Remark 1.1. When the fields $\psi_{(*)}$ and $\phi_{(*)}$ happen to be constant, then, by [6, Remark B.7], the equations (1.3) reduce to

$$\begin{aligned}
(a_n-\mu)\phi_* + \mathrm{v}\phi_*^2\phi &= a_n\psi_* \\
(a_n-\mu)\phi + \mathrm{v}\phi_*\phi^2 &= a_n\psi
\end{aligned} \tag{1.5}$$

where $\mathrm{v} = \int_{\mathcal{X}_n^3} dx_1\cdots dx_3\ V(0,x_1,x_2,x_3)$ is the average value of the kernel of $\mathcal{V}$. As long as $\mathrm{v}(|\psi_*|+|\psi|)^2$ is small enough, this system has a unique solution with

$$\phi_* = \tfrac{a_n}{a_n-\mu}\psi_* + O\big(\mathrm{v}(|\psi_*|+|\psi|)^3\big) \qquad \phi = \tfrac{a_n}{a_n-\mu}\psi + O\big(\mathrm{v}(|\psi_*|+|\psi|)^3\big)$$

If $\psi_* = \psi^*$, then the solution $\phi_* = \phi^*$.

[7] The number of RG steps, n_p, is chosen so that, for the chemical potentials μ under consideration, the operator $D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu$ is invertible.

In §3, we prove, in Proposition 3.1, bounds on maps which describe the variations of the background field with respect to ψ.

In §4, we consider variations of the background field with respect to the chemical potential μ and interaction $\mathcal{V}$. We prove, in Proposition 4.1, bounds on

$$\Delta\phi_{(*)n}(\psi_*,\psi,\mu,\delta\mu,\mathcal{V},\delta\mathcal{V}) = \phi_{(*)n}(\psi_*,\psi,\mu+\delta\mu,\mathcal{V}+\delta\mathcal{V}) - \phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})$$

as well as on ∂_ν and $D_n^{(*)}$ applied to these field maps.

Finally, in §5 we apply these results and [4, Proposition 4.a] to construct and bound the critical points, denoted ψ_{*n}, ψ_n, of the map

$$(\psi_*,\psi) \mapsto A_n(\psi_*,\psi,\phi_*,\phi,\mu,\mathcal{V})\Big|_{\substack{\phi_*=\phi_{*n}(\psi^*,\psi)\\ \phi=\phi_n(\psi^*,\psi)}}$$

The proofs and estimates in this paper depend heavily on bounds on operators like Q, Q_n and $S_n^{-1}(\mu)$, which in turn are developed in [2]. The size of an operator is formulated in terms of a norm on its kernel.

Definition 1.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be sublattices of a common lattice having metric d, with $\mathcal{X}$ having a "cell volume" $\mathrm{vol}_\mathcal{X}$ and with $\mathcal{Y}$ having a "cell volume" $\mathrm{vol}_\mathcal{Y}$. For any operator $A:\mathbb{C}^\mathcal{X}\to\mathbb{C}^\mathcal{Y}$, with kernel $A(y,x)$, and for any mass $m\ge 0$, we define the norm

$$\|A\|_m = \max\Big\{\sup_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}}\mathrm{vol}_\mathcal{X}\ e^{m|y-x|}|A(y,x)|\ ,\ \sup_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}\mathrm{vol}_\mathcal{Y}\ e^{m|y-x|}|A(y,x)|\Big\}$$

In the special case that $m=0$, this is just the usual ℓ^1–ℓ^∞ norm of the kernel.

Similarly, to measure the size of a function $f:\big(\mathcal{X}_j^{(n-j)}\big)^r\to\mathbb{C}$, we introduce the weighted ℓ^1–ℓ^∞ norm with mass $m\ge 0$

$$\|f\big(x_1,\cdots x_r\big)\|_m = \max_{i=1\cdots,r}\ \max_{x\in\mathcal{X}_j^{(n-j)}}\ \tfrac{1}{L^{5j}}\sum_{\substack{x_1,\cdots,x_r\in\mathcal{X}_j^{(n-j)}\\ x_i=x}}|f(x_1,\cdots,x_r)|\,e^{m\tau(x_1,\cdots,x_r)} \qquad (1.6)$$

where the tree length $\tau(x_1,\cdots,x_r)$ is the minimal length of a tree in $\mathcal{X}_j^{(n-j)}$ that has $x_1,\cdots,x_r$ among its vertices.

We use the terminology "field map" to designate an analytic map that assigns to one or more fields on a finite set $\mathcal{X}$ another field on a finite set $\mathcal{Y}$. The most prominent examples of field maps in this paper are the background fields $\phi_{(*)n}(\psi_*,\psi)$. In Appendix A, we define norms on field maps that are constructed by summing norms, like

(1.6), of the kernels in their power series expansions. The kernel of a monomial, for example of degree n in a field ψ, is weighted by κ^n, where κ is a "weight factor" assigned to ψ. For example, if $\phi(\psi)(y) = \sum_{n=0}^{\infty} \sum_{x_1,\cdots,x_n\in\mathcal{X}} \mathrm{vol}_{\mathcal{X}}^n\ \phi_n\big(y;x_1,\cdots,x_n\big)\psi(x_1)\cdots\psi(x_n)$

$$|||\phi||| = \sum_n \|\phi_n\|_m\ \kappa^n$$

For full definitions of our norms, see [5, §A.5].

In this paper, we fix masses $\bar{\mathfrak{m}} > \mathfrak{m} > 0$ and generic weight factors $\mathfrak{k}, \mathfrak{k}', \mathfrak{k}_{\mathfrak{l}} \geq 1$ and use the norm $|||F|||$ with mass $\mathfrak{m}$ and these weight factors to measure field maps F. The weight factor $\mathfrak{k}$ is used for the $\psi_{(*)}$'s, the weight factor $\mathfrak{k}'$ is used for the derivative fields $\psi_{(*)\nu}$ and the weight factor $\mathfrak{k}_{\mathfrak{l}}$ is used for the fluctuation fields $z_{(*)}$. See Appendix A.

Convention 1.3. The (finite number of) constants that appear in the bounds of this paper are consecutively labelled $K_1, K_2, \cdots$ or $\rho_1, \rho_2, \cdots$. All of the constants K_j, ρ_j are independent of L and the scale index n. They depend only on the masses $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ and the constant Γ_{op} of [2, Convention 1.2] (with mass $m = \bar{\mathfrak{m}}$) and, for the ρ_j's, the μ_{up} of [2, Proposition 5.1]. We define K_{bg} to be the maximum of the K_j's and ρ_{bg} to be the minimum of $\frac{1}{8}$ and the ρ_j's. We shall refer only to K_{bg} and ρ_{bg}, as opposed to the K_j's and ρ_j's, in [5, 6].

2 The Background Field

The main existence result for the background field, which was summarized in [5, Proposition 1.14], is

Proposition 2.1 (Existence of the background field)**.** *There are constants* K_1*,* $\rho_1 > 0$ *such that, if* $\|V\|_{\mathfrak{m}}\mathfrak{k}^2 + |\mu| \le \rho_1$*, the following hold.*

(a) There exist solutions to the equations (1.3) *for the background field. Precisely, there are field maps* $\phi^{(\ge 3)}_{(*)n}$ *such that*

$$\phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V}) = S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n \, \psi_{(*)} + \phi^{(\ge 3)}_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})$$

solves (1.3) *and*

$$\big|\!\big|\!\big| \phi^{(\ge 3)}_{(*)n} \big|\!\big|\!\big| \le K_1 \|V\|_{\mathfrak{m}} \mathfrak{k}^3$$

Furthermore $\phi^{(\ge 3)}_{*n}$ *is of degree at least one in* ψ_* *and* $\phi^{(\ge 3)}_{n}$ *is of degree at least one in* ψ*. Both are of degree at least three in* (ψ_*, ψ)*.*

(b) Set

$$B^{(+)}_{n,\mu,\nu} = \big[D_n^* + Q^{(+)}_{n,\nu} \mathfrak{Q}_n Q^{(-)}_{n,\nu} - \mu\big]^{-1} Q^{(+)}_{n,\nu} \mathfrak{Q}_n$$
$$B^{(-)}_{n,\mu,\nu} = \big[D_n + Q^{(+)}_{n,\nu} \mathfrak{Q}_n Q^{(-)}_{n,\nu} - \mu\big]^{-1} Q^{(+)}_{n,\nu} \mathfrak{Q}_n$$

where $Q^{(+)}_{n,\nu}, Q^{(-)}_{n,\nu}$ *were defined in [2, (2.11)]. There are, for each* $0 \le \nu \le 3$*, field maps* $\phi^{(\ge 3)}_{(*)n,\nu} = \phi^{(\ge 3)}_{(*)n,\nu}\big(\psi_*, \psi, \psi_{*\nu}, \psi_\nu, \mu, \mathcal{V}\big)$ *such that*

$$\partial_\nu \phi_{*n}(\psi_*, \psi, \mu, \mathcal{V}) = B^{(+)}_{n,\mu,\nu} \partial_\nu \psi_* + \phi^{(\ge 3)}_{*n,\nu}\big(\psi_*, \psi, \partial_\nu \psi_*, \partial_\nu \psi, \mu, \mathcal{V}\big)$$
$$\partial_\nu \phi_{n}(\psi_*, \psi, \mu, \mathcal{V}) = B^{(-)}_{n,\mu,\nu} \partial_\nu \psi + \phi^{(\ge 3)}_{n,\nu}\big(\psi_*, \psi, \partial_\nu \psi_*, \partial_\nu \psi, \mu, \mathcal{V}\big)$$

and

$$\big|\!\big|\!\big| \phi^{(\ge 3)}_{(*)n,\nu} \big|\!\big|\!\big| \le K_1 \, \|V\|_{\mathfrak{m}} \mathfrak{k}^2 \, \mathfrak{k}'$$

Furthermore $\partial_\nu \phi^{(\ge 3)}_{(*)n}(\psi_*, \psi, \mu, \mathcal{V}) = \phi^{(\ge 3)}_{(*)n,\nu}\big(\psi_*, \psi, \partial_\nu \psi_*, \partial_\nu \psi, \mu, \mathcal{V}\big)$*, and* $\phi^{(\ge 3)}_{*n,\nu}$ *and* $\phi^{(\ge 3)}_{n,\nu}$ *are each of degree precisely one in* $\psi_{(*)\nu}$ *and of degree at least two in* $\big(\psi_*, \psi\big)$*.*

(c) Set

$$B^{(+)}_{n,\mu,D} = \big[\mathbb{1} - (Q_n^*\mathfrak{Q}_n Q_n - \mu) S_n(\mu)^*\big] Q_n^*\mathfrak{Q}_n$$
$$B^{(-)}_{n,\mu,D} = \big[\mathbb{1} - (Q_n^*\mathfrak{Q}_n Q_n - \mu) S_n(\mu)\big] Q_n^*\mathfrak{Q}_n$$

There are field maps $\phi^{(\geq 3)}_{(*)n,D}$ *such that*

$$D_n^*\phi_{*n}(\psi_*,\psi,\mu,\mathcal{V}) = B^{(+)}_{n,\mu,D}\,\psi_* + \phi^{(\geq 3)}_{*n,D}\big(\psi_*,\psi,\mu,\mathcal{V}\big)$$
$$D_n\phi_n(\psi_*,\psi,\mu,\mathcal{V}) = B^{(-)}_{n,\mu,D}\,\psi + \phi^{(\geq 3)}_{n,D}\big(\psi_*,\psi,\mu,\mathcal{V}\big)$$

and

$$\big|\big|\big|\phi^{(\geq 3)}_{(*)n,D}\big|\big|\big| \leq K_1 \,\|V\|_{\mathfrak{m}}\mathfrak{k}^3$$

Furthermore $\phi^{(\geq 3)}_{(*)n,D}$ *are of degree at least three in* $\big(\psi_*,\psi\big)$*.*

Proof. (a) We shall write the equations (1.3) for $\phi_{(*)}(\psi_*,\psi,\mu,\mathcal{V})$ in the form

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha},\vec{\gamma}) + \vec{B}\big(\vec{\alpha}\,;\,\vec{\gamma}\big) \tag{2.1}$$

as in Appendix A or in [3, (4.1.b)] with $X = \mathcal{X}_n$. In particular, we shall use Proposition A.1 to supply solutions to those equations. Substituting

$$\alpha_* = Q_n^*\mathfrak{Q}_n\psi_* \qquad \alpha = Q_n^*\mathfrak{Q}_n\psi \qquad \vec{\alpha} = \big(\alpha_1,\alpha_2\big) = \big(\alpha_*,\alpha\big)$$
$$\phi_* = S_n(\mu)^*\big(\alpha_* + \gamma_*\big) \qquad \phi = S_n(\mu)\big(\alpha+\gamma\big) \qquad \vec{\gamma} = \big(\gamma_1,\gamma_2\big) = \big(\gamma_*,\gamma\big)$$

into (1.3) gives

$$\gamma_* + \mathcal{V}_*'\big(S_n(\mu)^*(\alpha_*+\gamma_*)\,,\,S_n(\mu)(\alpha+\gamma)\,,\,S_n(\mu)^*(\alpha_*+\gamma_*)\big) = 0$$
$$\gamma + \mathcal{V}'\big(S_n(\mu)(\alpha+\gamma)\,,\,S_n(\mu)^*(\alpha_*+\gamma_*)\,,\,S_n(\mu)(\alpha+\gamma)\big) = 0$$

We have the desired form with

$$\vec{f}(\vec{\alpha})(u) = \begin{bmatrix} -\mathcal{V}'_*\big(u; S_n(\mu)^*\alpha_*, S_n(\mu)\alpha, S_n(\mu)^*\alpha_*\big) \\ -\mathcal{V}'\big(u; S_n(\mu)\alpha, S_n(\mu)^*\alpha_*, S_n(\mu)\alpha\big) \end{bmatrix}$$

$$\vec{L}(\vec{\alpha};\vec{\gamma})(u) = \begin{bmatrix} -\mathcal{V}'_*\big(u; S_n(\mu)^*\alpha_*, S_n(\mu)\gamma, S_n(\mu)^*\alpha_*\big) \\ \qquad -2\mathcal{V}'_*\big(u; S_n(\mu)^*\alpha_*, S_n(\mu)\alpha, S_n(\mu)^*\gamma_*\big) \\ -\mathcal{V}'\big(u; S_n(\mu)\alpha, S_n(\mu)^*\gamma_*, S_n(\mu)\alpha\big) \\ \qquad -2\mathcal{V}'\big(u; S_n(\mu)\alpha, S_n(\mu)^*\alpha_*, S_n(\mu)\gamma\big) \end{bmatrix}$$

$$\vec{B}(\vec{\alpha};\vec{\gamma})(u) = \begin{bmatrix} -\mathcal{V}'_*\big(u; S_n(\mu)^*\gamma_*, S_n(\mu)\alpha, S_n(\mu)^*\gamma_*\big) \\ \qquad -2\mathcal{V}'_*\big(u; S_n(\mu)^*\gamma_*, S_n(\mu)\gamma, S_n(\mu)^*\alpha_*\big) \\ \qquad -\mathcal{V}'_*\big(u; S_n(\mu)^*\gamma_*, S_n(\mu)\gamma, S_n(\mu)^*\gamma_*\big) \\ -\mathcal{V}'\big(u; S_n(\mu)\gamma, S_n(\mu)^*\alpha_*, S_n(\mu)\gamma\big) \\ \qquad -2\mathcal{V}'\big(u; S_n(\mu)\gamma, S_n(\mu)^*\gamma_*, S_n(\mu)\alpha\big) \\ \qquad -\mathcal{V}'\big(u; S_n(\mu)\gamma, S_n(\mu)^*\gamma_*, S_n(\mu)\gamma\big) \end{bmatrix}$$

Here $V(u_1,u_2,u_3,u_4)$ is the kernel of $\mathcal{V}$ that has the symmetries

$$V(u_1,u_2,u_3,u_4) = V(u_3,u_2,u_1,u_4) = V(u_1,u_4,u_3,u_2) \tag{2.2}$$

Now apply [3, Proposition 4.1.a and Remark 3.5.a], or Proposition A.1, with $r = s = 2$ and

$$d_{\max} = 3 \qquad \mathfrak{c} = \tfrac{1}{2} \qquad \kappa_1 = \kappa_2 = \|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\,\mathfrak{k} \qquad \lambda_1 = \lambda_2 = \mathfrak{k}$$

(and the metric on X being $\mathfrak{m}$ times the metric on $\mathcal{X}_n$). Since

$$\begin{aligned} |||f_j|||_w &\le \|S_n(\mu)\|_{\mathfrak{m}}^3 \|V\|_{\mathfrak{m}} \kappa_1\kappa_2\kappa_j \\ &\le 8\|S_n\|_{\mathfrak{m}}^3 \|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}^3 \, \|V\|_{\mathfrak{m}} \mathfrak{k}^3 \\ |||L_j|||_{w_{\kappa,\lambda}} &\le \|S_n(\mu)\|_{\mathfrak{m}}^3 \|V\|_{\mathfrak{m}} \big(2\kappa_1\kappa_2\lambda_j + \kappa_j^2\lambda_{3-j}\big) \\ &\le 24\|S_n\|_{\mathfrak{m}}^3 \|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}^2 \, \|V\|_{\mathfrak{m}} \mathfrak{k}^3 \\ |||B_j|||_{w_{\kappa,\lambda}} &\le \|S_n(\mu)\|_{\mathfrak{m}}^3 \, \|V\|_{\mathfrak{m}} \big[\kappa_{3-j}\lambda_j^2 + 2\kappa_j\lambda_j\lambda_{3-j} + \lambda_j^2\lambda_{3-j}\big] \\ &\le 8\|S_n\|_{\mathfrak{m}}^3 \big(3\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}} + 1\big) \, \|V\|_{\mathfrak{m}} \mathfrak{k}^3 \end{aligned}$$

assuming that ρ_1 has been chosen small enough that $\|S_n(\mu)\|_{\mathfrak{m}} \le 2\|S_n\|_{\mathfrak{m}}$. By hypothesis, $|||f_j|||_w$, $|||L_j|||_{w_{\kappa,\lambda}}$, $|||B_j|||_{w_{\kappa,\lambda}} < \frac{1}{8}\lambda_j$ and [3, Proposition 4.1.a] gives a solution $\vec{\Gamma}(\vec{\alpha})$ to (2.1) that obeys the bound

$$|||\Gamma_j|||_w \le 16\|S_n\|_{\mathfrak{m}}^3 \|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}^3 \, \|V\|_{\mathfrak{m}} \mathfrak{k}^3$$

Hence

$$\begin{aligned}
\phi_* = \phi_*(\psi_*, \psi, \mu, \mathcal{V}) &= S_n(\mu)^* \alpha_*(\psi_*) + S_n(\mu)^* \Gamma_1\big(\alpha_*(\psi_*), \alpha(\psi)\big) \\
&= S_n(\mu)^* Q_n^* \mathfrak{Q}_n \psi_* + S_n(\mu)^* \Gamma_1\big(Q_n^* \mathfrak{Q}_n \psi_*, Q_n^* \mathfrak{Q}_n \psi\big) \\
\phi = \phi(\psi_*, \psi, \mu, \mathcal{V}) &= S_n(\mu) \alpha(\psi) + S_n(\mu) \Gamma_2\big(\alpha_*(\psi_*), \alpha(\psi)\big) \\
&= S_n(\mu) Q_n^* \mathfrak{Q}_n \psi + S_n(\mu) \Gamma_2\big(Q_n^* \mathfrak{Q}_n \psi_*, Q_n^* \mathfrak{Q}_n \psi\big)
\end{aligned}$$

and [3, Corollary 3.3] yields all of the claims.

