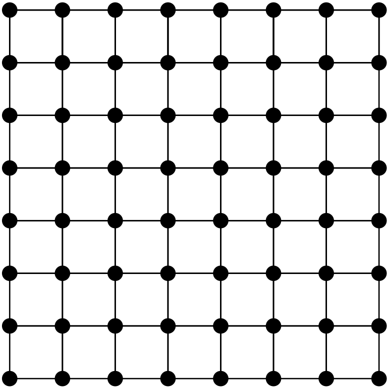
Perfect Matchings in Grid Graphs after Vertex Deletions

Richard Anstee
Jonathan Blackman
Gavin Yang
UBC

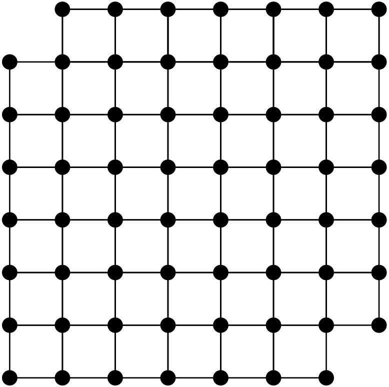
SIAM, June 14, 2010
Austin, Texas

Introduction

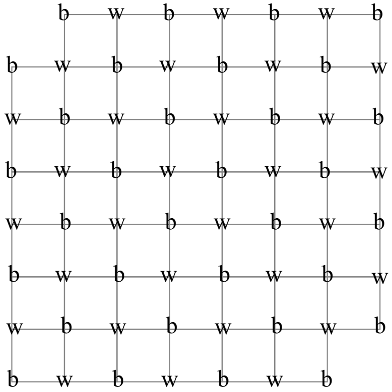
A *perfect matching* in a graph is a set of edges such that each vertex in the graph is incident with one edge of the matching.



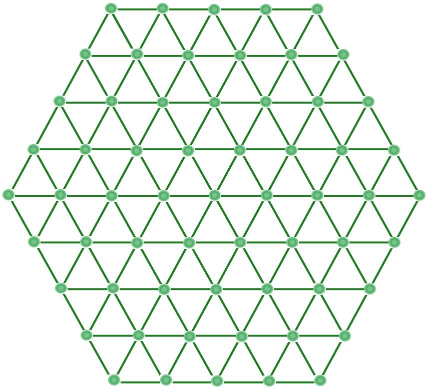
The 8×8 grid.
This graph has many perfect matchings.



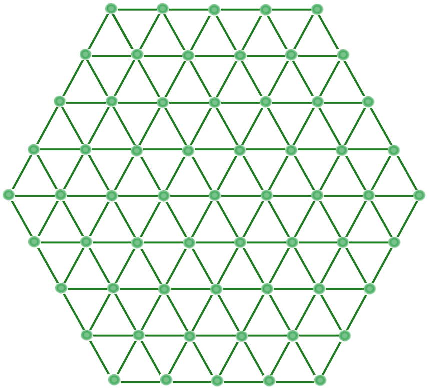
The 8×8 grid with two deleted vertices.



The black/white colouring revealed:
No perfect matching in the remaining graph.



A convex portion of the triangular grid

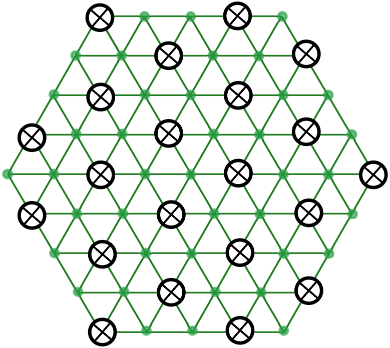


A convex portion of the triangular grid

A *near perfect matching* in a graph is a set of edges such that all but one vertex in the graph is incident with one edge of the matching. Our convex portion of the triangular grid has 61 vertices and many near perfect matchings.

