Forbidden Submatrices

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Subsets of a Finite Set and Simple Matrices

Consider the following family of subsets of $\{1,2,3,4\}$:

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

The incidence matrix A of the family $\mathcal{A}$ of subsets of $\{1,2,3,4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{A}|$$

Definition Let F and m be given.

$$\mathit{Avoids}(m, F) = \{A : A \text{ is } m\text{-rowed, simple, no submatrix } F\}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is submatrix of} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 1 & \boxed{0} \\ 0 & \boxed{0} & 0 & 0 & 1 & \boxed{1} \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Note that $B \notin \mathit{Avoids}(4, F)$.

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Note that $B \notin Avoids(4, F)$.

We consider the property of forbidding a submatrix F in A.

Definition Let

$$fs(m, F) = \max_A \{\|A\| : A \in Avoids(m, F)\}$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

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Example: Submatrices vs. Configurations

Proposition $fs(m, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = m+1$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix} \in Avoids(m, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

Proposition $forb(m, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 2$

(column of 0's and column of 1's)

Some Main Results

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles, Shelah 72, Vapnik, Chervonenkis 71) Let A be an $m \times n$ simple matrix with no configuration K_k. Then

$$n \le forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Theorem Let α be a $k \times 1$ column. Then

$$fs(m, \alpha) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Note that $fs(m, \alpha) \le forb(m, K_k)$. Thus we need a construction of an $m \times forb(m, K_k)$ simple matrix with no submatrix α.

Theorem (A 85) Let F be a $k \times \ell$ matrix. Then F is a submatrix of $K_{13k\log\ell}$. Thus $fs(m,F)$ is $O(m^{13k\log\ell-1})$

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Conjecture (A, Frankl, Füredi, Pach 86,87) If F is k-rowed then $fs(m,F)$ is $O(m^k)$.

This is our motivating conjecture

Theorem (Frankl, Füredi, Pach 87)

$$\text{Let } F = k\left\{\begin{bmatrix}1 & 0\\ \vdots & \vdots\\ 1 & 0\\ 1 & 0\end{bmatrix}\right.$$

$$\text{Then } fs(m, F) \text{ is } (1 + o(1))\binom{m}{k}.$$

New Results

Let ℓ, k be given.

Theorem (A, Chen 12) $fs(m, \left[\overbrace{\begin{array}{c} 1\,0\,1\,0\cdots \\ 0\,1\,0\,1\cdots \end{array}}^{\ell}\right])$ is $O(m^2)$.

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We conjecture $fs(m, \left[\overbrace{\begin{array}{c|c|c}1\,0\,1\,0 & 1\,0\,1\,0 & \cdots\\0\,1\,1\,0 & 0\,1\,1\,0 & \cdots\end{array}}^{4\ell}\right])$ is $O(m^2)$.

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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If A has no submatrix F, then # contributions at most $2\binom{m}{2}$ which is $\Theta(m^2)$.

Amortization idea

Adding something to a bucket is called a contribution.
Assume F is $2 \times \ell$.

For $A \in Avoids(m, F)$, we know that each bucket can receive at most $\ell - 1$ contributions and so total # contributions is $O(m^2)$.

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(some may make more, some may make no contributions).

A column that makes $k > 0$ contributions can be considered to yield k credits with one credit to pay for the column and $k - 1$ credits saved to pay for future columns. Columns that make no contributions are called filler columns. We use our saved credits to pay for these columns.

Theorem Let F be the $2 \times \ell$ $(0,1)$-matrix

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix}.$$

Then

$$fs(m, F) \leq (\ell - 1)\binom{m}{2} + m + 1. \quad \blacksquare$$

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We need to pay for the possible $m + 1$ initial filler columns, before any contributions are made.

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After those possible initial filler columns, each new contributing column pays for all new filler columns it 'creates'.

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Initially, the filler columns are

$$\left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \dots \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right\}$$

because the next column of F is $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ on every pair of rows.

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Introduce a digraph (tournament) on rows $1, 2, \ldots, m$ where

$$i \longrightarrow j$$

if the next column of F we are looking for to add to the bucket is

$$\begin{matrix} i \\ j \end{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ on rows } i, j.$$

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

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Example

$$A = \begin{bmatrix} \color{red}{0} & 1 & 0 & 0 & 1 & 1 \\ \color{red}{1} & 1 & 0 & 1 & 0 & 0 \\ \color{red}{0} & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

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$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Example

$$A = \begin{bmatrix} 0 & \color{red}{1} & 0 & 0 & 1 & 1 \\ 1 & \color{red}{1} & 0 & 1 & 0 & 0 \\ 0 & \color{red}{0} & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & {\color{red}0} & 1 & 1 \\ 1 & 1 & 0 & {\color{red}1} & 0 & 0 \\ 0 & 0 & 1 & {\color{red}1} & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \left[\begin{matrix} \\ \end{matrix} \right. \qquad \begin{matrix} \mathbf{1} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} \mathbf{1} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} \mathbf{1} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Claim: the digraph is always a total order

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Claim: the digraph is always a total order

Initially, all edges point down, filler columns as follows

$$\left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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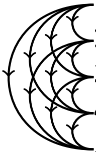
After processing a contributing column, "reverse shuffle" rows with 0's and 1's to make the contributing column look like one of the above

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Claim: the digraph is always a total order

Example



$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{matrix} \begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Claim: the digraph is always a total order

Example

$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{matrix} \begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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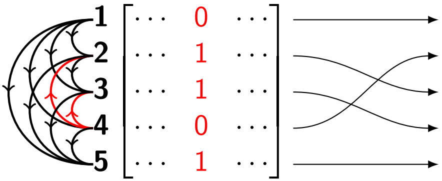
Example