(b) We denote $\phi_{(*)} = \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})$. Set

$$S^{(+)} = \big[D_n^* + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu\big]^{-1} \qquad S^{(-)} = \big[D_n + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu\big]^{-1}$$

By [2, Proposition 5.1], with $S^{(\pm)} = S_{n,\nu}^{(\pm)}(\mu)$, we have $\|S^{(\pm)}\|_{\mathfrak{m}} \le \Gamma_{\rm op}$, assuming that ρ_1 has been chosen small enough. By [2, (5.1) and Remark 2.5], applying ∂_ν to (1.3), and then replacing $\partial_\nu \phi_{(*)}$ by $\phi_{(*)\nu}$ and $\partial_\nu \psi_{(*)}$ by $\psi_{(*)\nu}$ gives

$$\begin{aligned}
\big(S^{(+)}\big)^{-1} \phi_{*\nu} + \mathcal{V}_*'\big(\phi_{*\nu}\,,\, T_\nu^{-1}\phi\,,\, \phi_* + T_\nu^{-1}\phi_*\big) + \mathcal{V}_*'\big(\phi_*\,,\, \phi_\nu\,,\, \phi_*\big) &= Q_{n,\nu}^{(+)} \mathfrak{Q}_n \psi_{*\nu} \\
\big(S^{(-)}\big)^{-1} \phi_\nu + \mathcal{V}'\big(\phi_\nu\,,\, T_\nu^{-1}\phi_*\,,\, \phi + T_\nu^{-1}\phi\big) + \mathcal{V}'\big(\phi\,,\, \phi_{*\nu}\,,\, \phi\big) &= Q_{n,\nu}^{(+)} \mathfrak{Q}_n \psi_\nu
\end{aligned} \tag{2.3}$$

with T_ν being the translation operator by the lattice basis vector in direction ν. Here we have used the translation invariance of V, the symmetries (2.2) and the "discrete product rule"

$$\partial_\nu\big(fg\big) = (\partial_\nu f)(T_\nu^{-1} g) + f \partial_\nu g \tag{2.4}$$

in the forms

$$\begin{aligned}
\partial_\nu(fgh) &= (\partial_\nu f)(T_\nu^{-1} g)(T_\nu^{-1} h) + f(\partial_\nu g)(T_\nu^{-1} h) + fg(\partial_\nu h) \\
\partial_\nu(fgf) &= (\partial_\nu f)(T_\nu^{-1} g)(T_\nu^{-1} f) + f(T_\nu^{-1} g)(\partial_\nu f) + f(\partial_\nu g) f
\end{aligned} \tag{2.5}$$

The equations (2.3) are of the form

$$\vec\gamma = \vec f(\vec\alpha) + \vec L(\vec\alpha, \vec\gamma) + \vec B\big(\vec\alpha\,;\, \vec\gamma\big) \tag{2.6}$$

as in [3, (4.1.b)], with

$$\begin{aligned}
\alpha_* = \phi_* \quad \alpha = \phi \quad \alpha_{*\nu} &= Q_{n,\nu}^{(+)} \mathfrak{Q}_n \psi_{*\nu} & \alpha_\nu &= Q_{n,\nu}^{(+)} \mathfrak{Q}_n \psi_\nu & \vec\alpha &= \big(\alpha_*, \alpha, \alpha_{*\nu}, \alpha_\nu\big) \\
\phi_{*\nu} &= S^{(+)}(\alpha_{*\nu} + \gamma_*) & \phi_\nu &= S^{(-)}(\alpha_\nu + \gamma) & \vec\gamma &= \big(\gamma_*, \gamma\big)
\end{aligned}$$

and

$$\vec{f}(\vec{\alpha}) = -\begin{bmatrix} \mathcal{V}_*'\big(S^{(+)}\alpha_{*\nu}\,,\, T_\nu^{-1}\alpha\,,\, \alpha_* + T_\nu^{-1}\alpha_*\big) + \mathcal{V}_*'\big(\alpha_*\,,\, S^{(-)}\alpha_\nu\,,\, \alpha_*\big) \\ \mathcal{V}'\big(S^{(-)}\alpha_\nu\,,\, T_\nu^{-1}\alpha_*\,,\, \alpha + T_\nu^{-1}\alpha\big) + \mathcal{V}'\big(\alpha\,,\, S^{(+)}\alpha_{*\nu}\,,\, \alpha\big) \end{bmatrix}$$

$$\vec{L}(\vec{\alpha};\vec{\gamma}) = -\begin{bmatrix} \mathcal{V}_*'\big(S^{(+)}\gamma_*\,,\, T_\nu^{-1}\alpha\,,\, \alpha_* + T_\nu^{-1}\alpha_*\big) + \mathcal{V}_*'\big(\alpha_*\,,\, S^{(-)}\gamma\,,\, \alpha_*\big) \\ \mathcal{V}'\big(S^{(-)}\gamma\,,\, T_\nu^{-1}\alpha_*\,,\, \alpha + T_\nu^{-1}\alpha\big) + \mathcal{V}'\big(\alpha\,,\, S^{(+)}\gamma_*\,,\, \alpha\big) \end{bmatrix}$$

$$\vec{B}(\vec{\alpha};\vec{\gamma}) = 0$$

Now apply [3, Proposition 4.1.a] with $\mathfrak{c} = \frac{1}{2}$ and

$$\kappa_1 = \kappa_2 = \Gamma_{\rm op}\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k} + K_1\|V_n\|_{\mathfrak{m}}\mathfrak{k}^3$$
$$\lambda_1 = \lambda_2 = \kappa_3 = \kappa_4 = \|Q_{n,\nu}^{(+)}\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k}'$$

Since

$$|||f_j|||_w \le \max_{\sigma=+,-}\|S_\mu^{(\sigma)}\|_{\mathfrak{m}}\,\|V_n\|_{\mathfrak{m}}\Big[2e^{2\varepsilon_n\mathfrak{m}}\kappa_1\kappa_2\kappa_{2+j} + \kappa_j^2\kappa_{5-j}\Big] \le b\lambda_j$$
$$|||L_j|||_{w_{\kappa,\lambda}} \le \max_{\sigma=+,-}\|S_\mu^{(\sigma)}\|_{\mathfrak{m}}\,\|V_n\|_{\mathfrak{m}}\Big[2e^{2\varepsilon_n\mathfrak{m}}\kappa_1\kappa_2\lambda_j + \kappa_j^2\lambda_{3-j}\Big] \le b\lambda_j$$
$$|||B_j|||_{w_{\kappa,\lambda}} = 0$$

where $\varepsilon_n = \frac{1}{L^n}$ and

$$b = 3\max_{\sigma=+,-}\|S_\mu^{(\sigma)}\|_{\mathfrak{m}}\; e^{2\varepsilon_n\mathfrak{m}}\Big[\Gamma_{\rm op}\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}} + K_1\|V\|_{\mathfrak{m}}\mathfrak{k}^2\Big]^2\|V\|_{\mathfrak{m}}\mathfrak{k}^2 \le {\rm const}\|V\|_{\mathfrak{m}}\mathfrak{k}^2 \le \tfrac{1}{4}$$

by the hypotheses, [3, Propositions 4.1.a] gives a solution $\vec{\Gamma}(\vec{\alpha})$ to (2.6) with

$$|||\Gamma_1|||_{w_{\kappa,\lambda}}, |||\Gamma_2|||_{w_{\kappa,\lambda}} \le K_1'\|V\|_{\mathfrak{m}}\mathfrak{k}^2\mathfrak{k}'$$

As (2.6) is a linear system of equations and $b \le \frac{1}{4}$, the solution is unique. Correspondingly

$$\phi_{*\nu} = B_{n,\mu,\nu}^{(+)}\psi_{*\nu} + S^{(+)}\Gamma_1\big(\alpha_*(\phi_*), \alpha(\phi), \alpha_{*\nu}(\psi_{*\nu}), \alpha_\nu(\psi_\nu)\big)$$
$$\phi_\nu = B_{n,\mu,\nu}^{(-)}\psi_\nu + S^{(-)}\Gamma_2\big(\alpha_*(\phi_*), \alpha(\phi), \alpha_{*\nu}(\psi_{*\nu}), \alpha_\nu(\psi_\nu)\big)$$

solves (2.3). The conclusion now follows by part (a) and [3, Corollary 3.3].

That $\partial_\nu\phi_{(*)n}^{(\ge 3)}(\psi_*, \psi, \mu, \mathcal{V}) = \phi_{(*)n,\nu}^{(\ge 3)}\big(\psi_*,\psi, \partial_\nu\psi_*, \partial_\nu\psi, \mu, \mathcal{V}\big)$ follows from the observation that $\partial_\nu S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n = B_{n,\mu,\nu}^{(\pm)}$, by [2, (5.1) and Remark 2.5].

(c) From (1.3) we see

$$\begin{aligned} D_n^* \phi_* &= Q_n^* \mathfrak{Q}_n \psi_* - \big(Q_n^* \mathfrak{Q}_n Q_n - \mu \big) \phi_* - \mathcal{V}_*'(\phi_*, \phi, \phi_*) \\ D_n \phi &= Q_n^* \mathfrak{Q}_n \psi - \big(Q_n^* \mathfrak{Q}_n Q_n - \mu \big) \phi - \mathcal{V}'(\phi, \phi_*, \phi) \end{aligned}$$

with $\phi_{(*)} = \phi_{(*)n}$. Now just substitute for $\phi_{(*)n}$ using part (a). □

Remark 2.2 (The complex conjugate of the background field)**.** Assume that the constants K_1, $\rho_1 > 0$ of Proposition 2.1 are chosen big enough and small enough, respectively, and fulfil its hypotheses. Let $\psi(x)$ be a field on $\mathcal{X}_0^{(n)}$ such that $|\psi(x)| < \mathfrak{k}$ and $|\partial_\nu \psi(x)| < \mathfrak{k}'$ for all $x \in \mathcal{X}_0^{(n)}$ and $0 \le \nu \le 3$. Then

$$\big| \phi_{*n}(\psi^*, \psi, \mu, \mathcal{V})^*(u) - \phi_n(\psi^*, \psi, \mu, \mathcal{V})(u) \big| \le K_1 \mathfrak{k}' \qquad \text{for all } u \in \mathcal{X}_n$$

Proof. Write $\phi_{(*)} = \phi_{(*)n}(\psi^*, \psi, \mu, \mathcal{V})$. By Proposition 2.1 and [3, Lemma 2.5.b]

$$|\phi(u)| \le K_1 \mathfrak{k} \quad \text{and} \quad |\partial_\nu \phi(u)| \le K_1 \mathfrak{k}' \qquad \text{for all } u \in \mathcal{X}_n,\ 0 \le \nu \le 3 \tag{2.7}$$

By (1.3) and the fact that $S_n^{-1}(\mu) - S_n^{-1}(\mu)^\dagger = D_n - D_n^\dagger$ (see the definition of $S_n(\mu)$ after (1.3))

$$S_n^{-1}(\mu)(\phi_*^* - \phi) + \mathcal{V}_*'(\phi_*, \phi, \phi_*)^* - \mathcal{V}'(\phi, \phi_*, \phi) = (D_n - D_n^\dagger) \phi_*^*$$

where † refers to the adjoint. Localizing as in [6, Corollary B.2],

$$S_n^{-1}(\mu)(\phi_*^* - \phi) + \mathrm{v}\phi^* (\phi_*^* + \phi)\,(\phi_*^* - \phi) - \mathrm{v}\phi^2\,(\phi_*^* - \phi)^* = (D_n - D_n^\dagger)\,\phi_*^* + \mathcal{V}_{\mathrm{loc}}(\phi_*, \phi) \tag{2.8}$$

where $\mathrm{v} = \int V(0, u_1, u_2, u_3)\; du_1\, du_2\, du_3$ and $\mathcal{V}_{\mathrm{loc}}(\phi_*, \phi)$ is a field such that

$$\big| \mathcal{V}_{\mathrm{loc}}(\phi_*, \phi)(u) \big| \le \mathrm{const}\, \mathfrak{k}' \qquad \text{for all } u \in \mathcal{X}_n$$

By [2, (3.1)],

$$\begin{aligned} D_n - D_n^\dagger &= L^{2n}\, \mathbb{L}_*^{-n} e^{-h_0} \big(\partial_0^\dagger - \partial_0 \big) \mathbb{L}_*^n \\ &= e^{-\mathbb{L}_*^{-n} h_0 \mathbb{L}_*^n}\ \big(\partial_0^\dagger - \partial_0 \big) \end{aligned}$$

Beware that in the first line ∂_0 acts on the $\mathcal{H}_0^{(n)}$, while in the second line ∂_0 acts on $\mathcal{H}_n$. Hence, by (2.7)

$$\big| (D_n - D_n^\dagger) \phi_*^*(u) \big| \le \mathrm{const}\, \mathfrak{k}' \qquad \text{for all } u \in \mathcal{X}_n \tag{2.9}$$

Also considering the complex conjugate, we see that $\sigma = \phi_*^* - \phi$ fulfils the equations

$$\begin{aligned}\big[\mathbb{1} + S_n(\mu)\, \mathrm{v}\phi^*(\phi_*^* + \phi)\big]\sigma - S_n(\mu)\, \mathrm{v}\phi^2\, \sigma^* &= S_n(\mu)\big[(D_n - D_n^\dagger)\, \phi_*^* + \mathcal{V}_{\rm loc}(\phi_*, \phi)\big] \\ \big[\mathbb{1} + \overline{S_n(\mu)}\, \mathrm{v}\phi(\phi_* + \phi^*)\big]\sigma^* - \overline{S_n(\mu)}\, \mathrm{v}\phi^{*2}\, \sigma &= \overline{S_n(\mu)}\, \big[(\overline{D_n} - D_n^*)\, \phi_* + \mathcal{V}_{\rm loc}(\phi_*, \phi)^*\big]\end{aligned} \tag{2.10}$$

where, in the square brackets on the left hand side, $\phi^*(\phi_*^* + \phi)$ and $\phi(\phi_* + \phi^*)$, respectively, are viewed as multiplication operators. By [2, Proposition 5.1] and (2.7), the L^1–L^∞ norm of the operators $S_n(\mu)\, \mathrm{v}\phi^*(\phi_*^* + \phi)$ and $S_n(\mu)\, \mathrm{v}\phi^2$ is bounded by $2K_{\rm op}\, \rho_1 \le \frac{1}{4}$. Hence, one can solve (2.10) for σ and σ^*, and the estimates (2.8) and (2.9) for the terms on the right hand side give the desired estimate. □

Remark 2.3 (Third order terms of the background field)**.** Proposition 2.1.a states that the linear part of the background field $\phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})$ is

$$\phi_{(*)n}^{(1)}(\psi_*, \psi, \mu, \mathcal{V}) = S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n\, \psi_{(*)}$$

and that the higher order terms $\phi_{(*)n}^{(\ge 3)}$ are of degree at least three in ψ_*, ψ. In fact, the term of degree exactly three can be described easily. There is a constant $\hat{K}_1$ and there are field maps $\phi_{(*)n}^{(\ge 5)}$ such that

$$\begin{aligned}\phi_{*n}^{(\ge 3)}(\psi_*, \psi, \mu, \mathcal{V}) &= -S_n(\mu)\, \mathcal{V}_*'\big(\Phi_*, \Phi, \Phi_*\big)\big|_{\Phi_{(*)} = \phi_{(*)n}^{(1)}(\psi_*, \psi, \mu, \mathcal{V})} \quad + \phi_{(*)n}^{(\ge 5)}(\psi_*, \psi, \mu, \mathcal{V}) \\ \phi_n^{(\ge 3)}(\psi_*, \psi, \mu, \mathcal{V}) &= -S_n(\mu)\, \mathcal{V}'\big(\Phi, \Phi_*, \Phi\big)\big|_{\Phi_{(*)} = \phi_{(*)n}^{(1)}(\psi_*, \psi, \mu, \mathcal{V})} \quad + \phi_{(*)n}^{(\ge 5)}(\psi_*, \psi, \mu, \mathcal{V})\end{aligned}$$

and $\big|\big|\big|\phi_{(*)n}^{(\ge 5)}\big|\big|\big| \le \hat{K}_1 \|V\|_{\mathfrak{m}}^2\, \mathfrak{k}^5$.

Proof. We prove the statement about $\phi_n^{(\ge 3)}$. Write $\phi_{(*)} = \phi_{(*)n}(\psi^*, \psi, \mu, \mathcal{V})$ and $\Phi_{(*)} = \phi_{(*)n}^{(1)}(\psi_*, \psi, \mu, \mathcal{V})$. By (1.3),

$$\begin{aligned}\phi &= S_n(\mu) Q_n^* \mathfrak{Q}_n \psi - S_n(\mu) \mathcal{V}'(\phi, \phi_*, \phi) \\ &= \Phi - S_n(\mu) \mathcal{V}'(\Phi + \phi^{(\ge 3)}, \Phi_* + \phi_*^{(\ge 3)}, \Phi + \phi^{(\ge 3)}) \\ &= \Phi - S_n(\mu)\, \mathcal{V}'\big(\Phi, \Phi_*, \Phi\big) + \phi_{(*)n}^{(\ge 5)}(\psi_*, \psi, \mu, \mathcal{V})\end{aligned}$$

with

$$\phi_n^{(\ge 5)}(\psi^*, \psi) = -S_n(\mu) \big\{\mathcal{V}'(\Phi + \phi^{(\ge 3)}, \Phi_* + \phi_*^{(\ge 3)}, \Phi_* + \phi_*^{(\ge 3)}) - \mathcal{V}'(\Phi, \Phi_*, \Phi)\big\}$$

The estimate on $\phi_n^{(\ge 5)}$ follows from Proposition 2.1.a and [3, Lemma 3.1]. □

To derive a representation of the background fields of the form (1.4) from Proposition 2.1, we use

Lemma 2.4. *There are field maps $F_{\rm lb}\big(\{\psi_\nu\}\big)$ and $F_{\rm lb*}\big(\{\psi_{*\nu}\}\big)$ and a constant K_2 such that*

$$\big(S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\psi_{(*)}\big)(u) = \tfrac{a_n}{a_n-\mu}\psi_{(*)}\big(X(u)\big) + F_{\rm lb(*)}(\{\partial_\nu\psi_{(*)}\})(u)$$

and

$$|||F_{\rm lb(*)}||| \le K_2\,\mathfrak{k}'\,\|S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\|_{\bar{\mathfrak{m}}}$$

Furthermore, the maps $F_{\rm lb(*)}$ are of degree precisely one.