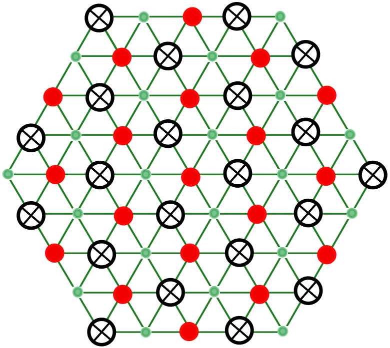
Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



We have deleted 21 vertices from the 61 vertex graph, many at distance 2.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



We have chosen 19 red vertices S from the remaining 40 vertices and discover that the other 21 vertices are now all isolated and so the 40 vertex graph has no perfect matching.

Definition We define a *d-dimensional grid graph G_m^d* as follows: Let $[m] = \{1, 2, \ldots, m\}$. Define

$$V(G_m^d) = \{(x_1, x_2, \ldots, x_d) \, : \, x_i \in [m] \text{ for } i \in [d]\}$$

and then we join $(x_1, x_2, \ldots x_m)$ and $(y_1, y_2, \ldots, y_m)$ by an edge if

$$\sum_{i=1}^{d} |x_i - y_i| = 1.$$

Main Theorem

Theorem (Aldred, A., Locke 07 ($d = 2$), A., Blackman, Yang 10 ($d \geq 3$)).
Let m, d be given with m even and $d \geq 2$. Then there exist constants a_d and b_d (depending only on d) for which we set

$$k = \lceil a_d m^{1/d} + b_d \rceil \quad \left(k \text{ is } \Theta(m^{1/d}) \right).$$

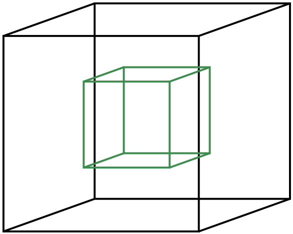
Let G_m^d have bipartition $V(G_m^d) = B \cup W$.
Then for $B' \subset B$ and $W' \subset W$ satisfying
i) $|B'| = |W'|$,
ii) For all $x, y \in B'$, $d(x, y) > 2k$,
iii) For all $x, y \in W'$, $d(x, y) > 2k$,
we may conclude that $G_m^d \backslash (B' \cup W')$ has a perfect matching.

Hall's Theorem

A bipartite graph has a perfect matching if for each choice of a subset A of one part, $|A| \leq |N(A)|$.

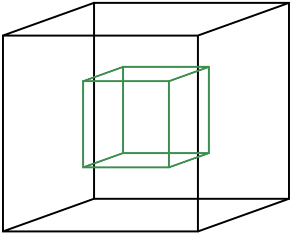
Hall's Theorem

A bipartite graph has a perfect matching if for each choice of a subset A of one part, $|A| \leq |N(A)|$. Now consider the grid G_m^3:



Hall's Theorem

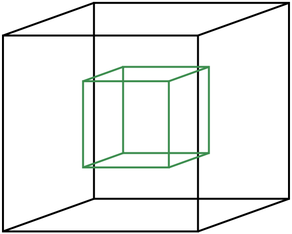
A bipartite graph has a perfect matching if for each choice of a subset A of one part, $|A| \leq |N(A)|$. Now consider the grid G_m^3:



If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.

Hall's Theorem

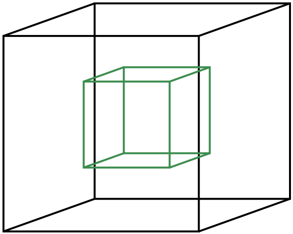
A bipartite graph has a perfect matching if for each choice of a subset A of one part, $|A| \leq |N(A)|$. Now consider the grid G_m^3:



If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.
If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube.

Hall's Theorem

A bipartite graph has a perfect matching if for each choice of a subset A of one part, $|A| \leq |N(A)|$. Now consider the grid G_m^3:



If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.
If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube.
We may choose c small enough so that we cannot find a perfect matching.

Hall's Theorem

Given a bipartite graph, then the graph has a perfect matching if for each choice of A a subset of one part

$$|A| \leq |N(A)|.$$

Given we have deleted white and black vertices W', B', we must have for each choice of $A \subset W \backslash W'$,

$$|A| \leq |N(A)| - |B' \cap N(A)|$$

Hall's Theorem

Given a bipartite graph, then the graph has a perfect matching if for each choice of A a subset of one part

$$|A| \leq |N(A)|.$$

Given we have deleted white and black vertices W', B', we must have for each choice of $A \subset W \backslash W'$,

$$|A| \leq |N(A)| - |B' \cap N(A)|$$

We may assume $|A \cup N(A)| \leq \frac{1}{2}m^d$.
We may also consider components $R = X \cup N(X)$ in G_m^d (with $X \subseteq A$).