$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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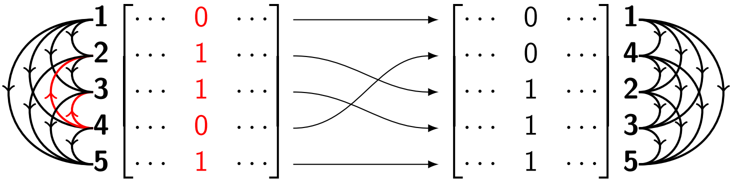
Example



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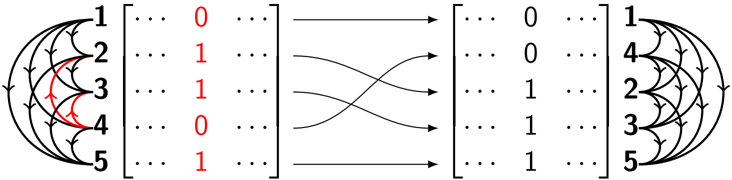

Flipped edges are unflipped by the shuffling, resulting in a total order.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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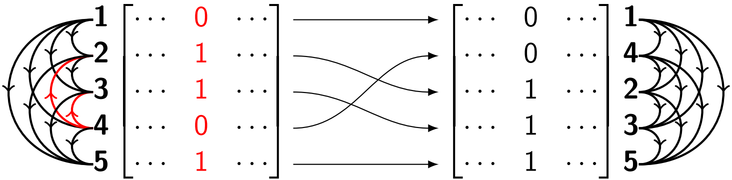
Example



Flipped edges are unflipped by the shuffling, resulting in a total order. Filler columns always look the same (0's above 1's) $\Rightarrow$ at most $m + 1$.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Credits (i.e. to pay for new fillers) come from excess contributions.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Credits (i.e. to pay for new fillers) come from excess contributions. After the row shuffling, every filler column looks like

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$F=\begin{bmatrix}1&0&1&0&\cdots\\0&1&0&1&\cdots\end{bmatrix}\quad(2\times\ell \text{ matrix})$$

Credits (i.e. to pay for new fillers) come from excess contributions. After the row shuffling, every filler column looks like

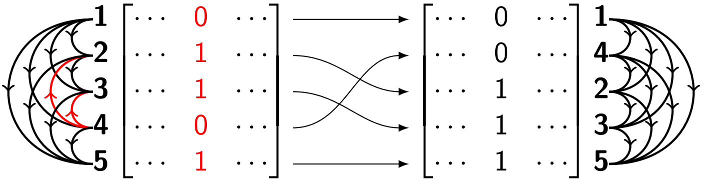
$$\left[\begin{array}{c}0\\ \vdots\\ 0\\ \hline 1\\ \vdots\\ 1\end{array}\right]$$

Depending on location of split between 0's and 1's, a filler column might have looked the same *before* the shuffling i.e. filler already paid for.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

Credits (i.e. to pay for new fillers) come from excess contributions.

Example



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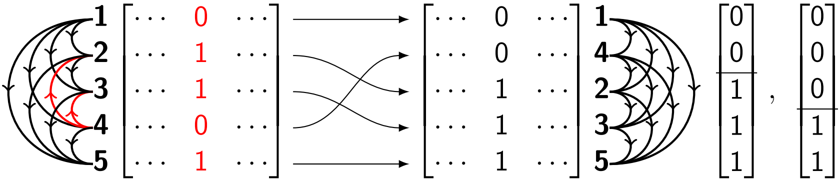
Example



$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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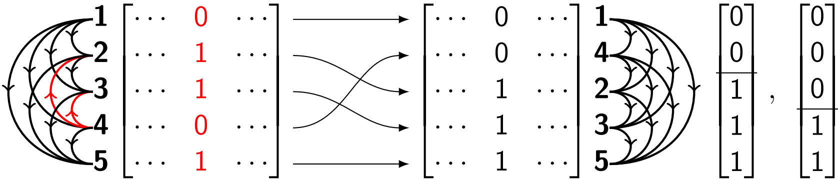
Example



In this case there is one new filler and the contributing column.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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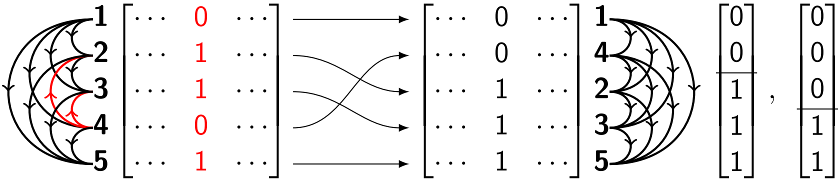

In this case there is one new filler and the contributing column.

In this case there are two contributions.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (2 \times \ell \text{ matrix})$$

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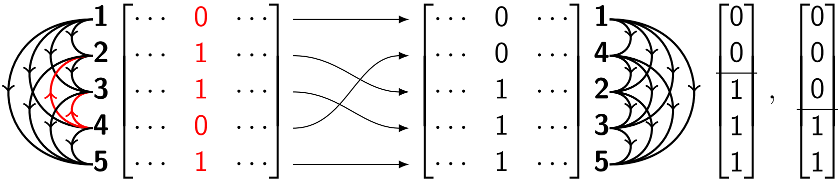
Example



In this case there is one new filler and the contributing column.
In this case there are two contributions.

$$\|A\| \leq \# \text{ contributions} + m + 1 \leq (\ell - 1)\binom{m}{2} + m + 1. \quad \blacksquare$$

We use other forms of amortization to handle other cases. In particular it seems reasonable to allow some columns to not be paid for right away but rather place them on a debt pile and pay for them later. We keep the debt pile small (somehow!). We hope to finish the 2-rowed case this summer.

THANKS to the organizers.
Historic Halifax has been wonderful.