Proof. We prove the lemma for $B = S_n(\mu)Q_n^*\mathfrak{Q}_n$. Denote by 1 and $1_{\rm fin}$ the constant fields on $\mathcal{X}_0^{(n)}$ and $\mathcal{X}_n$, respectively, that always take the value 1. By [6, Remark B.7], $Q_n 1_{\rm fin} = 1$, $Q_n^* 1 = 1_{\rm fin}$ and $\mathfrak{Q}_n 1 = a_n 1$. Since D_n annihilates constant fields,

$$B\,1 = S_n(\mu)Q_n^*\mathfrak{Q}_n 1 = (D_n + Q_n^*\mathfrak{Q}_nQ_n - \mu)^{-1}Q_n^*\mathfrak{Q}_n 1 = \tfrac{a_n}{a_n-\mu}1_{\rm fin}$$

Fix any $u\in\mathcal{X}_n$ and any field ψ on $\mathcal{X}_0^{(n)}$. Then

$$\begin{aligned}\big(B\psi\big)(u) &= \sum_{x\in\mathcal{X}_0^{(n)}} B(u,x)\,\psi(x)\\ &= \sum_{x\in\mathcal{X}_0^{(n)}} B(u,x)\,\psi\big(X(u)\big) + \sum_{x\in\mathcal{X}_0^{(n)}} B(u,x)\,\big[\psi(x)-\psi\big(X(u)\big)\big]\\ &= \tfrac{a_n}{a_n-\mu}\psi\big(X(u)\big) + \sum_{x\in\mathcal{X}_0^{(n)}} B(u,x)\,\big[\psi(x)-\psi\big(X(u)\big)\big]\end{aligned}$$

It now suffices to apply [6, Lemma B.1]. □

Corollary 2.5. *There are field maps $\breve\phi_{(*)n}$ and a constant K_3 such that, under the hypotheses of Proposition 2.1,*

$$\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})(u) = \tfrac{a_n}{a_n-\mu}\psi_{(*)}\big(X(u)\big) + \breve\phi_{(*)n}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\,,\,\mu,\mathcal{V}\big)(u)$$

and

$$|||\breve\phi_{(*)n}||| \le K_3\big(\mathfrak{k}' + \|V\|_{\mathfrak{m}}\mathfrak{k}^3\big)$$

Proof. Proposition 2.1.a and Lemma 2.4 imply that

$$\begin{aligned}
\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})(u) &= \big(S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\psi_{(*)}\big)(u) + \phi_{(*)n}^{(\ge 3)}(\psi_*,\psi,\mu,\mathcal{V})(u) \\
&= \tfrac{a_n}{a_n-\mu}\psi_{(*)}\big(X(u)\big) + F_{\mathrm{lb}(*)}(\{\partial_\nu\psi_{(*)}\})(u) + \phi_{(*)n}^{(\ge 3)}(\psi_*,\psi,\mu,\mathcal{V})(u) \\
&= \tfrac{a_n}{a_n-\mu}\psi_{(*)}\big(X(u)\big) + \breve{\phi}_{(*)n}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\,,\,\mu,\mathcal{V}\big)(u)
\end{aligned}$$

with

$$|||\breve{\phi}_{(*)n}||| \le K_2\|S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\|_{\bar{\mathfrak{m}}}\ \mathfrak{k}' + K_1\|V\|_{\mathfrak{m}}\mathfrak{k}^3 \le K_3\big(\mathfrak{k}' + \|V\|_{\mathfrak{m}}\mathfrak{k}^3\big)$$

□

3 Variations of the Background Field with Respect to ψ

Recall from [5, (4.7)] that

$$\delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z) = \mathbb{S}\big[\delta\check{\phi}_{(*)n+1}\big(\mathbb{S}^{-1}\psi_*\,,\,\mathbb{S}^{-1}\psi_*\,,\,D^{(n)*}\mathbb{L}_*z_*\,,\,D^{(n)}\mathbb{L}_*z\,,\,\mu,\mathcal{V}\big)\big] \tag{3.1}$$

where

- the fields

$$\delta\check{\phi}_{(*)n+1}(\theta_*,\theta,\delta\psi_*,\delta\psi,\mu,\mathcal{V}) = \Big[\phi_{(*)n}\big(\psi_*+\delta\psi_*,\psi+\delta\psi,\mu,\mathcal{V}\big) - \phi_{(*)n}\big(\psi_*,\psi,\mu,\mathcal{V}\big)\Big]_{\psi_{(*)}=\psi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V})}$$

  were defined in [5, Definition 3.5.a],
- the scaling operators $\mathbb{S}$ and $\mathbb{L}_*$ were defined in [5, Appendix A.2], and
- the operator square root $D^{(n)}$ of the fluctuation field covariance $C^{(n)}$ was defined just before [5, (1.15)].

The fields $\delta\hat{\phi}_{(*)n+1}$ also depend implicitly on μ and $\mathcal{V}$. Proposition 3.1, below, implies that $\delta\hat{\phi}_{(*)n+1}$ are analytic maps in (ψ_*,ψ,z_*,z) from a neighborhood of the origin in $\mathcal{H}_0^{(n+1)}\times\mathcal{H}_0^{(n+1)}\times\mathcal{H}_1^{(n)}\times\mathcal{H}_1^{(n)}$ to $\mathcal{H}_{n+1}^{(0)}$. As in [6, §5], we define, on the space of field maps $F(\psi_*,\psi,z_*,z)$, the projections

- P_2^ψ which extracts the part of degree exactly one in each of ψ_* and ψ, and of arbitrary degree in $z_{(*)}$ and
- P_1^ψ which extracts the part of degree exactly one in $\psi_{(*)}$, and of arbitrary degree in $z_{(*)}$ and
- P_0^ψ which extracts the part of degree zero in $\psi_{(*)}$ and of arbitrary degree in $z_{(*)}$.

Proposition 3.1. *There are constants*[8] K_4 *and* $\rho_2>0$ *such that the following hold, if*

$$\max\big\{\ L^2|\mu|\ ,\ \|V\|_{\mathfrak{m}}(\mathfrak{k}+L^9\mathfrak{k}_{\mathfrak{l}})(\mathfrak{k}+\mathfrak{k}'+L^9\mathfrak{k}_{\mathfrak{l}})\ \big\}\le\rho_2$$

- *The field maps* $\delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z)$ *obey* $\quad|||\delta\hat{\phi}_{(*)n+1}|||\le L^{11}K_4\,\mathfrak{k}_{\mathfrak{l}}$.
- *Write, as in [5, (4.9)]*

$$\delta\hat{\phi}^{(+)}_{(*)n+1}(\psi_*,\psi,z_*,z) = \delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z) - L^{3/2}\mathbb{S}S_n^{(*)}Q_n^*\mathfrak{Q}_nD^{(n)(*)}\mathbb{S}^{-1}z_{(*)}$$

  It obeys $\quad\big|\big|\big|\delta\hat{\phi}^{(+)}_{(*)n+1}\big|\big|\big|\le L^{29}K_4\,\{\|V\|_{\mathfrak{m}}(\mathfrak{k}+\mathfrak{k}_{\mathfrak{l}})^2+|\mu|\}\mathfrak{k}_{\mathfrak{l}}$.

[8] Recall Convention 1.3.

- *The part,* $\delta\hat{\phi}^{(\geq 2)}_{(*)n+1}$*, of* $\delta\hat{\phi}^{(+)}_{(*)n+1}$ *that is of degree at least two in* $z_{(*)}$*, fulfils the bound*

$$\left|\!\left|\!\left|\delta\hat{\phi}^{(\geq 2)}_{(*)n+1}\right|\!\right|\!\right| \leq L^{29} K_4 \, \|V\|_{\mathfrak{m}} (\mathfrak{k} + \mathfrak{k}_{\mathfrak{l}}) \mathfrak{k}_{\mathfrak{l}}^2$$

- *Using the notation of [5, Definition 3.1], we have*

$$\begin{aligned}
\delta\hat{\phi}^{(+)}_{*n+1}(\psi_*, \psi, z_*, z) &= L^{3/2} \mathbb{S}[S_n(\mu)^* - S_n^*] Q_n^* \mathfrak{Q}_n D^{(n)*} \mathbb{S}^{-1} z_* \\
&\qquad - L^{\frac{3}{2}} \mathbb{L}_*^{-1} S_n(\mu)^* \mathcal{V}'_*(\varphi_*, \varphi, \varphi_*) \Big|^{\varphi_* = \phi_* + \delta\phi_* }_{\varphi_* = \phi_* }{}^{\varphi = \phi + \delta\phi}_{\varphi = \phi} + \delta\hat{\phi}^{(\mathrm{h.o.})}_* \\
\delta\hat{\phi}^{(+)}_{n+1}(\psi_*, \psi, z_*, z) &= L^{3/2} \mathbb{S}[S_n(\mu) - S_n] Q_n^* \mathfrak{Q}_n D^{(n)} \mathbb{S}^{-1} z \\
&\qquad - L^{\frac{3}{2}} \mathbb{L}_*^{-1} S_n(\mu) \mathcal{V}'(\varphi, \varphi_*, \varphi) \Big|^{\varphi_* = \phi_* + \delta\phi_* }_{\varphi_* = \phi_* }{}^{\varphi = \phi + \delta\phi}_{\varphi = \phi} + \delta\hat{\phi}^{(\mathrm{h.o.})}
\end{aligned}$$

with the substitutions

$$\begin{aligned}
\phi_{(*)} &= \mathbb{S}^{-1} S_{n+1}(L^2\mu)^{(*)} Q_{n+1}^* \mathfrak{Q}_{n+1} \, \psi_{(*)} \\
\delta\phi_{(*)} &= S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n \, L^{3/2} D^{(n)(*)} \mathbb{S}^{-1} z_{(*)}
\end{aligned}$$

and with the contributions in $\delta\hat{\phi}^{(\mathrm{h.o.})}_{(*)}$ *being of degree at least five in* $(\psi_{(*)}, z_{(*)})$ *and obeying*

$$\left|\!\left|\!\left| P_j^{\psi} \delta\hat{\phi}^{(\mathrm{h.o.})}_{(*)} \right|\!\right|\!\right| \leq L^{(1-j)3/2} L^{9(5-j)} K_4 \|V\|^2_{\mathfrak{m}} \mathfrak{k}^j \mathfrak{k}_{\mathfrak{l}}^{5-j} \qquad \text{for } j = 0, 1, 2$$

- *There are field maps* $\delta\hat{\phi}_{(*)n+1,\nu}\big(\psi_*, \psi, \psi_{*\nu}, \psi_\nu, z_*, z\big)$*,* $0 \leq \nu \leq 3$*, such that*

$$\big(\partial_\nu \delta\hat{\phi}_{(*)n+1}\big)(\psi_*, \psi, z_*, z) = \delta\hat{\phi}_{(*)n+1,\nu}\big(\psi_*, \psi, \partial_\nu\psi_*, \partial_\nu\psi, z_*, z\big)$$

and

$$|||\delta\hat{\phi}_{(*)n+1,\nu}||| \leq L_\nu L^{11} K_4 \, \mathfrak{k}_{\mathfrak{l}}$$

where $L_0 = L^2$ *and* $L_\nu = L$ *for* $\nu = 1, 2, 3$*.*

This Proposition will be proven following the proof of Lemma 3.3. Recall, from (3.1), that $\delta\hat{\phi}_{(*)n+1}$ is defined in terms of $\delta\check{\phi}_{(*)n+1}$. Also recall, from [5, Remark 3.6.c and Definition 3.1], that $\delta\check{\phi}_{(*)n+1}$ is obtained from the solution $\delta\phi_{(*)} = \delta\varphi_{(*)}(\phi_*, \phi, \delta\psi_*, \delta\psi)$ of

$$\begin{aligned}
\delta\phi_* &= S_n^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* + \mu S_n^* \delta\phi_* - S_n^* \mathcal{V}'(\varphi_*, \varphi, \varphi_*) \Big|^{\varphi_* = \phi_* + \delta\phi_* }_{\varphi_* = \phi }{}^{\varphi = \phi + \delta\phi}_{\varphi = \phi} \\
\delta\phi &= S_n Q_n^* \mathfrak{Q}_n \, \delta\psi + \mu S_n \delta\phi - S_n \mathcal{V}'_*(\varphi, \varphi_*, \varphi) \Big|^{\varphi_* = \phi_* + \delta\phi_* }_{\varphi_* = \phi_* }{}^{\varphi = \phi + \delta\phi}_{\varphi = \phi}
\end{aligned} \tag{3.2}$$

by substituting $\phi_{(*)} = \check{\phi}_{(*)n+1}(\theta_*, \theta, \mu, \mathcal{V})$. So we first prove the existence of and develop bounds on $\delta\varphi_{(*)}(\phi_*, \phi, \delta\psi_*, \delta\psi)$. We fix any $\mathfrak{k}_\phi, \mathfrak{k}'_\phi$, $\mathfrak{k}_{\delta\psi} \ge 1$ and denote by $|||\ \cdot\ |||_\phi$ the (auxiliary) norm with mass $\mathfrak{m}$ that assigns the weight factors $\mathfrak{k}_\phi$ to the fields $\phi_{(*)}$, $\mathfrak{k}'_\phi$ to the fields $\phi_{(*)\nu}$ and $\mathfrak{k}_{\delta\psi}$ to the fields $\delta\psi_{(*)}$.

Lemma 3.2. *There are constants* K'_4 *and* $\rho'_2 > 0$ *such that the following hold, if*

$$\max\Big\{\ |\mu|\ ,\ \|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})(\mathfrak{k}_\phi + \mathfrak{k}'_\phi + \mathfrak{k}_{\delta\psi})\ \Big\} \le \rho'_2$$

- *There are field maps* $\delta\varphi_{(*)}(\phi_*, \phi, \delta\psi_*, \delta\psi)$ *that obey* $\big|\big|\big|\delta\varphi_{(*)}\big|\big|\big|_\phi \le K'_4\, \mathfrak{k}_{\delta\psi}$ *and solve* (3.2). *Write*

$$\delta\varphi_{(*)} = S_n^{(*)} Q_n^* \mathfrak{Q}_n \delta\psi_{(*)} + \delta\varphi_{(*)}^{(+)}$$

  and denote by $\delta\varphi_{(*)}^{(\ge 2)}$ *the part of* $\delta\varphi_{(*)}^{(+)}$ *that is of degree at least two in* $\delta\psi_{(*)}$. *They obey*

$$\begin{aligned} \big|\big|\big|\delta\varphi_{(*)}^{(+)}\big|\big|\big|_\phi &\le K'_4\, \{\|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})^2 + |\mu|\}\mathfrak{k}_{\delta\psi} \\ \big|\big|\big|\delta\varphi_{(*)}^{(\ge 2)}\big|\big|\big|_\phi &\le K'_4\, \|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})\mathfrak{k}_{\delta\psi}^2 \end{aligned}$$

- *There are field maps* $\delta\varphi_{(*)\nu}\big(\phi_*, \phi, \phi_{*\nu}, \phi_\nu, \delta\psi_*, \delta\psi\big)$, $0 \le \nu \le 3$, *such that*

$$\big(\partial_\nu \delta\varphi_{(*)}\big)(\phi_*, \phi, \delta\psi_*, \delta\psi) = \delta\varphi_{(*)\nu}\big(\phi_*, \phi, \partial_\nu\phi_*, \partial_\nu\phi, \delta\psi_*, \delta\psi\big)$$

  and $\quad |||\delta\varphi_{(*)\nu}|||_\phi \le K'_4\, \mathfrak{k}_{\delta\psi}$.

Proof. (a) The equations (3.2), for $\delta\varphi_{(*)}$, are of the form

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha}, \vec{\gamma}) + \vec{B}\big(\vec{\alpha}\,;\,\vec{\gamma}\big)$$

as in [3, (4.1.b)], with $X = \mathcal{X}_n$ and

$$\begin{array}{lllll} \alpha_* = \phi_* & \alpha = \phi & \delta\alpha_* = Q_n^* \mathfrak{Q}_n \delta\psi_* & \delta\alpha = Q_n^* \mathfrak{Q}_n \delta\psi & \vec{\alpha} = \big(\alpha_*, \alpha, \delta\alpha_*, \delta\alpha\big) \\ & & \delta\phi_* = S_n^* \gamma_* & \delta\phi = S_n \gamma & \vec{\gamma} = \big(\gamma_*, \gamma\big) \end{array}$$

and

$$\vec{f}(\vec{\alpha})(u) = \begin{bmatrix}\delta\alpha_*(u)\\ \delta\alpha(u)\end{bmatrix}$$

$$\vec{L}(\vec{\alpha};\vec{\gamma})(u) = \begin{bmatrix}\mu(S_n^*\gamma_*)(u) - \mathcal{V}_*'\big(u;\alpha_*,S_n\gamma,\alpha_*\big) - 2\mathcal{V}_*'\big(u;\alpha_*,\alpha,S_n^*\gamma_*\big)\\ \mu(S_n\gamma)(u) - \mathcal{V}'\big(u;\alpha,S_n^*\gamma_*,\alpha\big) - 2\mathcal{V}'\big(u;\alpha,\alpha_*,S_n\gamma\big)\end{bmatrix}$$

$$\vec{B}(\vec{\alpha};\vec{\gamma})(u) = \begin{bmatrix}-\mathcal{V}_*'\big(u;S_n^*\gamma_*,\alpha,S_n^*\gamma_*\big) - 2\mathcal{V}_*'\big(u;S_n^*\gamma_*,S_n\gamma,\alpha_*\big)\\ \qquad\qquad -\mathcal{V}_*'\big(u;S_n^*\gamma_*,S_n\gamma,S_n^*\gamma_*\big)\\ -\mathcal{V}'\big(u;S_n\gamma,\alpha_*,S_n\gamma\big) - 2\mathcal{V}'\big(u;S_n\gamma,S_n^*\gamma_*,\alpha\big)\\ \qquad\qquad -\mathcal{V}'\big(u;S_n\gamma,S_n^*\gamma_*,S_n\gamma\big)\end{bmatrix}$$

Now apply [3, Proposition 4.1.a and Remark 3.5.a] with $d_{\max} = 3$, $\mathfrak{c} = \frac{1}{2}$ and

$$\kappa_1 = \kappa_2 = \mathfrak{k}_\phi \qquad \kappa_3 = \kappa_4 = \|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k}_{\delta\psi} \qquad \lambda_1 = \lambda_2 = 4\kappa_4$$

Since

$$\begin{aligned}
|||f_j|||_w &\le \kappa_{2+j} = \tfrac{1}{4}\lambda_j\\
|||L_j|||_{w_{\kappa,\lambda}} &\le \|S_n\|_{\mathfrak{m}}(|\mu| + 2\|V\|_{\mathfrak{m}}\kappa_1\kappa_2)\lambda_j + \|S_n\|_{\mathfrak{m}}\ \|V\|_{\mathfrak{m}}\kappa_j^2\lambda_{3-j}\\
&\le \|S_n\|_{\mathfrak{m}}\Big\{|\mu| + 3\|V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\Big\}\lambda_j\\
|||B_j|||_{w_{\kappa,\lambda}} &\le \|S_n\|_{\mathfrak{m}}^2\ \|V\|_{\mathfrak{m}}\big[\kappa_{3-j}\lambda_j^2 + 2\kappa_j\lambda_j\lambda_{3-j} + \|S_n\|_{\mathfrak{m}}\lambda_j^2\lambda_{3-j}\big]\\
&\le \|S_n\|_{\mathfrak{m}}^2\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\|V\|_{\mathfrak{m}}\Big\{12\mathfrak{k}_\phi\mathfrak{k}_{\delta\psi} + 16\|S_n\|_{\mathfrak{m}}\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k}_{\delta\psi}^2\Big\}\lambda_j
\end{aligned}$$

[3, Proposition 4.1.a] gives

$$\delta\varphi_{(*)}(\phi_*,\phi,\delta\psi_*,\delta\psi) = \delta\phi_{(*)} = S_n^{(*)}\Gamma_{(*)}\big(\phi_*,\phi,Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big)$$

with $|||\Gamma_{(*)}|||_{w_{\kappa,\lambda}} \le 2\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k}_{\delta\psi}$. The first conclusion now follows.