Main Inequalities

We assume $R = X \cup N(X)$ is a connected component of G_m^d for some $X \subseteq W \backslash W'$ and $|R| \leq \frac{1}{2}m^d$. There are constants c, c', c'' depending only on d so that

$$|N^k(R)| \leq |R| + ck^{d-1}|\partial R|$$

$$(2^d/d!)k^d \leq |N^k(x)| \leq 2^d k^d$$

$$|N(X)| - |X| \geq c'|\partial R|$$

$$|N(X)| - |X| \geq c''\frac{|R|}{m}$$

How many deleted blacks in a region?

If $x, y \in B'$, then because $d(x,y) > 2k$ we deduce that

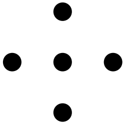
$$N^k(x) \cap N^k(y) = \emptyset.$$

We obtain the estimate $(R = X \cup N(X))$

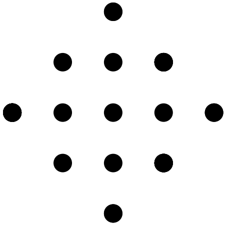
$$|B' \cap N(X)| \leq \frac{|N^k(R)|}{|N^k(x)|}$$

Let $f(k, d)$ denote $|N^k(x)|$ in d dimensions.

Let $f(k,d)$ denote $|N^k(x)|$ in d dimensions.

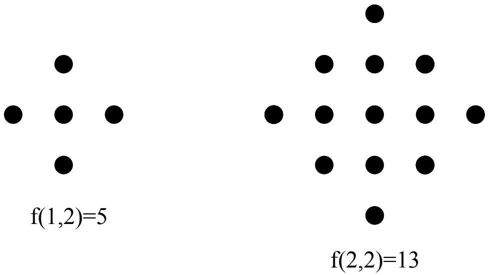


f(1,2)=5



f(2,2)=13

Let $f(k, d)$ denote $|N^k(x)|$ in d dimensions.

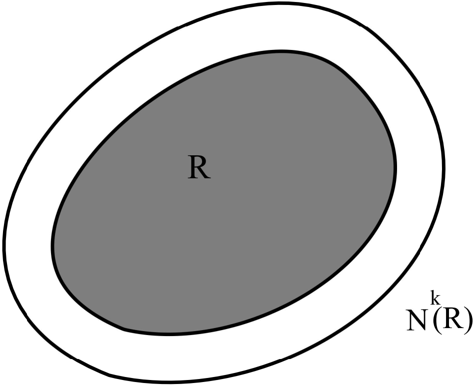


We discover $f(k, d) = f(d, k)$. Also $f(1, 1) = 3$, $f(2, 2) = 13$ and $f(3, 3) = 63$. From these three terms we may access Sloane's Catalog of Integer Sequences and discover that $f(k, d)$ is a *Delannoy* number. We only need an estimate:

$$(2^d/d!)k^d \leq |N^k(x)| \leq 2^d k^d.$$

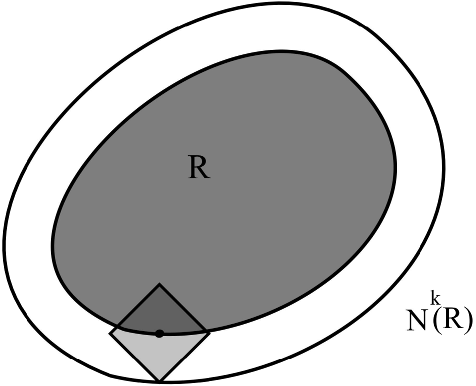
$$N^k(R) = R \cup \left(\cup_{x \in \partial R} N^k(x) \right)$$