Denote by $\delta\varphi_{(*)}^{(1)}$ the part of $\delta\varphi_{(*)}$ that is of degree precisely one in $\delta\psi_{(*)}$ and decompose

$$\delta\varphi_{(*)}^{(1)} = S_n^{(*)}Q_n^*\mathfrak{Q}_n\delta\psi_{(*)} + \delta\varphi_{(*)}^{(1)\sim}$$

In the notation of [3, Proposition 4.1.b], $\vec{\Gamma}^{(1)}$ is the part of $\vec{\Gamma}$ that is of degree precisely 1 in $\vec{f}$. In our application, $\vec{f}$ is homogeneous of degree one in $\delta\psi_{(*)}$, and $\delta\psi_{(*)}$ does not appear in either $\vec{L}$ or $\vec{B}$, so

$$\begin{aligned}
\delta\varphi_{(*)}^{(1)} &= S_n^{(*)}\Gamma_{(*)}^{(1)}\big(\phi_*,\phi,Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big)\\
\delta\varphi_{(*)}^{(1)\sim} &= S_n^{(*)}\big\{\Gamma_{(*)}^{(1)}\big(\phi_*,\phi,Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big) - f_{(*)}\big(Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big)\big\}\\
\delta\varphi_{(*)}^{(\ge 2)} &= S_n^{(*)}\big\{\Gamma_{(*)}\big(\phi_*,\phi,Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big) - \Gamma_{(*)}^{(1)}\big(\phi_*,\phi,Q_n^*\mathfrak{Q}_n\delta\psi_*,Q_n^*\mathfrak{Q}_n\delta\psi\big)\big\}
\end{aligned}$$

Hence the bounds on $\delta\varphi_{(*)}^{(\ge 2)}$ and $\delta\varphi_{(*)}^{(+)} = \delta\varphi_{(*)}^{(1)\sim} + \delta\varphi_{(*)}^{(\ge 2)}$ follows from [3, Proposition 4.1.b and Remark 3.5.a] with $d_{\max} = 3$ and

$$\begin{aligned}
&\max_{1\le j\le r} \tfrac{1}{\lambda_j}|||B_j|||_{w_{\kappa,\lambda}} \le \tilde{K}_4'\|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})\mathfrak{k}_{\delta\psi}\\
&\mathfrak{c} = \max_{1\le j\le r} \tfrac{1}{\lambda_j}|||L_j|||_{w_{\kappa,\lambda}} + 3\max_{1\le j\le r} \tfrac{1}{\lambda_j}|||B_j|||_{w_{\kappa,\lambda}} \le \tilde{K}_4'\{\|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})^2 + |\mu|\}
\end{aligned}$$

(b) We follow the same strategy as in Proposition 2.1.b. That is, we apply ∂_ν to (3.2) and use the "discrete product rule" (2.5) and

$$\partial_\nu S_n^* = S_{n,\nu}^{(+)}\partial_\nu \qquad \partial_\nu S_n = S_{n,\nu}^{(-)}\partial_\nu \qquad \partial_\nu Q_n^*\mathfrak{Q}_n = Q_{n,\nu}^{(+)}\mathfrak{Q}_n\partial_\nu \tag{3.3}$$

where $Q_{n,\nu}^{(+)}$ was defined in [2, (2.11)] and $S_{n,\nu}^{(\pm)}$ was defined in [2, (5.2) and (5.3)]. (See [2, Remark 2.5 and (5.1)].) Denoting $\delta\phi_{(*)} = \delta\varphi_{(*)}(\phi_*,\phi,\delta\psi_*,\delta\psi)$, this gives

$$\begin{aligned}
\partial_\nu\delta\phi_* &+ S_{n,\nu}^{(+)}L_{11}(\partial_\nu\delta\phi_*) + S_{n,\nu}^{(+)}L_{12}(\partial_\nu\delta\phi) &= S_{n,\nu}^{(+)}f_*\\
\partial_\nu\delta\phi &+ S_{n,\nu}^{(-)}L_{21}(\partial_\nu\delta\phi_*) + S_{n,\nu}^{(-)}L_{22}(\partial_\nu\delta\phi) &= S_{n,\nu}^{(-)}f
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
L_{11}(\partial_\nu\delta\phi_*) &= -\mu\partial_\nu\delta\phi_* + 2\mathcal{V}_*'\big(\phi_*,\phi,\partial_\nu\delta\phi_*\big) + \mathcal{V}_*'\big(\partial_\nu\delta\phi_*,T_\nu^{-1}\phi,\delta\phi_* + T_\nu^{-1}\delta\phi_*\big)\\
&\quad + 2\mathcal{V}_*'\big(\partial_\nu\delta\phi_*,T_\nu^{-1}\delta\phi,T_\nu^{-1}\phi_*\big) + \mathcal{V}_*'\big(\partial_\nu\delta\phi_*,T_\nu^{-1}\delta\phi,\delta\phi_* + T_\nu^{-1}\delta\phi_*\big)\\
L_{12}(\partial_\nu\delta\phi) &= \mathcal{V}_*'\big(\phi_*,\partial_\nu\delta\phi,\phi_*\big) + 2\mathcal{V}_*'\big(\delta\phi_*,\partial_\nu\delta\phi,T_\nu^{-1}\phi_*\big) + \mathcal{V}_*'\big(\delta\phi_*,\partial_\nu\delta\phi,\delta\phi_*\big)\\
L_{21}(\partial_\nu\delta\phi_*) &= \mathcal{V}'\big(\phi,\partial_\nu\delta\phi_*,\phi\big) + 2\mathcal{V}'\big(\delta\phi,\partial_\nu\delta\phi_*,T_\nu^{-1}\phi\big) + \mathcal{V}'\big(\delta\phi,\partial_\nu\delta\phi_*,\delta\phi\big)\\
L_{22}(\partial_\nu\delta\phi) &= -\mu\partial_\nu\delta\phi + 2\mathcal{V}'\big(\phi,\phi_*,\partial_\nu\delta\phi\big) + \mathcal{V}'\big(\partial_\nu\delta\phi,T_\nu^{-1}\phi_*,\delta\phi + T_\nu^{-1}\delta\phi\big)\\
&\quad + 2\mathcal{V}'\big(\partial_\nu\delta\phi,T_\nu^{-1}\delta\phi_*,T_\nu^{-1}\phi\big) + \mathcal{V}'\big(\partial_\nu\delta\phi,T_\nu^{-1}\delta\phi_*,\delta\phi + T_\nu^{-1}\delta\phi\big)\\
f_* &= Q_{n,\nu}^{(+)}\mathfrak{Q}_n[T_\nu\delta\psi_* - \delta\psi_*] - \mathcal{V}_*'\big(\partial_\nu\phi_*,T_\nu^{-1}\delta\phi,\phi_* + T_\nu^{-1}\phi_*\big)\\
&\quad - 2\mathcal{V}_*'\big(\partial_\nu\phi_*,T_\nu^{-1}\phi,T_\nu^{-1}\delta\phi_*\big) - 2\mathcal{V}_*'\big(\phi_*,\partial_\nu\phi,T_\nu^{-1}\delta\phi_*\big)\\
&\quad - \mathcal{V}_*'\big(\delta\phi_*,\partial_\nu\phi,\delta\phi_*\big) - 2\mathcal{V}_*'\big(\delta\phi_*,\delta\phi,\partial_\nu\phi_*\big)\\
f &= Q_{n,\nu}^{(+)}\mathfrak{Q}_n[T_\nu\delta\psi - \delta\psi] - \mathcal{V}'\big(\partial_\nu\phi,T_\nu^{-1}\delta\phi_*,\phi + T_\nu^{-1}\phi\big)\\
&\quad - 2\mathcal{V}'\big(\partial_\nu\phi,T_\nu^{-1}\phi_*,T_\nu^{-1}\delta\phi\big) - 2\mathcal{V}'\big(\phi,\partial_\nu\phi_*,T_\nu^{-1}\delta\phi\big)\\
&\quad - \mathcal{V}'\big(\delta\phi,\partial_\nu\phi_*,\delta\phi\big) - 2\mathcal{V}'\big(\delta\phi,\delta\phi_*,\partial_\nu\phi\big)
\end{aligned}$$

The system of equations (3.4) is of the form

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha}, \vec{\gamma}) + \vec{B}\big(\vec{\alpha}\,;\,\vec{\gamma}\big)$$

as in [3, (4.1.b)], with $\vec{\alpha} = (\alpha_1, \cdots, \alpha_8)$, $\vec{\gamma} = (\gamma_*, \gamma)$ and

$$\begin{array}{llll} \alpha_1 = \phi_* & \alpha_2 = \phi & \alpha_3 = \partial_\nu \phi_* & \alpha_4 = \partial_\nu \phi \\ \alpha_5 = \delta\phi_* & \alpha_6 = \delta\phi & \alpha_7 = \delta\psi_* & \alpha_8 = \delta\psi \\ \partial_\nu \delta\phi_* = S^{(+)}_{n,\nu}\gamma_* & \partial_\nu \delta\phi = S^{(-)}_{n,\nu}\gamma & & \end{array}$$

and $\vec{B}(\vec{\alpha};\vec{\gamma}) = 0$ and

$$\vec{L}(\vec{\alpha};\vec{\gamma}) = -\begin{bmatrix} L_{11}\,(S^{(+)}_{n,\nu}\gamma_*) \;+\; L_{12}\,(S^{(-)}_{n,\nu}\gamma) \\ L_{21}\,(S^{(+)}_{n,\nu}\gamma_*) \;+\; L_{22}\,(S^{(-)}_{n,\nu}\gamma) \end{bmatrix}$$

Now apply [3, Proposition 4.1.a] with $\mathfrak{c} = \frac{1}{2}$ and

$$\begin{array}{llll} \kappa_1 = \kappa_2 = \mathfrak{k}_\phi & \kappa_3 = \kappa_4 = \mathfrak{k}'_\phi & \kappa_5 = \kappa_6 = K'_4\mathfrak{k}_{\delta\psi} & \kappa_7 = \kappa_8 = \mathfrak{k}_{\delta\psi} \\ \lambda_1 = \lambda_2 = 4\mathfrak{k}_f & & & \end{array}$$

with

$$\begin{aligned} \mathfrak{k}_f &= (e^{\mathfrak{m}} + 1)\|Q^{(+)}_{n,\nu}\mathfrak{Q}_n\|_{\mathfrak{m}}\kappa_7 + e^{2\varepsilon_n\mathfrak{m}}\|V\|_{\mathfrak{m}}\big\{6\kappa_1\kappa_3\kappa_5 + 3\kappa_3\kappa_5^2\big\} \\ &= \Big[(e^{\mathfrak{m}} + 1)\|Q^{(+)}_{n,\nu}\mathfrak{Q}_n\|_{\mathfrak{m}} + K'_4 e^{2\varepsilon_n\mathfrak{m}}\|V\|_{\mathfrak{m}}\big\{6\mathfrak{k}_\phi\mathfrak{k}'_\phi + 3\mathfrak{k}'_\phi\,K'_4\mathfrak{k}_{\delta\psi}\big\}\Big]\mathfrak{k}_{\delta\psi} \\ &\le \tfrac{1}{2}K'_4\mathfrak{k}_{\delta\psi} \end{aligned}$$

for a new K'_4 and $\varepsilon_n = \frac{1}{L^n}$. Since $|||f_j|||_w \le \mathfrak{k}_f = \frac{1}{4}\lambda_j$, $|||B_j|||_{w_{\kappa,\lambda}} = 0$ and

$$\begin{aligned} |||L_j|||_{w_{\kappa,\lambda}} &\le \max_{\sigma=+,-}\|S^{(\sigma)}_{n,\nu}\|_{\mathfrak{m}}\Big[|\mu|\lambda_j + \|V\|_{\mathfrak{m}}e^{2\varepsilon_n\mathfrak{m}}\big\{2\kappa_1^2 + 4\kappa_1\kappa_5 + 2\kappa_5^2\big\}\lambda_j \\ &\qquad\qquad + \|V\|_{\mathfrak{m}}e^{\varepsilon_n\mathfrak{m}}\big\{\kappa_1^2 + 2\kappa_1\kappa_5 + \kappa_5^2\big\}\lambda_{3-j}\Big] \\ &\le \max_{\sigma=+,-}\|S^{(\sigma)}_{n,\nu}\|_{\mathfrak{m}}\Big[|\mu| + 3e^{2\varepsilon_n\mathfrak{m}}\|V\|_{\mathfrak{m}}(\mathfrak{k}_\phi + K'_4\mathfrak{k}_{\delta\psi})^2\Big]\lambda_j \end{aligned}$$

[3, Propositions 4.1.a] gives

$$\begin{aligned} \partial_\nu\delta\phi_* &= S^{(+)}_{n,\nu}\Gamma_1\big(\alpha_1, \cdots, \alpha_8\big) \\ \partial_\nu\delta\phi &= S^{(-)}_{n,\nu}\Gamma_2\big(\alpha_1, \cdots, \alpha_8\big) \end{aligned}$$

with

$$|||\Gamma_1|||_{w_{\kappa,\lambda}}, |||\Gamma_2|||_{w_{\kappa,\lambda}} \le 2 \max_{j=1,2} |||f_j|||_w \le 2\mathfrak{k}_f \le K_4' \mathfrak{k}_{\delta\psi}$$

The conclusion now follows by [3, Corollary 3.3].

□

We define, on the space of field maps $F(\phi_*, \phi, \delta\psi_*, \delta\psi)$, the projections

- P_2^ϕ which extracts the part of degree exactly one in each of ϕ_* and ϕ, and of arbitrary degree in $\delta\psi_{(*)}$ and
- P_1^ϕ which extracts the part of degree exactly one in $\phi_{(*)}$, and of arbitrary degree in $\delta\psi_{(*)}$ and
- P_0^ϕ which extracts the part of degree zero in $\phi_{(*)}$ and of arbitrary degree in $\delta\psi_{(*)}$.

Lemma 3.3. *Under the hypothesis of Lemma 3.2, there is a constant K_4'' such that the field maps $\delta\varphi_{(*)}(\phi_*, \phi, \delta\psi_*, \delta\psi)$ of Lemma 3.2 have the form*

$$\delta\varphi_* = S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* - S_n(\mu)^* \mathcal{V}_*'(\varphi_*, \varphi, \varphi_*)\Big|_{\substack{\varphi_*=\phi \\ \varphi=\phi}}^{\substack{\varphi_*=\phi_*+S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* \\ \varphi=\phi+S_n(\mu) Q_n^* \mathfrak{Q}_n \, \delta\psi}} + \delta\varphi_*^{(\ge 5)}$$

$$\delta\varphi = S_n(\mu) Q_n^* \mathfrak{Q}_n \, \delta\psi - S_n(\mu) \mathcal{V}'(\varphi, \varphi_*, \varphi)\Big|_{\substack{\varphi_*=\phi_* \\ \varphi=\phi}}^{\substack{\varphi_*=\phi_*+S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* \\ \varphi=\phi+S_n(\mu) Q_n^* \mathfrak{Q}_n \, \delta\psi}} + \delta\varphi^{(\ge 5)}$$

with $\delta\varphi_{(*)}^{(\ge 5)}$ being of order at least five in $(\phi_{(*)}, \delta\psi_{(*)})$ and obeying

$$\big|\big|\big| P_j^\phi \delta\varphi_{(*)}^{(\ge 5)} \big|\big|\big|_\phi \le K_4'' \|V\|_{\mathfrak{m}}^2 (\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})^j \mathfrak{k}_{\delta\psi}^{5-j} \qquad \text{for } j = 0, 1, 2$$

Proof. Rewrite the equations (3.2) for $\delta\varphi_{(*)}(\phi_*, \phi, \delta\psi_*, \delta\psi)$ in the form

$$\delta\varphi_* = S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* - S_n(\mu)^* \mathcal{V}_*'(\varphi_*, \varphi, \varphi_*)\Big|_{\substack{\varphi_*=\phi \\ \varphi=\phi}}^{\substack{\varphi_*=\phi_*+\delta\varphi_* \\ \varphi=\phi+\delta\varphi}}$$

$$\delta\varphi = S_n(\mu) Q_n^* \mathfrak{Q}_n \, \delta\psi - S_n(\mu) \mathcal{V}'(\varphi, \varphi_*, \varphi)\Big|_{\substack{\varphi_*=\phi_* \\ \varphi=\phi}}^{\substack{\varphi_*=\phi_*+\delta\varphi_* \\ \varphi=\phi+\delta\varphi}}$$

We see from these equations that $\delta\varphi_{(*)} = S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n \, \delta\psi_{(*)} + \delta\varphi_{(*)}^{(\ge 3)}$, with $\delta\varphi_{(*)}^{(\ge 3)}$ being of order at least three in $(\phi_{(*)}, \delta\psi_{(*)})$ and obeying $\big|\big|\big| \delta\varphi_{(*)}^{(\ge 3)} \big|\big|\big|_{\mathfrak{m}} \le \tilde{K}_4'' \|V\|_{\mathfrak{m}}$

$$\big|\big|\big| P_j^\phi \delta\varphi_{(*)}^{(\ge 3)} \big|\big|\big|_\phi \le \tilde{K}_4'' \|V\|_{\mathfrak{m}} (\mathfrak{k}_\phi + \mathfrak{k}_{\delta\psi})^j \mathfrak{k}_{\delta\psi}^{3-j} \qquad \text{for } j = 0, 1, 2$$

Hence

$$\delta\varphi_* = S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* - S_n(\mu)^* \mathcal{V}_*'(\varphi_*, \varphi, \varphi_*)\Big|_{\substack{\varphi_* = \phi_* + S_n(\mu)^* Q_n^* \mathfrak{Q}_n \, \delta\psi_* + \delta\varphi_*^{(\ge 3)} \\ \varphi = \phi + S_n(\mu) Q_n^* \mathfrak{Q}_n \, \delta\psi + \delta\varphi^{(\ge 3)} \\ \varphi_* = \phi \\ \varphi = \phi}}$$

The claim follows immediately from this and the corresponding equation for $\delta\varphi$. □

Proof of Proposition 3.1. Parts (a) and (e): By (3.2)

$$\begin{aligned} &\delta\check{\phi}_{(*)n+1}(\theta_*, \theta, \delta\psi_*, \delta\psi, \mu, \mathcal{V}) \\ &\qquad = \delta\varphi_{(*)}\big(\check{\phi}_{*n+1}(\theta_*, \theta, \mu, \mathcal{V}) \, , \, \check{\phi}_{n+1}(\theta_*, \theta, \mu, \mathcal{V}) \, , \, \delta\psi_* \, , \, \delta\psi\big) \end{aligned}$$

so that, by (3.1) and [5, Definition 3.2],

$$\begin{aligned} \delta\hat{\phi}_{(*)n+1}(\psi_*, \psi, z_*, z) &= \mathbb{S}\Big[\delta\varphi_{(*)}\big(\phi_* \, , \, \phi \, , \, \delta\psi_* \, , \, \delta\psi\big)\Big]_{\substack{\phi_{(*)} = \check{\phi}_{(*)n+1}(\mathbb{S}^{-1}\psi_*, \mathbb{S}^{-1}\psi, \mu, \mathcal{V}) \\ \delta\psi_{(*)} = D^{(n)(*)} \mathbb{L}_* z_{(*)}}} \\ &= L^{\frac{3}{2}} \mathbb{L}_*^{-1} \Big[\delta\varphi_{(*)}\big(\mathbb{S}^{-1}\Phi_* \, , \, \mathbb{S}^{-1}\Phi \, , \, \mathbb{S}^{-1}\delta\Psi_* \, , \, \mathbb{S}^{-1}\delta\Psi\big)\Big]_{\substack{\Phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu, \mathbb{S}\mathcal{V}) \\ \delta\Psi_{(*)} = L^{3/2} \mathbb{S} D^{(n)(*)} \mathbb{S}^{-1} z_{(*)}}} \\ &= L^{\frac{3}{2}} \delta\varphi_{(*)}^{(s)}\big(\Phi_* \, , \, \Phi \, , \, \delta\Psi_* \, , \, \delta\Psi\big)\Big|_{\substack{\Phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu, \mathbb{S}\mathcal{V}) \\ \delta\Psi_{(*)} = L^{3/2} \mathbb{S} D^{(n)(*)} \mathbb{S}^{-1} z_{(*)}}} \end{aligned} \tag{3.5}$$

in the notation of [5, (C.1)]. Similarly, using [5, Remark 2.2.b],

$$\begin{aligned} &\big(\partial_\nu \delta\hat{\phi}_{(*)n+1}\big)(\psi_*, \psi, z_*, z) = \mathbb{S}_\nu \partial_\nu \mathbb{S}^{-1} \delta\hat{\phi}_{(*)n+1}(\psi_*, \psi, z_*, z) \\ &\quad = L_\nu L^{\frac{3}{2}} \mathbb{L}_*^{-1} \Big[\partial_\nu \delta\varphi_{(*)}\big(\mathbb{S}^{-1}\Phi_* \, , \, \mathbb{S}^{-1}\Phi \, , \, \mathbb{S}^{-1}\delta\Psi_* \, , \, \mathbb{S}^{-1}\delta\Psi\big)\Big]_{\substack{\Phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu, \mathbb{S}\mathcal{V}) \\ \delta\Psi_{(*)} = L^{3/2} \mathbb{S} D^{(n)(*)} \mathbb{S}^{-1} z_{(*)}}} \\ &\quad = L_\nu L^{\frac{3}{2}} \mathbb{L}_*^{-1} \Big[\delta\varphi_{(*)\nu}\big(\mathbb{S}^{-1}\Phi_* \, , \, \mathbb{S}^{-1}\Phi \, , \, \mathbb{S}_\nu^{-1}\partial_\nu\Phi_* \, , \, \mathbb{S}_\nu^{-1}\partial_\nu\Phi \, , \, \mathbb{S}^{-1}\delta\Psi_* \, , \, \mathbb{S}^{-1}\delta\Psi\big)\Big]_{\substack{\Phi_{(*)} = \cdots \\ \delta\Psi_{(*)} = \cdots}} \\ &\quad = \delta\hat{\phi}_{(*)n+1,\nu}\big(\psi_*, \psi, \partial_\nu\psi_*, \partial_\nu\psi, z_*, z\big) \end{aligned}$$

where $L_0 = L^2$ and $L_\nu = L$ for $1 \le \nu \le 3$, and we have set

$$\begin{aligned} &\delta\hat{\phi}_{(*)n+1,\nu}\big(\psi_*, \psi, \psi_{*\nu}, \psi_\nu, z_*, z\big) \\ &\quad = L_\nu L^{\frac{3}{2}} \delta\varphi_{(*)\nu}^{(s)}\big(\Phi_*, \Phi, \Phi_{*\nu}, \Phi_\nu, \delta\Psi_*, \delta\Psi\big)\Big|_{\substack{\Phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu, \mathbb{S}\mathcal{V}) \\ \Phi_{(*)\nu} = B^{(\pm)}_{n+1, L^2\mu, \nu} \psi_{(*)\nu} + \phi^{(\ge 3)}_{(*)n+1,\nu}(\psi_*, \psi, \psi_{*\nu}, \psi_\nu, L^2\mu, \mathbb{S}\mathcal{V}) \\ \delta\Psi_{(*)} = L^{3/2} \mathbb{S} D^{(n)(*)} \mathbb{S}^{-1} z_{(*)}}} \end{aligned}$$

We shall bound $\delta\varphi^{(s)}_{(*)}$ and $\delta\varphi^{(s)}_{(*)\nu}$ using the norm $|||\,\cdot\,|||_\Phi$ with mass $\mathfrak{m}$ and weight factors

$$
\begin{aligned}
\mathfrak{k}_\Phi &= \|S_{n+1}(L^2\mu)\|_\mathfrak{m}\|Q^*_{n+1}\mathfrak{Q}_{n+1}\|_\mathfrak{m}\mathfrak{k} + K_1\|\mathbb{S}V\|_\mathfrak{m}\mathfrak{k}^3 \\
\mathfrak{k}'_\Phi &= \max_{\sigma=+,-}\|B^{(\sigma)}_{n+1,L^2\mu,\nu}\|_\mathfrak{m}\mathfrak{k}' + K_1\|\mathbb{S}V\|_\mathfrak{m}\mathfrak{k}^2\mathfrak{k}' \\
\mathfrak{k}_{\delta\Psi} &= L^{3/2}\|\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_\mathfrak{m}\ \mathfrak{k}_\mathfrak{l}
\end{aligned}
$$

By [3, Corollary 3.3] and Proposition 2.1, with n replaced by $n+1$, μ replaced by $L^2\mu$ and $V=\mathbb{S}\mathcal{V}$,

$$
|||\delta\hat\phi_{(*)n+1}||| \le L^{3/2}\big|\big|\big|\delta\varphi^{(s)}_{(*)}\big|\big|\big|_\Phi \qquad |||\delta\hat\phi_{(*)n+1,\nu}||| \le L_\nu L^{3/2}\big|\big|\big|\delta\varphi^{(s)}_{(*)\nu}\big|\big|\big|_\Phi
$$

The hypothesis $\|\mathbb{S}V\|_\mathfrak{m}\mathfrak{k}^2 + L^2|\mu| \le \rho_1$ of Proposition 2.1 is satisfied if ρ_2 is small enough, since $\|\mathbb{S}V\|_\mathfrak{m} \le \frac{1}{L}\|V\|_\mathfrak{m}$, by [5, Lemma C.2.a]. By [5, Lemma C.2.c] with $\mathfrak{k}=\mathfrak{k}_\Phi$, $\mathfrak{k}'=\mathfrak{k}'_\Phi$, $\mathfrak{k}_\mathfrak{l}=\mathfrak{k}_{\delta\Psi}$, and $\check{\mathfrak{m}}=\mathfrak{m}$, $\check{\mathfrak{k}}=\mathfrak{k}_\phi$, $\check{\mathfrak{k}}'=\mathfrak{k}'_\phi$, $\check{\mathfrak{k}}_\mathfrak{l}=\mathfrak{k}_{\delta\psi}$, with the choice

$$
\begin{aligned}
\mathfrak{k}_\phi &= L^{-3/2}\mathfrak{k}_\Phi = L^{-3/2}\big[\|S_{n+1}(L^2\mu)\|_\mathfrak{m}\|Q^*_{n+1}\mathfrak{Q}_{n+1}\|_\mathfrak{m} + K_1\|\mathbb{S}V\|_\mathfrak{m}\mathfrak{k}^2\big]\mathfrak{k} \\
\mathfrak{k}'_\phi &= L^{-5/2}\mathfrak{k}'_\Phi = L^{-5/2}\big[\max_{\sigma=+,-}\|B^{(\sigma)}_{n+1,L^2\mu,\nu}\|_\mathfrak{m} + K_1\|\mathbb{S}V\|_\mathfrak{m}\mathfrak{k}^2\big]\mathfrak{k}' \\
\mathfrak{k}_{\delta\psi} &= L^{-3/2}\mathfrak{k}_{\delta\Psi} = L^9\big(\tfrac{1}{L^9}\|\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_\mathfrak{m}\big)\ \mathfrak{k}_\mathfrak{l}
\end{aligned}
$$

we have $|||\delta\varphi^{(s)}_{(*)}|||_\Phi \le |||\delta\varphi_{(*)}|||_\phi$ and $\big|\big|\big|\delta\varphi^{(s)}_{(*)\nu}\big|\big|\big|_\Phi \le \big|\big|\big|\delta\varphi_{(*)\nu}\big|\big|\big|_\phi$ so that

$$
|||\delta\hat\phi_{(*)n+1}||| \le L^{3/2}|||\delta\varphi_{(*)}|||_\phi \qquad |||\delta\hat\phi_{(*)n+1,\nu}||| \le L_\nu L^{3/2}\big|\big|\big|\delta\varphi_{(*)\nu}\big|\big|\big|_\phi \tag{3.6}
$$

So, by Lemma 3.2,

$$
\begin{aligned}
|||\delta\hat\phi_{(*)n+1}||| &\le L^{3/2}K'_4\mathfrak{k}_{\delta\psi} \le L^{11}K_4\mathfrak{k}_\mathfrak{l} \\
|||\delta\hat\phi_{(*)n+1,\nu}||| &\le L_\nu L^{11}K_4\mathfrak{k}_\mathfrak{l}
\end{aligned}
$$

The hypothesis $\max\big\{\ |\mu|\ ,\ \|V\|_\mathfrak{m}(\mathfrak{k}_\phi+\mathfrak{k}_{\delta\psi})(\mathfrak{k}_\phi+\mathfrak{k}'_\phi+\mathfrak{k}_{\delta\psi})\ \big\} \le \rho'_2$ of Lemma 3.2 is satisfied if ρ_2 is small enough.

Parts (b) and (c): As in (3.6),

$$
\begin{aligned}
|||\delta\hat\phi^{(+)}_{(*)n+1}||| &\le L^{3/2}|||\delta\varphi^{(+)}_{(*)}|||_\phi\ \le L^{29}K_4\left\{\|V\|_\mathfrak{m}(\mathfrak{k}+\mathfrak{k}_\mathfrak{l})^2+|\mu|\right\}\mathfrak{k}_\mathfrak{l} \\
|||\delta\hat\phi^{(\ge 2)}_{(*)n+1}||| &\le L^{3/2}|||\delta\varphi^{(\ge 2)}_{(*)}|||_\phi \le L^{29}K_4\,\|V\|_\mathfrak{m}(\mathfrak{k}+\mathfrak{k}_\mathfrak{l})\mathfrak{k}_\mathfrak{l}^2
\end{aligned}
$$

by Lemma 3.2.a.

Part (d): By (3.5) and Lemma 3.3,

$$\begin{aligned}\delta\hat{\phi}^{(+)}_{n+1}(\psi_*,\psi,z_*,z) &= L^{3/2}\mathbb{S}[S_n(\mu)-S_n]Q_n^*\mathfrak{Q}_n D^{(n)}\mathbb{S}^{-1}z \\ &\quad - L^{\frac{3}{2}}\mathbb{L}_*^{-1}S_n(\mu)\mathcal{V}'(\varphi,\varphi_*,\varphi)\Big|^{\varphi_{(*)}=\phi_{(*)}+S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\,\delta\psi_{(*)}}_{\varphi_{(*)}=\phi_{(*)}} \\ &\quad + L^{3/2}\mathbb{L}_*^{-1}\delta\varphi^{(\geq 5)}(\phi_*,\phi,\delta\psi_*,\delta\psi)\end{aligned}$$

with the substitutions

$$\phi_{(*)} = \mathbb{S}^{-1}\phi_{(*)n+1}(\psi_*,\psi,L^2\mu,\mathbb{S}\mathcal{V}) \tag{3.7.a}$$

$$\delta\psi_{(*)} = L^{3/2}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)} \tag{3.7.b}$$

In the substitution, we expand, by Proposition 2.1.a,

$$\phi_{(*)} = \mathbb{S}^{-1}S_{n+1}(L^2\mu)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\,\psi_{(*)} + \mathbb{S}^{-1}\phi^{(\geq 3)}_{(*)n+1}(\psi_*,\psi,L^2\mu,\mathbb{S}\mathcal{V}) \tag{3.8}$$

to get the statement of the proposition with $\delta\hat{\phi}^{(\mathrm{h.o.})}$ being the sum of

$$-L^{\frac{3}{2}}\mathbb{L}_*^{-1}S_n(\mu)\mathcal{V}'(\varphi,\varphi_*,\varphi)\Big|^{\varphi_{(*)}=\phi_{(*)}+S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\,\delta\psi_{(*)}}_{\varphi_{(*)}=\mathbb{S}^{-1}S_{n+1}(L^2\mu)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\,\psi_{(*)}+S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\,\delta\psi_{(*)}}$$

and

$$L^{3/2}\mathbb{L}_*^{-1}\delta\varphi^{(\geq 5)}(\phi_*,\phi,\delta\psi_*,\delta\psi)$$

with the substitutions (3.7.b) and (3.8). As in (3.6), the specified properties of $\delta\hat{\phi}^{(\mathrm{h.o.})}$ follow from [5, Lemma C.2.c], the properties of $\phi^{(\geq 3)}_{(*)n+1}$ in Proposition 2.1.a and the properties of $\delta\varphi^{(\geq 5)}$ in Lemma 3.3.

□

4 Variations of the Background Field with Respect to the Chemical Potential μ and the Interaction V

Proposition 4.1. *There are constants*[9] $\rho_3 > 0$ *and* K_5, *such that, if*

$$\max\left\{ |\mu| \,,\, |\delta\mu| \,,\, \|V\|_{\mathfrak{m}}\mathfrak{k}^2 \,,\, \|\delta V\|_{\mathfrak{m}}\mathfrak{k}^2 \right\} \le \rho_3$$

then there are field maps $\Delta\phi_{(*)n}$, $\Delta\phi_{(*)n,\nu}$ *and* $\Delta\phi_{(*)n,D}$ *such that*

$$\begin{aligned}
\phi_{(*)n}(\psi_*,\psi,\mu+\delta\mu,\mathcal{V}+\delta\mathcal{V}) &= \phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) + \Delta\phi_{(*)n}(\psi_*,\psi,\mu,\delta\mu,\mathcal{V},\delta\mathcal{V}) \\
\partial_\nu\phi_{(*)n}(\psi_*,\psi,\mu+\delta\mu,\mathcal{V}+\delta\mathcal{V}) &= \partial_\nu\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) \\
&\qquad + \Delta\phi_{(*)n,\nu}(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi,\mu,\delta\mu,\mathcal{V},\delta\mathcal{V}) \\
D_n^{(*)}\phi_{(*)n}(\psi_*,\psi,\mu+\delta\mu,\mathcal{V}+\delta\mathcal{V}) &= D_n^{(*)}\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) \\
&\qquad + \Delta\phi_{(*)n,D}\big(\psi_*,\psi,\mu,\delta\mu,\mathcal{V},\delta\mathcal{V}\big)
\end{aligned}$$

The field maps fulfill the bounds

$$\begin{aligned}
|||\Delta\phi_{(*)n}||| &\le K_5\ \big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}^2\big)\mathfrak{k} \\
|||\Delta\phi_{(*)n,\nu}||| &\le K_5\ \big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}^2\big)\mathfrak{k}' \\
|||\Delta\phi_{(*)n,D}||| &\le K_5\big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}^2\big)\,\mathfrak{k}
\end{aligned}$$

Furthermore $\Delta\phi_{(*)n}$ *and* $\Delta\phi_{(*)n,D}$ *are of degree at least one in* $\psi_{(*)}$ *and each of* $\Delta\phi_{*n,\nu}$ *and* $\Delta\phi_{n,\nu}$ *are of degree precisely one in* $\psi_{(*)\nu}$. *Indeed,*

$$\begin{aligned}
\Delta\phi_{(*)n} &= \delta\mu\, B_{(*)n,\mu}\,\psi_{(*)} + \Delta\phi_{(*)n}^{(\ge 3)} \\
\Delta\phi_{(*)n,\nu} &= \delta\mu\, B_{(*)n,\mu,\nu}\,\psi_{(*)\nu} + \Delta\phi_{(*)n,\nu}^{(\ge 3)} \\
\Delta\phi_{(*)n,D} &= \delta\mu\, B_{(*)n,\mu,D}\,\psi_{(*)} + \Delta\phi_{(*)n,D}^{(\ge 3)}
\end{aligned}$$

where

$$\begin{aligned}
B_{(*)n,\mu} &= S_n^{(*)}\left[\mathbb{1} - (\mu+\delta\mu)S_n^{(*)}\right]^{-1} S_n(\mu)^{(*)} Q_n^*\mathfrak{Q}_n \\
B_{*n,\mu,\nu} &= S_{n,\nu}^{(+)}\left[\mathbb{1} - (\mu+\delta\mu)S_{n,\nu}^{(+)}\right]^{-1} B_{n,\nu,\mu}^{(+)} \\
B_{n,\mu,\nu} &= S_{n,\nu}^{(-)}\left[\mathbb{1} - (\mu+\delta\mu)S_{n,\nu}^{(-)}\right]^{-1} B_{n,\nu,\mu}^{(-)} \\
B_{(*)n,\mu,D} &= S_n(\mu)^{(*)} Q_n^*\mathfrak{Q}_n - \big(Q_n^*\mathfrak{Q}_n Q_n - \mu - \delta\mu\big) B_{(*)n,\mu}
\end{aligned}$$

[9] Recall Convention 1.3.

$\Delta\phi^{(\ge 3)}_{(*)n}$, $\Delta\phi^{(\ge 3)}_{(*)n,D}$ *are of degree at least three in* $\psi_{(*)}$. $\Delta\phi^{(\ge 3)}_{(*)n,\nu}$ *and* $\Delta\phi^{(\ge 3)}_{n,\nu}$ *are of degree precisely one in* $\psi_{(*),\nu}$ *and of degree at least two in* $\psi_{(*)}$. *They obey the bounds*

$$\big|\big|\big|\Delta\phi^{(\ge 3)}_{(*)n}\big|\big|\big|\ ,\ \big|\big|\big|\Delta\phi^{(\ge 3)}_{(*)n,D}\big|\big|\big| \ \le\ K_5\big(|\delta\mu|\|V\|_{\mathfrak{m}} + \|\delta V\|_{\mathfrak{m}}\big)\,\mathfrak{k}^3$$
$$\big|\big|\big|\Delta\phi^{(\ge 3)}_{(*)n,\nu}\big|\big|\big| \ \le\ K_5\big(|\delta\mu|\|V\|_{\mathfrak{m}} + \|\delta V\|_{\mathfrak{m}}\big)\,\mathfrak{k}^2\mathfrak{k}'$$

As in the proof of Lemma 3.2, we fix any $\mathfrak{k}_\phi$ and $\mathfrak{k}'$, $0 \le \nu \le 3$, and denote by $|||\ \cdot\ |||_\phi$ the (auxiliary) norm with mass $\mathfrak{m}$ that assigns the weight factors $\mathfrak{k}_\phi$ to the fields $\phi_{(*)}$ and $\mathfrak{k}'$ to the fields $\phi_{\nu(*)}$.