Assume we have a closed walk $x_1, x_2, x_3 \ldots x_n$ with $\cup_{i=1}^n x_i = \partial R$.
Then we can compute
$\cup_{x \in \partial R} N^k(x) = \cup_{i=2}^n N^k(x_{i+1}) \setminus N^k(x_i) \leq ck^{d-1} |\partial R|$



$$N^k(R) = R \cup \left(\cup_{x \in \partial R} N^k(x) \right)$$

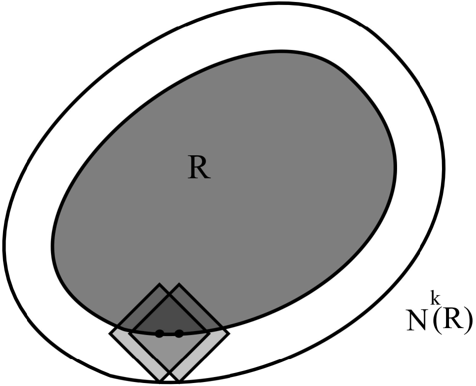
Assume we have a closed walk $x_1, x_2, x_3 \ldots x_n$ with $\cup_{i=1}^{n} x_i = \partial R$.
Then we can compute
$\cup_{x \in \partial R} N^k(x) = \cup_{i=2}^{n} N^k(x_{i+1}) \setminus N^k(x_i) \leq ck^{d-1}|\partial R|$



$$N^k(R) = R \cup \left(\cup_{x \in \partial R} N^k(x) \right)$$

Assume we have a closed walk $x_1, x_2, x_3 \dots x_n$ with $\cup_{i=1}^n x_i = \partial R$. Then we can compute

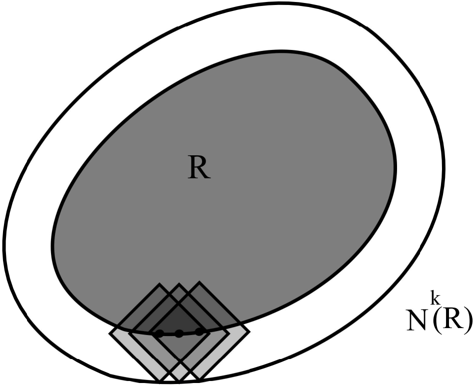
$\cup_{x \in \partial R} N^k(x) = \cup_{i=2}^n N^k(x_{i+1}) \backslash N^k(x_i) \leq ck^{d-1}|\partial R|$



$$N^k(R) = R \cup \left(\cup_{x\in\partial R} N^k(x) \right)$$

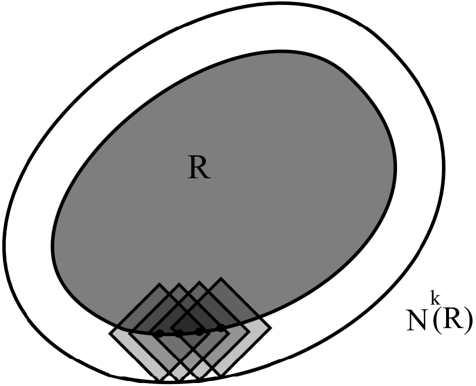
Assume we have a closed walk $x_1, x_2, x_3 \ldots x_n$ with $\cup_{i=1}^{n} x_i = \partial R$. Then we can compute

$\cup_{x\in\partial R} N^k(x) = \cup_{i=2}^{n} N^k(x_{i+1}) \backslash N^k(x_i) \leq ck^{d-1}|\partial R|$



$$N^k(R) = R \cup \left(\cup_{x \in \partial R} N^k(x) \right)$$

Assume we have a closed walk $x_1, x_2, x_3 \ldots x_n$ with $\cup_{i=1}^{n} x_i = \partial R$.
Then we can compute
$\cup_{x \in \partial R} N^k(x) = \cup_{i=2}^{n} N^k(x_{i+1}) \setminus N^k(x_i) \leq ck^{d-1}|\partial R|$



We would like to show ∂R is connected but that is problematical on two grounds.

We would like to show ∂R is connected but that is problematical on two grounds.

1. We would need that $R^c = V_\infty^d \backslash R$ is connected but in general $V_\infty^d \backslash R$ is a union of components $C_0, C_1, \ldots$. Thinking of C_0 as the infinite component, we think of the remaining components $C_1, C_2, \ldots$ as *holes* of R. We do have that $V_\infty^d \backslash C_i$ is connected.