Lemma 4.2. *There are constants* $\rho'_3 > 0$, K'_5, *such that, if*

$$\max\big\{\,|\mu|\,,\,|\delta\mu|\,,\,\|V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\,,\,\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\,\big\} \le \rho'_3$$

then the following are true.

- *There are field maps* $\Delta\varphi_{(*)n} = \Delta\varphi_{(*)n}\big(\phi_*, \phi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}\big)$ *such that*

$$\begin{aligned}\phi_{*n}(\psi_*, \psi, \mu{+}\delta\mu, \mathcal{V}{+}\delta\mathcal{V}) &= \phi_{*n}(\psi_*, \psi, \mu, \mathcal{V}) \\ &\quad + \Delta\varphi_{*n}\big(\phi_*, \phi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}\big)\Big|_{\phi_{(*)}=\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})} \\ \phi_{n}(\psi_*, \psi, \mu{+}\delta\mu, \mathcal{V}{+}\delta\mathcal{V}) &= \phi_{n}(\psi_*, \psi, \mu, \mathcal{V}) \\ &\quad + \Delta\varphi_{n}\big(\phi_*, \phi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}\big)\Big|_{\phi_{(*)}=\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})}\end{aligned}$$

  and

$$|||\Delta\varphi_{(*)n}|||_\phi \le 4\|S_n\|_{\mathfrak{m}}\big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\,\mathfrak{k}_\phi$$

  Furthermore $\Delta\varphi_{*n}$ *and* $\Delta\varphi_n$ *are of degree at least one in* $\phi_{(*)}$. *Indeed*

$$\Delta\varphi_{(*)n} = \delta\mu\, S_n^{(*)}\,\big[\mathbb{1} - (\mu+\delta\mu)S_n^{(*)}\big]^{-1}\phi_{(*)} + \Delta\varphi^{(\ge 3)}_{(*)n} \tag{4.1}$$

  where $\Delta\varphi^{(\ge 3)}_{(*)n}$ *is the part of* $\Delta\varphi_{(*)n}$ *that is of degree at least three in* $\phi_{(*)}$, *and*

$$\big|\big|\big|\Delta\varphi^{(\ge 3)}_{(*)n}\big|\big|\big|_\phi \le 4\|S_n\|_{\mathfrak{m}}\big\{\|\delta V\|_{\mathfrak{m}} + 16\|S_n\|_{\mathfrak{m}}\ \|V\|_{\mathfrak{m}}\,|\delta\mu|\big\}\mathfrak{k}_\phi^3$$

∘ *There are field maps* $\Delta\varphi_{(*)n,\nu} = \Delta\varphi_{(*)n,\nu}\big(\phi_*, \phi, \phi_{*\nu}, \phi_\nu, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}\big)$ *such that*

$$\begin{aligned}
&\partial_\nu \phi_{(*)n}(\psi_*, \psi, \mu+\delta\mu, \mathcal{V}+\delta\mathcal{V}) \\
&\qquad = \partial_\nu \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V}) \\
&\qquad\qquad + \Delta\varphi_{(*)n,\nu}\big(\phi_*, \phi, \partial_\nu\phi_*, \partial_\nu\phi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}\big)\Big|_{\phi_{(*)}=\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})}
\end{aligned}$$

and

$$|||\Delta\varphi_{(*)n,\nu}|||_\phi \le K_5'\ \big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\mathfrak{k}_\phi'$$

Furthermore $\Delta\varphi_{*n,\nu}$ *and* $\Delta\varphi_{n,\nu}$ *are both of degree precisely one in* $\phi_{(*)\nu}$*. Indeed*

$$\Delta\varphi_{(*)n,\nu} = \delta\mu\, S_{n,\nu}^{(\pm)}\,\big[\mathbb{1} - (\mu+\delta\mu)S_{n,\nu}^{(\pm)}\big]^{-1}\phi_{(*)\nu} + \Delta\varphi_{(*)n,\nu}^{(\ge 3)} \tag{4.2}$$

where $\Delta\varphi_{*n,\nu}^{(\ge3)}$ *and* $\Delta\varphi_{n,\nu}^{(\ge3)}$ *are both of degree precisely one in* $\phi_{(*)\nu}$ *and of degree at least two in* $\phi_{(*)}$*, and*

$$\Big|\Big|\Big|\Delta\varphi_{(*)n,\nu}^{(\ge3)}\Big|\Big|\Big|_\phi \le K_5'\ \big(|\delta\mu|\|V\|_{\mathfrak{m}} + \|\delta V\|_{\mathfrak{m}}\big)\,\mathfrak{k}_\phi^2\mathfrak{k}_\phi'$$

Proof. (a) Write

$$\begin{aligned}
\phi_{(*)n}(\psi_*, \psi, \mu+\delta\mu, \mathcal{V}+\delta\mathcal{V}) &= \phi_{(*)n}(\psi_*, \psi, \mu, \mathcal{V}) + \Delta\phi_{(*)}(\psi_*, \psi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V}) \\
&= \phi_{(*)} + \Delta\phi_{(*)}
\end{aligned}$$

Then, by (1.3), using the notation of [5, Definition 3.1],

$$\begin{aligned}
&S_n^{*-1}\big(\phi_* + \Delta\phi_*\big) + (\mathcal{V}_*' + \delta\mathcal{V}_*')(\phi_* + \Delta\phi\,,\ \phi + \Delta\phi\,,\ \phi_* + \Delta\phi_*) \\
&\qquad\qquad\qquad\qquad - (\mu+\delta\mu)[\phi_* + \Delta\phi_*] = Q_n^*\mathfrak{Q}_n\psi_* \\
&S_n^{-1}\big(\phi + \Delta\phi\big) + (\mathcal{V}' + \delta\mathcal{V}')(\phi + \Delta\phi\,,\ \phi_* + \Delta\phi_*\,,\ \phi\Delta\phi) \\
&\qquad\qquad\qquad\qquad - (\mu+\delta\mu)[\phi + \Delta\phi] \quad = Q_n^*\mathfrak{Q}_n\psi
\end{aligned}$$

Subtracting these equations but with $\delta\mu = \delta\mathcal{V} = \delta\mathcal{V}_{(*)}' = \Delta\phi_{(*)} = 0$, we see that $\Delta\phi_{(*)} = \Delta\phi_{(*)}(\psi_*, \psi, \mu, \delta\mu, \mathcal{V}, \delta\mathcal{V})$ is the solution to

$$\begin{aligned}
&\tilde S_n^{*-1}\Delta\phi_* + (\mathcal{V}_*' + \delta\mathcal{V}_*')(\phi_*{+}\Delta\phi_*\,,\ \phi + \Delta\phi\,,\ \phi_*{+}\Delta\phi_*) - \mathcal{V}_*'(\phi_*\,,\ \phi\,,\ \phi_*) = \delta\mu\,\phi_* \\
&\tilde S_n^{-1}\Delta\phi \quad + (\mathcal{V}' + \delta\mathcal{V}')(\phi{+}\Delta\phi\,,\ \phi_* + \Delta\phi_*\,,\ \phi{+}\Delta\phi) \quad - \mathcal{V}'(\phi\,,\ \phi_*\,,\ \phi) \ = \delta\mu\,\phi
\end{aligned} \tag{4.3}$$

when

$$\tilde S_n^{-1} = S_n^{-1} - \mu - \delta\mu \qquad \phi_* = \phi_{*n}(\mu, \mathcal{V}) \qquad \phi = \phi_n(\mu, \mathcal{V})$$

(Recall that $\tilde S_n^*$ is the transpose, rather than the adjoint, of $\tilde S_n$.) If ρ_3' is small enough $|\mu+\delta\mu|\,\|S_n\|_{\mathfrak m}\le \frac12$, and $\|\tilde S_n\|_{\mathfrak m}\le 2\|S_n\|_{\mathfrak m}$. Rewrite (4.3) as

$$\begin{aligned}
\tilde S_n^{*-1}\Delta\phi_* \quad &+(\mathcal V_*'+\delta\mathcal V_*')(\phi_*{+}\Delta\phi_*\,,\,\phi+\Delta\phi\,,\,\phi_*{+}\Delta\phi_*)-(\mathcal V_*'+\delta\mathcal V_*')(\phi_*\,,\,\phi\,,\,\phi_*)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \delta\mu\,\phi_*-\delta\mathcal V_*'(\phi_*\,,\,\phi\,,\,\phi_*)\\
\tilde S_n^{-1}\Delta\phi \quad &+(\mathcal V'+\delta\mathcal V')(\phi{+}\Delta\phi\,,\,\phi_*+\Delta\phi_*\,,\,\phi{+}\Delta\phi)-(\mathcal V'+\delta\mathcal V')(\phi\,,\,\phi_*\,,\,\phi)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \delta\mu\,\phi-\delta\mathcal V'(\phi\,,\,\phi_*\,,\,\phi)
\end{aligned}$$

This is of the form

$$\vec\gamma=\vec f(\vec\alpha)+\vec L(\vec\alpha,\vec\gamma)+\vec B\big(\vec\alpha\,;\,\vec\gamma\big)$$

as in [3, (4.1.b)], with $X=X_n$ and

$$\begin{aligned}
\alpha_*=\phi_*\qquad \alpha=\phi\qquad \delta\alpha_*&=\delta\mu\,\phi_*\qquad \delta\alpha=\delta\mu\,\phi\qquad \vec\alpha=\big(\alpha_*,\alpha,\delta\alpha_*,\delta\alpha\big)\\
\Delta\phi_*&=\tilde S_n^*\gamma_*\qquad \Delta\phi=\tilde S_n\gamma\qquad \vec\gamma=\big(\gamma_*,\gamma\big)
\end{aligned}$$

and

$$\vec f(\vec\alpha)(u)=\begin{bmatrix}\delta\alpha_*(u)-\delta\mathcal V_*'\big(u;\alpha_*,\alpha,\alpha_*\big)\\ \delta\alpha(u)-\delta\mathcal V'\big(u;\alpha,\alpha_*,\alpha\big)\end{bmatrix}$$

$$\vec L(\vec\alpha;\vec\gamma)(u)=\begin{bmatrix}-(\mathcal V_*'+\delta\mathcal V_*')\big(u;\alpha_*,\tilde S_n\gamma,\alpha_*\big)-2(\mathcal V_*'+\delta\mathcal V_*')\big(u;\alpha_*,\alpha,\tilde S_n^*\gamma_*\big)\\ -(\mathcal V'+\delta\mathcal V')\big(u;\alpha,\tilde S_n^*\gamma_*,\alpha\big)-2(\mathcal V'+\delta\mathcal V')\big(u;\alpha,\alpha_*,\tilde S_n\gamma\big)\end{bmatrix}$$

$$\vec B(\vec\alpha;\vec\gamma)(u)=\begin{bmatrix}-(\mathcal V_*'+\delta\mathcal V_*')\big(u;\tilde S_n^*\gamma_*,\alpha,\tilde S_n^*\gamma_*\big)-2(\mathcal V_*'+\delta\mathcal V_*')\big(u;\tilde S_n^*\gamma_*,\tilde S_n\gamma,\alpha_*\big)\\ \qquad -(\mathcal V_*'+\delta\mathcal V_*')\big(u;\tilde S_n^*\gamma_*,\tilde S_n\gamma,\tilde S_n^*\gamma_*\big)\\ -(\mathcal V'+\delta\mathcal V')\big(u;\tilde S_n\gamma,\alpha_*,\tilde S_n\gamma\big)-2(\mathcal V'+\delta\mathcal V')\big(u;\tilde S_n\gamma,\tilde S_n^*\gamma_*,\alpha\big)\\ \qquad -(\mathcal V'+\delta\mathcal V')\big(u;\tilde S_n\gamma,\tilde S_n^*\gamma_*,\tilde S_n\gamma\big)\end{bmatrix}$$

Now apply [3, Proposition 4.1.a and Remark 3.5.a] with $d_{\max}=3$, $\mathfrak c=\frac12$ and

$$\kappa_1=\kappa_2=\mathfrak k_\phi\qquad \kappa_3=\kappa_4=|\delta\mu|\,\mathfrak k_\phi\qquad \lambda_1=\lambda_2=4\kappa_f$$

with

$$\kappa_f=|\delta\mu|\,\mathfrak k_\phi+\|\delta V\|_{\mathfrak m}\mathfrak k_\phi^3$$

Since

$$
\begin{aligned}
|||f_j|||_w &\le \kappa_f = \tfrac{1}{4}\lambda_j \\
|||L_j|||_{w_{\kappa,\lambda}} &\le \|\tilde{S}_n\|_{\mathfrak{m}}\|V+\delta V\|_{\mathfrak{m}}\{\kappa_j^2\lambda_{3-j}+2\kappa_1\kappa_2\lambda_j\} \\
&\le 6\|S_n\|_{\mathfrak{m}}\|V+\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\lambda_j \\
|||B_j|||_{w_{\kappa,\lambda}} &\le \|\tilde{S}_n\|_{\mathfrak{m}}^2\ \|V+\delta V\|_{\mathfrak{m}}\ \big\{\kappa_{3-j}\lambda_j^2+2\kappa_j\lambda_1\lambda_2\big\} + \|\tilde{S}_n\|_{\mathfrak{m}}^3\|V+\delta V\|_{\mathfrak{m}}\lambda_j^2\lambda_{3-j} \\
&\le 4\|S_n\|_{\mathfrak{m}}^2\ \|V+\delta V\|_{\mathfrak{m}}\ \big\{3\kappa_j+2\|S_n\|_{\mathfrak{m}}\lambda_j\big\}\lambda_j^2 \\
&\le 4\|S_n\|_{\mathfrak{m}}^2\ \|V+\delta V\|_{\mathfrak{m}}\ \big\{3+8\|S_n\|_{\mathfrak{m}}\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\big\} \\
&\qquad\qquad 4\mathfrak{k}_\phi^2\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\lambda_j \\
&\le 50\|S_n\|_{\mathfrak{m}}^2\|V+\delta V\|_{\mathfrak{m}}\ \big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\ \mathfrak{k}_\phi^2\lambda_j
\end{aligned}
$$

[3, Proposition 4.1.a] gives

$$
\begin{aligned}
\Delta\phi_{*n} &= \tilde{S}_n^*\Gamma_1\big(\phi_*,\phi,\delta\mu\,\phi_*,\delta\mu\,\phi\big) \\
\Delta\phi_n &= \tilde{S}_n\Gamma_2\big(\phi_*,\phi,\delta\mu\,\phi_*,\delta\mu\,\phi\big)
\end{aligned}
$$

with

$$
|||\Gamma_1|||_w, |||\Gamma_2|||_w \le 2\kappa_f = 2\ \big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\ \mathfrak{k}_\phi
$$

Now we prove (4.1), using the same system of equations and the same $\vec{\alpha}$, $\vec{\gamma}$, κ's and λ's. But we apply [3, Proposition 4.1.b] with

$$
\mathfrak{c} = \max_{j=1,2}\tfrac{1}{\lambda_j}|||L_j|||_{w_{\kappa,\lambda}} + 3\max_{j=1,2}\tfrac{1}{\lambda_j}|||B_j|||_{w_{\kappa,\lambda}} \le 8\|S_n\|_{\mathfrak{m}}\|V+\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2
$$

which gives, for $j=1,2$,

$$
\begin{aligned}
|||\Gamma_j^{(1)}-f_j|||_w &\le \tfrac{\mathfrak{c}}{1-\mathfrak{c}}\max_{j'=1,2}|||f_{j'}|||_w \le 16\|S_n\|_{\mathfrak{m}}\|V+\delta V\|_{\mathfrak{m}}\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\ \mathfrak{k}_\phi^3 \\
|||\Gamma_j-\Gamma_j^{(1)}|||_w &\le \max_{j'=1,2}|||B_j|||_{w_{\kappa,\lambda}} \le 200\|S_n\|_{\mathfrak{m}}^2\|V+\delta V\|_{\mathfrak{m}}\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)^2\mathfrak{k}_\phi^3
\end{aligned}
$$

where $\vec{\Gamma}^{(1)}$ is the solution of $\vec{\gamma}=\vec{f}(\vec{\alpha})+\vec{L}(\vec{\alpha},\vec{\gamma})$. Since

$$
\vec{\Gamma}^{(1)} = \vec{f}(\vec{\alpha})+\vec{L}(\vec{\alpha},\vec{\Gamma}^{(1)}) \qquad \text{and} \qquad \vec{\Gamma} = \vec{f}(\vec{\alpha})+\vec{L}(\vec{\alpha},\vec{\Gamma})+\vec{B}\big(\vec{\alpha}\,;\vec{\Gamma}\big)
$$

and $\vec{\Gamma}$ is degree at least one in $\phi_{(*)}$ and $\vec{L}$ and $\vec{B}$ are of degree three in $(\vec{\alpha},\vec{\gamma})$, both $\vec{\Gamma}^{(1)}-\vec{f}$ and $\vec{\Gamma}-\vec{f}$ are of degree at least 3 in $\phi_{(*)}$. So is $\vec{f}-\vec{F}$ where

$$
\vec{F} = \begin{bmatrix}\delta\alpha_*\\ \delta\alpha\end{bmatrix} = \begin{bmatrix}\delta\mu\,\phi_*\\ \delta\mu\,\phi\end{bmatrix}
$$

Consequently

$$\Delta\varphi_{(*)n} = \delta\mu\, \tilde{S}_n^{(*)}\, \phi_{(*)} + \Delta\varphi_{(*)}^{(\geq 3)}$$

with

$$\begin{aligned}\Delta\varphi_{*n}^{(\geq 3)} &= \tilde{S}_n^{*}\big\{[f_1 - F_1] + [\Gamma_1^{(1)} - f_1] + [\Gamma_1 - \Gamma_1^{(1)}]\big\}\\ \Delta\varphi_{n}^{(\geq 3)} &= \tilde{S}_n\big\{[f_2 - F_2] + [\Gamma_2^{(1)} - f_2] + [\Gamma_2 - \Gamma_2^{(1)}]\big\}\end{aligned}$$

As $\tilde{S}_n^{(*)} = \big[{S_n^{(*)}}^{-1} - \mu - \delta\mu\big]^{-1} = S_n^{(*)}\,\big[\mathbb{1} - (\mu+\delta\mu)S_n^{(*)}\big]^{-1}$ and

$$\begin{aligned}&|||f_j - F_j|||_w + |||\Gamma_j^{(1)} - f_j|||_w + |||\Gamma_j - \Gamma_j^{(1)}|||_w\\ &\qquad\qquad \leq \|\delta V\|_{\mathfrak{m}}\, \mathfrak{k}_\phi^3 + 32\|S_n\|_{\mathfrak{m}}\|V+\delta V\|_{\mathfrak{m}}\big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\,\mathfrak{k}_\phi^3\\ &\qquad\qquad \leq 2\big\{\|\delta V\|_{\mathfrak{m}} + 16\|S_n\|_{\mathfrak{m}}\|V\|_{\mathfrak{m}}|\delta\mu|\big\}\mathfrak{k}_\phi^3\end{aligned}$$

the desired bound on $\big|\big|\big|\Delta\varphi_{(*)n}^{(\geq 3)}\big|\big|\big|_\phi$ follows by [3, Proposition 3.2.a].

(b) By (3.3), applying ∂_ν to (4.3) gives

$$\begin{aligned}\big(\tilde{S}_{n,\nu}^{(+)}\big)^{-1}\partial_\nu\Delta\phi_* &+ L_{11}(\partial_\nu\Delta\phi_*) + L_{12}(\partial_\nu\Delta\phi) = f_*\\ \big(\tilde{S}_{n,\nu}^{(-)}\big)^{-1}\partial_\nu\Delta\phi\ &+ L_{21}(\partial_\nu\Delta\phi_*) + L_{22}(\partial_\nu\Delta\phi) = f\end{aligned}\tag{4.4}$$

where $\big(\tilde{S}_{n,\nu}^{(\sigma)}\big)^{-1} = \big(S_{n,\nu}^{(\sigma)}\big)^{-1} - \mu - \delta\mu$ and

$$\begin{aligned}L_{11}(\partial_\nu\Delta\phi_*) &= 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\phi_*, \phi, \partial_\nu\Delta\phi_*\big) + (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\partial_\nu\Delta\phi_*, T_\nu^{-1}\phi, \Delta\phi_* + T_\nu^{-1}\Delta\phi_*\big)\\ &\quad + 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\partial_\nu\Delta\phi_*, T_\nu^{-1}\Delta\phi, T_\nu^{-1}\phi_*\big)\\ &\quad + (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\partial_\nu\Delta\phi_*, T_\nu^{-1}\Delta\phi, \Delta\phi_* + T_\nu^{-1}\Delta\phi_*\big)\\ L_{12}(\partial_\nu\Delta\phi) &= (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\phi_*, \partial_\nu\Delta\phi, \phi_*\big) + 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\Delta\phi_*, \partial_\nu\Delta\phi, T_\nu^{-1}\phi_*\big)\\ &\quad + (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\Delta\phi_*, \partial_\nu\Delta\phi, \Delta\phi_*\big)\\ L_{21}(\partial_\nu\Delta\phi_*) &= (\mathcal{V}' + \delta\mathcal{V}')\big(\phi, \partial_\nu\Delta\phi_*, \phi\big) + 2(\mathcal{V}' + \delta\mathcal{V}')\big(\Delta\phi, \partial_\nu\Delta\phi_*, T_\nu^{-1}\phi\big)\\ &\quad + (\mathcal{V}' + \delta\mathcal{V}')\big(\Delta\phi, \partial_\nu\Delta\phi_*, \Delta\phi\big)\\ L_{22}(\partial_\nu\Delta\phi) &= 2(\mathcal{V}' + \delta\mathcal{V}')\big(\phi, \phi_*, \partial_\nu\Delta\phi\big) + (\mathcal{V}' + \delta\mathcal{V}')\big(\partial_\nu\Delta\phi, T_\nu^{-1}\phi_*, \Delta\phi + T_\nu^{-1}\Delta\phi\big)\\ &\quad + 2(\mathcal{V}' + \delta\mathcal{V}')\big(\partial_\nu\Delta\phi, T_\nu^{-1}\Delta\phi_*, T_\nu^{-1}\phi\big)\\ &\quad + (\mathcal{V}' + \delta\mathcal{V}')\big(\partial_\nu\Delta\phi, T_\nu^{-1}\Delta\phi_*, \Delta\phi + T_\nu^{-1}\Delta\phi\big)\end{aligned}$$

$$
\begin{aligned}
f_* = \delta\mu\,\partial_\nu\phi_* &- (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\partial_\nu\phi_*, T_\nu^{-1}\Delta\phi, \phi_* + T_\nu^{-1}\phi_*\big)\\
&- 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\partial_\nu\phi_*, T_\nu^{-1}\phi, T_\nu^{-1}\Delta\phi_*\big)\\
&- 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\phi_*, \partial_\nu\phi, T_\nu^{-1}\Delta\phi_*\big)\\
&- (\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\Delta\phi_*, \partial_\nu\phi, \Delta\phi_*\big) - 2(\mathcal{V}_*' + \delta\mathcal{V}_*')\big(\Delta\phi_*, \Delta\phi, \partial_\nu\phi_*\big)\\
&- \delta\mathcal{V}_*'\big(\phi_*, \partial_\nu\phi, \phi_*\big) - \delta\mathcal{V}_*'\big(\partial_\nu\phi_*, T_\nu^{-1}\phi, \phi_* + T_\nu^{-1}\phi_*\big)\\
f = \delta\mu\,\partial_\nu\phi &- (\mathcal{V}' + \delta\mathcal{V}')\big(\partial_\nu\phi, T_\nu^{-1}\Delta\phi_*, \phi + T_\nu^{-1}\phi\big)\\
&- 2(\mathcal{V}' + \delta\mathcal{V}')\big(\partial_\nu\phi, T_\nu^{-1}\phi_*, T_\nu^{-1}\Delta\phi\big) - 2(\mathcal{V}' + \delta\mathcal{V}')\big(\phi, \partial_\nu\phi_*, T_\nu^{-1}\Delta\phi\big)\\
&- (\mathcal{V}' + \delta\mathcal{V}')\big(\Delta\phi, \partial_\nu\phi_*, \Delta\phi\big) - 2(\mathcal{V}' + \delta\mathcal{V}')\big(\Delta\phi, \Delta\phi_*, \partial_\nu\phi\big)\\
&- \delta\mathcal{V}'\big(\partial_\nu\phi, T_\nu^{-1}\phi_*, \phi + T_\nu^{-1}\phi\big)
\end{aligned}
$$

Here we have used the "discrete product rule" (2.5). Observe that, if ρ_3' is small enough, then $|\mu + \delta\mu|\, \|S_{n,\nu}^{(\sigma)}\|_{\mathfrak{m}} \le \frac{1}{2}$, and $\|\tilde{S}_{n,\nu}^{(\sigma)}\|_{\mathfrak{m}} \le 2\|S_{n,\nu}^{(\sigma)}\|_{\mathfrak{m}}$.

The system of equations (4.4) is of the form

$$
\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha}, \vec{\gamma}) + \vec{B}\big(\vec{\alpha}\,;\,\vec{\gamma}\big)
$$

as in [3, (4.1.b)], with $\vec{\alpha} = (\alpha_1, \cdots, \alpha_6)$, $\vec{\gamma} = (\gamma_*, \gamma)$ and

$$
\begin{array}{llllll}
\alpha_1 = \phi_* & \alpha_2 = \phi & \alpha_3 = \partial_\nu\phi_* & \alpha_4 = \partial_\nu\phi & \alpha_5 = \Delta\phi_* & \alpha_6 = \Delta\phi\\
\partial_\nu\Delta\phi_* = \tilde{S}_{n,\nu}^{(+)}\gamma_* & \partial_\nu\Delta\phi = \tilde{S}_{n,\nu}^{(-)}\gamma
\end{array}
$$

and $\vec{B}(\vec{\alpha};\vec{\gamma}) = 0$ and

$$
\vec{L}(\vec{\alpha};\vec{\gamma}) = -\begin{bmatrix} L_{11}\,(\tilde{S}_{n,\nu}^{(+)}\gamma_*) \;+\; L_{12}\,(\tilde{S}_{n,\nu}^{(-)}\gamma) \\ L_{21}\,(\tilde{S}_{n,\nu}^{(+)}\gamma_*) \;+\; L_{22}\,(\tilde{S}_{n,\nu}^{(-)}\gamma) \end{bmatrix}
$$

Now apply [3, Proposition 4.1.a] with $\mathfrak{c} = \frac{1}{2}$ and

$$
\begin{array}{lll}
\kappa_1 = \kappa_2 = \mathfrak{k}_\phi & \kappa_3 = \kappa_4 = \mathfrak{k}_\phi' & \kappa_5 = \kappa_6 = 4\|S_n\|_{\mathfrak{m}}\big(|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\,\mathfrak{k}_\phi\\
\lambda_1 = \lambda_2 = 4\kappa_f
\end{array}
$$

with

$$
\begin{aligned}
\kappa_f &= \kappa_3\big\{|\delta\mu| + \|V + \delta V\|_{\mathfrak{m}} e^{2\varepsilon_n\mathfrak{m}}\big(6\kappa_1 + 3\kappa_5\big)\kappa_5 + \|\delta V\|_{\mathfrak{m}} e^{2\varepsilon_n\mathfrak{m}} 3\kappa_1^2\big\}\\
&\le 8e^{2\varepsilon_n\mathfrak{m}}\big\{|\delta\mu| + \|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big\}\mathfrak{k}_\phi'
\end{aligned}
$$

and $\varepsilon_n = \frac{1}{L^n}$. Since $\|f_j\|_w \le \kappa_f = \frac{1}{4}\lambda_j$, $\|B_j\|_{w_{\kappa,\lambda}} = 0$ and

$$\begin{aligned}\|L_j\|_{w_{\kappa,\lambda}} &\le \max_{\sigma=+,-} \|\tilde S^{(\sigma)}_{n,\nu}\|_{\mathfrak m}\|V+\delta V\|_{\mathfrak m} e^{2\varepsilon_n \mathfrak m}\Big[\big\{2\kappa_1^2+4\kappa_1\kappa_5+2\kappa_5^2\big\}\lambda_j \\ &\qquad\qquad + \big\{\kappa_1^2+2\kappa_1\kappa_5+\kappa_5^2\big\}\lambda_{3-j}\Big] \\ &\le 3\max_{\sigma=+,-} \|\tilde S^{(\sigma)}_{n,\nu}\|_{\mathfrak m}\|V+\delta V\|_{\mathfrak m} e^{2\varepsilon_n \mathfrak m}\big\{\kappa_1+\kappa_5\big\}^2\lambda_j \\ &\le 3\max_{\sigma=+,-} \|\tilde S^{(\sigma)}_{n,\nu}\|_{\mathfrak m}\|V+\delta V\|_{\mathfrak m} e^{2\varepsilon_n \mathfrak m}\big\{1+4\|S_n\|_{\mathfrak m}\big(|\delta\mu|+\|\delta V\|_{\mathfrak m}\mathfrak k_\phi^2\big)\big\}^2\mathfrak k_\phi^2\lambda_j\end{aligned}$$

[3, Propositions 4.1.a] gives

$$\begin{aligned}\partial_\nu\Delta\phi_* &= \tilde S^{(+)}_{n,\nu}\Gamma_1\big(\alpha_1,\cdots,\alpha_6\big) \\ \partial_\nu\Delta\phi &= \tilde S^{(-)}_{n,\nu}\Gamma_2\big(\alpha_1,\cdots,\alpha_6\big)\end{aligned}$$

with

$$|||\Gamma_1|||_{w_{\kappa,\lambda}}, |||\Gamma_2|||_{w_{\kappa,\lambda}} \le 16 e^{2\varepsilon_n\mathfrak m}\big\{|\delta\mu|+\|\delta V\|_{\mathfrak m}\mathfrak k_\phi^2\big\}\mathfrak k'_\phi$$

The conclusions, except for (4.2) now follow by [3, Corollary 3.3].

To prove (4.2), write

$$\begin{bmatrix}\partial_\nu\Delta\varphi_{*n}\big(\phi_*,\phi,\mu,\delta\mu\big)\\ \partial_\nu\Delta\varphi_n\big(\phi_*,\phi,\mu,\delta\mu\big)\end{bmatrix} = \begin{bmatrix}\tilde S^{(+)}_{n,\nu} & 0\\ 0 & \tilde S^{(-)}_{n,\nu}\end{bmatrix}\Big\{\vec f(\vec\alpha)+\vec L\Big(\vec\alpha\,,\,\big[\Gamma_1(\vec\alpha),\Gamma_2(\vec\alpha)\big]\Big)\Big\}$$

with

$$\begin{aligned}&\alpha_1=\phi_* &&\alpha_2=\phi &&\alpha_3=\partial_\nu\phi_* &&\alpha_4=\partial_\nu\phi\\ &\alpha_5=\Delta\varphi_{*n}\big(\phi_*,\phi,\mu,\delta\mu,\mathcal V,\delta\mathcal V\big) && && \alpha_6=\Delta\varphi_n\big(\phi_*,\phi,\mu,\delta\mu,\mathcal V,\delta\mathcal V\big)\end{aligned}$$

Observe that the right hand side is of the form

$$\delta\mu\begin{bmatrix}\tilde S^{(+)}_{n,\nu}\partial_\nu\phi_*\\ \tilde S^{(-)}_{n,\nu}\partial_\nu\phi\end{bmatrix} + \begin{bmatrix}\Delta\varphi^{(\ge3)}_{*n,\nu}\big(\phi_*,\phi,\partial_\nu\phi_*,\partial_\nu\phi,\mu,\delta\mu,\mathcal V,\delta\mathcal V\big)\\ \Delta\varphi^{(\ge3)}_{n,\nu}\big(\phi_*,\phi,\partial_\nu\phi_*,\partial_\nu\phi,\mu,\delta\mu,\mathcal V,\delta\mathcal V\big)\end{bmatrix}$$

with $\Delta\varphi^{(\ge3)}_{(*)n,\nu}\big(\phi_*,\phi,\partial_\nu\phi_*,\partial_\nu\phi,\mu,\delta\mu,\mathcal V\big)$ a finite sum of terms each of which is either of the form $\pm\tilde S^{(\pm)}_{n,\nu}(\mathcal V'_{(*)}+\delta\mathcal V'_{(*)})(\zeta_1,\zeta_2,\zeta_3)$ with

- ◦ exactly one of ζ_1, ζ_2, ζ_3 being one of $\partial_\nu\phi_{(*)}$, $\partial_\nu\Delta\phi_{(*)}$ (which are of degree precisely one in $\partial_\nu\phi_{(*)}$) and

◦ each of the remaining two ζ_j's being one of $\phi_{(*)}$, $\Delta\phi_{(*)}$, possibly translated by T_ν^{-1}, (which are of degree at least one in $\phi_{(*)}$) and

◦ at least one of ζ_1, ζ_2, ζ_3 being one of $\Delta\phi_{(*)}$, $\partial_\nu\Delta\phi_{(*)}$, possibly translated by T_ν^{-1}.

or of the form $\pm\tilde{S}_{n,\nu}^{(\pm)}\delta\mathcal{V}'_{(*)}(\zeta_1,\zeta_2,\zeta_3)$ with

◦ exactly one of ζ_1, ζ_2, ζ_3 being a $\partial_\nu\phi_{(*)}$ and

◦ the remaining two ζ_j's being a $\phi_{(*)}$, possibly translated by T_ν^{-1}.

The degree properties and bounds on $\Delta\varphi_{(*)n,\nu}^{(\geq 3)}$ follow, with, in the bound,

◦ a factor of $\|V+\delta V\|_{\mathfrak{m}}$ coming from the kernel of $\mathcal{V}'_{(*)}+\delta\mathcal{V}'_{(*)}$,

◦ a factor of $\|\delta V\|_{\mathfrak{m}}$ coming from the kernel of $\delta\mathcal{V}'_{(*)}$,

◦ $\partial_\nu\phi_{(*)}$ contributing a factor $\mathfrak{k}'_\phi$,

◦ $\partial_\nu\Delta\phi_{(*)}$ contributing a factor of $\mathrm{const}\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\mathfrak{k}'_\phi\leq\mathrm{const}\,\mathfrak{k}'_\phi$,

◦ each $\phi_{(*)}$, possibly translated by T_ν^{-1}, contributing a factor of $\mathrm{const}\,\mathfrak{k}_\phi$, and

◦ each $\Delta\phi_{(*)}$, possibly translated by T_ν^{-1}, giving a factor of

$$\mathrm{const}\big(|\delta\mu|+\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\big)\mathfrak{k}_\phi\leq\mathrm{const}\,\mathfrak{k}_\phi$$

since $\|V\|_{\mathfrak{m}}\|\delta V\|_{\mathfrak{m}}\mathfrak{k}_\phi^2\leq\mathrm{const}\|\delta V\|_{\mathfrak{m}}$. □

Proof of Proposition 4.1. We apply Lemma 4.2 with

$$\begin{aligned}\mathfrak{k}_\phi&=2\|S_n\|_{\mathfrak{m}}\|Q_n^*\mathfrak{Q}_n\|_{\mathfrak{m}}\mathfrak{k}+K_1\|V\|_{\mathfrak{m}}\mathfrak{k}^3\\ \mathfrak{k}'_\phi&=\max_{\sigma=\pm}\|B_{n,\mu,\nu}^{(\sigma)}\|_{\mathfrak{m}}\mathfrak{k}'+K_1\|V\|_{\mathfrak{m}}\mathfrak{k}^2\ \mathfrak{k}'\end{aligned}$$

First observe that $\mathfrak{k}_\phi$ and $\mathfrak{k}'_\phi$ are each bounded by a constant times $\mathfrak{k}$ and $\mathfrak{k}'$, respectively. So for a suitable choice of ρ_3, the hypothesis of Lemma 4.2 is satisfied. The claims concerning $\Delta\phi_{(*)n}$ and $\Delta\phi_{(*)n,\nu}$ now follow by substituting

$$\begin{aligned}\phi_{(*)}=\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})&=S_n(\mu)^{(*)}Q_n^*\mathfrak{Q}_n\,\psi_{(*)}+\phi_{(*)n}^{(\geq 3)}(\psi_*,\psi,\mu,\mathcal{V})\\ \partial_\nu\phi_*=\partial_\nu\phi_{*n}(\psi_*,\psi,\mu,\mathcal{V})&=B_{n,\nu,\mu}^{(+)}\partial_\nu\psi_*+\phi_{*n,\nu}^{(\geq 3)}\big(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi,\mu,\mathcal{V}\big)\\ \partial_\nu\phi=\partial_\nu\phi_n(\psi_*,\psi,\mu,\mathcal{V})&=B_{n,\nu,\mu}^{(-)}\partial_\nu\psi+\phi_{n,\nu}^{(\geq 3)}\big(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi,\mu,\mathcal{V}\big)\end{aligned}$$

into the conclusions of Lemma 4.2, using Proposition 2.1 and [3, Corollary 3.3].

From (4.3) we see

$$
\begin{aligned}
D_n^*\Delta\phi_* &= \delta\mu\,\phi_* - \big(Q_n^*\mathfrak{Q}_nQ_n - \mu - \delta\mu\big)\Delta\phi_* - (\mathcal{V}_*' + \delta\mathcal{V}_*')(\Phi_*, \Phi, \Phi_*)\Big|_{\Phi(*)=\phi_{(*)}}^{\Phi(*)=\phi_{(*)}+\Delta\phi_{(*)}} \\
&\qquad - \delta\mathcal{V}_*'(\phi_*, \phi, \phi_*) \\
D_n\Delta\phi &= \delta\mu\,\phi - \big(Q_n^*\mathfrak{Q}_nQ_n - \mu - \delta\mu\big)\Delta\phi - (\mathcal{V}' + \delta\mathcal{V}')(\Phi, \Phi_*, \Phi)\Big|_{\Phi(*)=\phi_{(*)}}^{\Phi(*)=\phi_{(*)}+\Delta\phi_{(*)}} \\
&\qquad - \delta\mathcal{V}'(\phi, \phi_*, \phi)
\end{aligned}
$$

with $\phi_{(*)} = \phi_{(*)n}(\mu, \mathcal{V})$ and $\Delta\phi_{(*)} = \phi_{(*)n}(\mu + \delta\mu, \mathcal{V} + \delta\mathcal{V}) - \phi_{(*)n}(\mu, \mathcal{V})$. Now just substitute for $\phi_{(*)n}$ using Proposition 2.1 and for $\Delta\phi_{(*)n}$ using the first part of this proposition.