We would like to show ∂R is connected but that is problematical on two grounds.

1. We would need that $R^c = V_\infty^d \backslash R$ is connected but in general $V_\infty^d \backslash R$ is a union of components $C_0, C_1, \ldots$. Thinking of C_0 as the infinite component, we think of the remaining components $C_1, C_2, \ldots$ as *holes* of R. We do have that $V_\infty^d \backslash C_i$ is connected.

2. We would like to deduce that $\partial(V_\infty^d \backslash C_i) = \partial^+ C_i$ is connected for each i but this is not true in G_∞^d. This is easy enough to overcome namely we can deduce that $\partial^+ C_i$ is α_d-connected. We need to extend G_∞^d to include all diagonals (of each unit hypercube) and α_d-connectivity is defined in terms of this extended edge set. (Deuschel, Pisztora 96, Hermann 98)

Recall $k = \lceil a_d m^{1/d} + b_d \rceil$ i.e. k is $\Theta(m^{1/d})$.

We must establish the following inequality:

$$|N(X)| - |X| \geq \frac{|N^k(R)|}{|N^k(x)|} \qquad (R = X \cup N(X))$$

Recall $k = \lceil a_d m^{1/d} + b_d \rceil$ i.e. k is $\Theta(m^{1/d})$.

We must establish the following inequality:

$|N(X)| - |X| \geq \frac{|N^k(R)|}{|N^k(x)|}$ $(R = X \cup N(X))$

or establish $|\frac{2^d}{d!}k^d||(|N(X)| - |X|) \geq |R| + ck^{d-1}|\partial R|$

using our inequalities $|N^k(R)| \leq |R| + ck^{d-1}|\partial R|$ and $|N^k(x)| \geq \frac{2^d}{d!}k^d$

Recall $k = \lceil a_d m^{1/d} + b_d \rceil$ i.e. k is $\Theta(m^{1/d})$.

We must establish the following inequality:

$|N(X)| - |X| \geq \frac{|N^k(R)|}{|N^k(x)|}$ $(R = X \cup N(X))$

or establish $|\frac{2^d}{d!}k^d||(|N(X)| - |X|) \geq |R| + ck^{d-1}|\partial R|$

or establish $|\frac{2^d}{d!}k^d - \frac{c}{c'}k^{d-1}||(|N(X)| - |X|) \geq |R|$

using our inequality $|N(X)| - |X| \geq c'|\partial R|$

Recall $k = \lceil a_d m^{1/d} + b_d \rceil$ i.e. k is $\Theta(m^{1/d})$.

We must establish the following inequality:

$|N(X)| - |X| \geq \frac{|N^k(R)|}{|N^k(x)|}$ $(R = X \cup N(X))$

or establish $|\frac{2^d}{d!} k^d| (|N(X)| - |X|) \geq |R| + ck^{d-1}|\partial R|$

or establish $|\frac{2^d}{d!} k^d - \frac{c}{c'} k^{d-1}| (|N(X)| - |X|) \geq |R|$

or establish $(\frac{2^d}{d!} k^d - \frac{c}{c'} k^{d-1})(c'' \frac{1}{m}|R|) \geq |R|$

using our inequality $|N(X)| - |X| \geq c'' \frac{1}{m}|R|$

Recall $k = \lceil a_d m^{1/d} + b_d \rceil$ i.e. k is $\Theta(m^{1/d})$.

We must establish the following inequality:

$|N(X)| - |X| \geq \frac{|N^k(R)|}{|N^k(x)|}$ $(R = X \cup N(X))$

or establish $|\frac{2^d}{d!}k^d||(|N(X)| - |X|) \geq |R| + ck^{d-1}|\partial R|$

or establish $|\frac{2^d}{d!}k^d - \frac{c}{c'}k^{d-1}||(|N(X)| - |X|) \geq |R|$

or establish $(\frac{2^d}{d!}k^d - \frac{c}{c'}k^{d-1})(c''\frac{1}{m}|R|) \geq |R|$

Using $k^d \approx (a_d)^d m$, we can choose a_d large enough so that the final inequality is true.

THANKS TO THE ORGANIZERS!