□

5 The Critical Field

In this subsection we formulate and prove a precise version of [5, Proposition 1.15]. Recall from [5, (4.3)] that

$$\hat\psi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) = \mathbb{S}\big[\psi_{(*)n}(\mathbb{S}^{-1}\psi_*,\mathbb{S}^{-1}\psi,\mu,\mathcal{V})\big] \tag{5.1}$$

is a rescaled version of the critical field $\psi_{(*)n}$.

Proposition 5.1. *Let $n\ge 1$. There are constants*[10] K_6*,* $\rho_4>0$ *such that the following hold if* $\frac{1}{L}\|V\|_{\mathfrak{m}}\mathfrak{k}^2 + L^2|\mu| \le \rho_4$.

There are field maps $\hat\psi^{(\ge 3)}_{(*)n}\big(\psi_*,\psi,\mu\big)$ *such that*

$$\hat\psi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) = \tfrac{a}{L^2}\mathbb{S}C^{(n)}(\mu)^{(*)}Q^*\mathbb{S}^{-1}\psi_{(*)} + \hat\psi^{(\ge 3)}_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})$$

where

$$\begin{aligned} C^{(n)}(\mu) &= \big(\tfrac{a}{L^2}Q^*Q + \Delta^{(n)}(\mu)\big)^{-1} \\ \Delta^{(n)}(\mu) &= \begin{cases} \mathfrak{Q}_n - \mathfrak{Q}_n\,Q_nS_n(\mu)Q_n^*\mathfrak{Q}_n & \text{if } n\ge 1 \\ D_0-\mu & \text{if } n=0 \end{cases} \end{aligned}$$

and

$$|||\hat\psi_{(*)n}||| \le K_6\mathfrak{k} \qquad \big|\big|\big|\hat\psi^{(\ge 3)}_{(*)n}\big|\big|\big| \le K_6\tfrac{1}{L}\|V\|_{\mathfrak{m}}\mathfrak{k}^3$$

Furthermore $\hat\psi^{(\ge 3)}_{(*)n}$ *is of degree at least one in* $\psi_{(*)}$ *and is of degree at least three in* (ψ_*,ψ).

There are also field maps $\hat\psi_{(*)n,\nu}\big(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu,\mathcal{V}\big)$ *and* $\hat\psi^{(\ge 3)}_{(*)n,\nu}\big(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu,\mathcal{V}\big)$ *and a linear operator* $B_{\psi_{(*)},n,\nu}(\mu)$ *such that*

$$\begin{aligned} \partial_\nu\hat\psi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) &= \hat\psi_{(*)n,\nu}\big(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi,\mu,\mathcal{V}\big) \\ &= B_{\psi_{(*)},n,\nu}(\mu)\,\partial_\nu\psi_{(*)} + \hat\psi^{(\ge 3)}_{(*)n,\nu}\big(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi,\mu,\mathcal{V}\big) \end{aligned}$$

and

$$|||\hat\psi_{(*)n,\nu}||| \le K_6\mathfrak{k}' \qquad \big|\big|\big|\hat\psi^{(\ge 3)}_{(*)n,\nu}\big|\big|\big| \le K_6\tfrac{1}{L}\|V\|_{\mathfrak{m}}\mathfrak{k}^2\mathfrak{k}'$$

Furthermore $\hat\psi^{(\ge 3)}_{*n,\nu}$ *and* $\hat\psi^{(\ge 3)}_{n,\nu}$ *are each of degree precisely one in* $\psi_{(*)\nu}$ *and of degree at least two in* $\big(\psi_*,\psi\big)$.

[10] Recall Convention 1.3.

Proof. Set

$$\check{S}_{n+1}(\mu) = L^2\,\mathbb{S}^{-1}S_{n+1}(L^2\mu)\mathbb{S} = \big\{D_n - \mu + \check{Q}_{n+1}^*\check{\mathfrak{Q}}_{n+1}\check{Q}_{n+1}\big\}^{-1} : \mathcal{H}_n \to \mathcal{H}_n \qquad (5.2)$$

where, as in [5, Lemma 2.4], $\check{Q}_n = \mathbb{S}^{-1}Q_n\mathbb{S}$ and $\check{\mathfrak{Q}}_n = \frac{1}{L^2}\mathbb{S}^{-1}\mathfrak{Q}_n\mathbb{S}$. Observe that, by [4, Remark 10.e] and the fact that under the substitutions [5, (3.3)], $\check{\mathfrak{Q}} = \check{\mathfrak{Q}}_{n+1}$, $\check{Q}_- = \check{Q}_{n+1}$ and $\check{S} = \check{S}_{n+1}(\mu)$,

$$\tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^* = \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)^{-1}\big\{\tfrac{a}{L^2}Q^* + \mathfrak{Q}_nQ_n\check{S}_{n+1}(\mu)^{(*)}\check{Q}_{n+1}^*\check{\mathfrak{Q}}_{n+1}\big\} \qquad (5.3)$$

By [5, Definition 3.2] and Proposition 2.1 with n replaced by $n+1$,

$$\check{\phi}_{(*)n+1}(\theta_*,\theta,\mu,\mathcal{V}) = \mathbb{S}^{-1}S_{n+1}(L^2\mu)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\,\mathbb{S}\theta_{(*)} + \mathbb{S}^{-1}\big[\phi_{(*)n+1}^{(\ge 3)}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V})\big]$$

Hence, by the definition of $\psi_{(*)n}$ in [5, Proposition 3.4, Lemma 2.4.b], (5.2) and (5.3),

$$\begin{aligned}
\psi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) &= \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)^{-1}\big\{\tfrac{a}{L^2}Q^*\theta_{(*)} + \mathfrak{Q}_nQ_n\check{\phi}_{(*)n+1}(\theta_*,\theta,\mu,\mathcal{V})\big\}\\
&= \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)^{-1}\big\{\tfrac{a}{L^2}Q^* + \mathfrak{Q}_nQ_n\check{S}_{n+1}(\mu)^{(*)}\check{Q}_{n+1}^*\check{\mathfrak{Q}}_{n+1}\big\}\theta_{(*)}\\
&\qquad + \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)^{-1}\mathfrak{Q}_nQ_n\mathbb{S}^{-1}\big[\phi_{(*)n+1}^{(\ge 3)}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V})\big]\\
&= \tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^*\theta_{(*)} + A_{\psi,\phi}\mathbb{S}^{-1}\big[\phi_{(*)n+1}^{(\ge 3)}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V})\big]
\end{aligned} \qquad (5.4)$$

where

$$A_{\psi,\phi} = (\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n)^{-1}\mathfrak{Q}_nQ_n$$

So, by (5.1),

$$\begin{aligned}
\hat{\psi}_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) &= \mathbb{S}\big[\psi_{(*)n}(\mathbb{S}^{-1}\psi_*,\mathbb{S}^{-1}\psi,\mu,\mathcal{V})\big]\\
&= \tfrac{a}{L^2}\mathbb{S}C^{(n)}(\mu)^{(*)}Q^*\mathbb{S}^{-1}\psi_{(*)} + \mathbb{S}A_{\psi,\phi}\mathbb{S}^{-1}\phi_{(*)n+1}^{(\ge 3)}(\psi_*,\psi,L^2\mu,\mathbb{S}\mathcal{V})
\end{aligned}$$

Defining

$$\hat{\psi}_{(*)n}^{(\ge 3)}(\psi_*,\psi,\mu,\mathcal{V}) = \mathbb{S}A_{\psi,\phi}\mathbb{S}^{-1}\phi_{(*)n+1}^{(\ge 3)}(\psi_*,\psi,L^2\mu,\mathbb{S}\mathcal{V})$$

we have the specified bounds on $\hat{\psi}_{(*)n}(\psi_*,\psi,\mu)$, by [2, Propostion 6.1], Proposition 2.1.a and the fact that the kernel, $V^{(s)}$, of $\mathbb{S}\mathcal{V}$ obeys

$$\|V^{(s)}\|_{\mathfrak{m}} \le \tfrac{1}{L}\|V\|_{\mathfrak{m}}$$

by [5, Lemma C.2.a].

For $\partial_\nu \psi_{(*)n}$ we use that, by [2, Proposition 6.1.b],

$$\begin{aligned}\partial_\nu \hat\psi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V}) &= \partial_\nu \tfrac{a}{L^2} \mathbb{S} C^{(n)}(\mu)^{(*)} Q^* \mathbb{S}^{-1} \psi_{(*)} + \partial_\nu \mathbb{S} A_{\psi,\phi} \mathbb{S}^{-1} \phi^{(\geq 3)}_{(*)n+1}\big(\psi_*,\psi, L^2\mu, \mathbb{S}\mathcal{V}\big) \\ &= \mathbb{S} A_{\psi_{(*)}\theta_{(*)}\nu}(\mu) \mathbb{S}^{-1}\ \partial_\nu \psi_{(*)} + \mathbb{S} A_{\psi,\phi,\nu} \mathbb{S}^{-1} \phi^{(\geq 3)}_{(*)n+1,\nu}\big(\psi_*,\psi,\partial_\nu\psi_*,\partial_\nu\psi, L^2\mu, \mathbb{S}\mathcal{V}\big)\end{aligned}$$

Now apply [2, Proposition 6.1.b] and, for the second term, Proposition 2.1.b. □

Remark 5.2. By (5.1), the definition of $\psi_{(*)0}$ in [5, Proposition 3.4 and Definition 3.2], we have

$$\hat\psi_{(*)0}(\psi_*,\psi,\mu,\mathcal{V}) = \phi_{(*)1}(\psi_*,\psi,L^2\mu,\mathbb{S}\mathcal{V})$$

Hence Proposition 2.1 provides the existence of, properties of, and bounds on $\hat\psi_{(*)0}$.

Remark 5.3. [5, Proposition 1.15] follows from [5, Proposition 3.4]. To get bounds on $\psi_{(*)n}$, write, by (5.1), $\psi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) = \mathbb{S}^{-1}\big[\hat\psi_{(*)n}(\mathbb{S}\theta_*,\mathbb{S}\theta,\mu,\mathcal{V})\big]$ and apply Proposition 5.1.

Remark 5.4 (The complex conjugate of the critical field)**.** There exists a constant K_7 such that the following holds for all $n \geq 1$. Let $\theta(y)$ be a field on $\mathcal{X}^{(n+1)}_{-1}$ such that[11] $|\theta(y)| < \frac{1}{L^{3/2}}\mathfrak{k}$ and $|\partial_\nu\theta(y)| < \frac{1}{L^{3/2}L_\nu}\mathfrak{k}'$ for all $y \in \mathcal{X}^{(n+1)}_{-1}$ and $0 \leq \nu \leq 3$. Then

$$\big|\psi_{*n}(\theta^*,\theta,\mu,\mathcal{V})^*(x) - \psi_n(\theta^*,\theta,\mu,\mathcal{V})(x)\big| \leq K_7\mathfrak{k}' \qquad \text{for all } x \in \mathcal{X}^{(n)}_0$$

Proof. By [5, Proposition 3.4],

$$\begin{aligned}&\psi_{*n}(\theta^*,\theta,\mu,\mathcal{V})^* - \psi_n(\theta^*,\theta,\mu,\mathcal{V}) \\ &\qquad = A_{\psi,\phi}\mathbb{S}^{-1}\big[\phi^*_{*n+1}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V}) - \phi_{n+1}(\mathbb{S}\theta_*,\mathbb{S}\theta,L^2\mu,\mathbb{S}\mathcal{V})\big]\end{aligned}$$

with $A_{\psi,\phi}$ as after (5.4). Now apply Remark 2.2. □

[11] Recall that $L_0 = L^2$ and $L_\nu = L$ for $\nu = 1,2,3$.

A Norms and a Fixed Point Theorem

We use the terminology "field map" to designate an analytic map that assigns to one or more fields on a finite set X another field on a finite set Y. We assume that X and Y are equipped with volume factors (like the volume of a fundamental cell in a finite lattice) vol_X and vol_Y. Then such a field map $\phi(\psi_1, \cdots, \psi_n)$ has a unique representation as a power series

$$\phi(\psi_1, \cdots, \psi_n)(y) = \sum_{r_1,\cdots,r_n \geq 0} \mathrm{vol}_X^{r_1+\cdots+r_n} \sum_{\substack{\vec{x}_i \in X^{r_i} \\ 1 \leq i \leq n}} \phi_{r_1,\cdots,r_n}\big(y; \vec{x}_1, \cdots, \vec{x}_n\big)\, \psi_1(\vec{x}_1) \cdots \psi_n(\vec{x}_n)$$

where the coefficients $\phi_{r_1,\cdots,r_n}\big(y; \vec{x}_1, \cdots, \vec{x}_n\big)$ are invariant under permutations of the components of each vector $\vec{x}_i$ and where, for $\vec{x} = (x_1, \cdots, x_r) \in X^r$ we set $\psi(\vec{x}) = \prod_{i=1}^r \psi(x_i)$.

To measure the size of field maps, we assume that X and Y are both subsets of a common metric space with metric d. As in [3, §2], we introduce norms whose finiteness implies that all the kernels in its power series representation are small and decay exponentially as their arguments separate. The norm of ϕ with mass $\mathfrak{m}$ and weight factors $\kappa_1, \cdots, \kappa_n > 0$ is defined to be

$$|||\phi||| = \sum_{r_1,\cdots,r_n \geq 0} \big\|\phi_{r_1,\cdots,r_n}\big\|_{\mathfrak{m}} \prod_{i=1}^{r} \kappa_i^{r_i}$$

where

$$\big\|\phi_{r_1,\cdots,r_n}\big\|_{\mathfrak{m}} = \max\big\{L_{\mathfrak{m}}(\phi_{r_1,\cdots,r_n}),\, R_{\mathfrak{m}}(\phi_{r_1,\cdots,r_n})\big\}$$

and

$$L_{\mathfrak{m}}(\phi_{r_1,\cdots,r_n}) = \max_{y\in Y}\ \mathrm{vol}_X^{r_1+\cdots+r_n} \sum_{\substack{\vec{x}_i \in X^{r_i} \\ 1 \leq i \leq n}} \big|\phi_{r_1,\cdots,r_n}\big(y; \vec{x}_1, \cdots, \vec{x}_n\big)\big| e^{\mathfrak{m}\tau_d(y,\vec{x}_1,\cdots,\vec{x}_n)}$$

$$R_{\mathfrak{m}}(\phi_{r_1,\cdots,r_n}) = \max_{x'\in X} \max_{\substack{1\leq j\leq n \\ r_j \neq 0 \\ 1 \leq i \leq r_j}}\ \mathrm{vol}_Y \sum_{y \in Y}\ \mathrm{vol}_X^{r_1+\cdots+r_n-1} \sum_{\substack{\vec{x}_\ell \in X^{r_\ell} \\ 1 \leq \ell \leq n \\ (\vec{x}_j)_i = x'}} \big|\phi_{r_1,\cdots,r_n}\big(y; \vec{x}_1, \cdots, \vec{x}_n\big)\big|\, e^{\mathfrak{m}\tau_d(y;\vec{x}_1,\cdots,\vec{x}_n)}$$

where the tree length $\tau_d(x_1, \cdots, x_p)$ is the minimal length of a tree in the common metric space that has $x_1, \cdots, x_p$ among its vertices.

The main tool that we use in the proof of the existence of and bounds on the background field is [3, Proposition 4.1], which provides solutions $\vec{\gamma} = \vec{\Gamma}(\vec{\alpha})$ to equations of the form

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}\big(\vec{\alpha}, \vec{\gamma}\big) + \vec{B}\big(\vec{\alpha}, \vec{\gamma}\big)$$

Here

- $\vec{f}(\vec{\alpha}) = \big(f_1(\vec{\alpha}), \cdots, f_s(\vec{\alpha})\big)$ is an s–tuple of field maps with each $f_j(\vec{\alpha})$ mapping the r–tuple of fields $\vec{\alpha} = \big(\alpha_1, \cdots, \alpha_r\big)$ on X to the field $f_j(\vec{\alpha})$ on Y.
- $\vec{L}$ and $\vec{B}$ are both s–tuples of field maps with each $j^{\rm th}$ component mapping the $(r+s)$–tuple of fields $(\vec{\alpha}, \vec{\gamma})$ on X and Y to the field $L_j(\vec{\alpha}, \vec{\gamma})$, respectively $B_j(\vec{\alpha}, \vec{\gamma})$, on Y.
- Each L_j is linear in $\vec{\gamma}$. Each B_j is of degree at least two and at most $d_{\max}$ in $\vec{\gamma}$.

For the readers convenience, here is the basic statement of [3, Proposition 4.1].

Proposition A.1. *Let $\kappa_1, \cdots, \kappa_s$ and $\lambda_1, \cdots, \lambda_r$ be weight factors for the fields $\alpha_1, \cdots, \alpha_s$, on X, and $\gamma_1, \cdots, \gamma_r$, on Y, respectively. For s–tuples of field maps $\vec{\Gamma}(\vec{\alpha}) = \big(\Gamma_1(\vec{\alpha}), \cdots, \Gamma_s(\vec{\alpha})\big)$, we introduce the norm*

$$\|\vec{\Gamma}\| = \max_{1\le j\le r} \tfrac{1}{\lambda_j} |||\Gamma_j|||$$

where $|||\ \cdot\ |||$ is the norm with mass $\mathfrak{m}$ and weight factors $\kappa_1, \cdots, \kappa_s$. Denote by $\mathcal{B}_1 = \big\{\ \vec{\Gamma}\ \big|\ \|\vec{\Gamma}\| \le 1\ \big\}$ the closed unit ball.

Let $0 < \mathfrak{c} < 1$. Assume that, in the notation above,

$$\begin{aligned} |||f_j||| + |||L_j||| + |||B_j||| &\le \lambda_j \\ |||L_j||| + d_{\max}|||B_j||| &\le \mathfrak{c}\lambda_j \end{aligned}$$

for $1 \le j \le r$. Then there is a unique $\vec{\Gamma} \in \mathcal{B}_1$ for which

$$\vec{\Gamma}(\vec{\alpha}) = \vec{f}(\vec{\alpha}) + \vec{L}\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big) + \vec{B}\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big)$$

Furthermore

$$\max_j \tfrac{1}{\lambda_j}|||\Gamma_j||| \le \tfrac{1}{1-\mathfrak{c}} \max_j \tfrac{1}{\lambda_j}|||f_j||| \qquad \max_j \tfrac{1}{\lambda_j}|||\Gamma_j - f_j||| \le \tfrac{\mathfrak{c}}{1-\mathfrak{c}} \max_j \tfrac{1}{\lambda_j}|||f_j|||$$

There are more refined statements in [3, Proposition 4.1].

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