Mathematics Undergraduate Research at UBC

Richard Anstee
UBC, Vancouver

ASEE Physics and Mathematics Business Meeting
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Introduction

I'm Richard Anstee, a Professor of Mathematics. My research area is Discrete Mathematics and this summer I am working with one second year undergraduate student (Ronnie Chen) and one Ph.D. student (Miguel Raggi) on research. An M.Sc. student (Christina Koch) will start this fall.

NSERC USRA program

NSERC is in many ways the Canadian alternative to NSF and is virtually the sole funding source for most Mathematics. There are not the same array of funding sources as in the USA. The Undergraduate Summer Research Assistantship (USRA) is a kind of Canadian equivalent of REU programs in the USA (Research Experiences for Undergraduates). There are a number of differences. Typically the USRA students works with a single faculty on research and is supported by the supervisors grant ($4500 from NSERC, $1500 from supervisors grant). In some sense we are expecting productive research that is in support of the research proposal of the supervisor. The Canadian system is quite flexible.

Typical Summer

A selection process occurs in February and some outstanding students (fourteen this year) are selected. The student is assigned to work 16 (consecutive) weeks. Our program at UBC runs May through mid August. The student learns about a research area through readings, meetings with supervisor, fellow students, and perhaps seminars. The students is typically given some research project to provide focus. If things work out, actual research will get done. In any event, the students is expected to provide a write up of work done in the final week or so of the summer. In addition a weekly seminar is run with two 1/2 hour talks each week giving each student a chance to present twice.

Seminar

We decided to have a seminar for all the USRA students so that they will meet each other. Last year the students chose a 2 pm time (somehow 10 am wasn't as popular). Two 30 minute student presentations are given. The faculty member(s) in attendance ask lots of questions to aid the presentation and encourage the students not to be passive participants. The 30 minute time per student is relatively strict. The students were happy with the 30 minute maximum partly because it meant each talk was over in 30 minutes. Remember that students are working in a variety of areas (quite different from an REU). Afterwards we retreat for coffee/tea/soft drinks and an informal chat.

Hike to Black Mountain 2010

Hike

Each of the last 4 years I've organized a hike for the participants up to Black Mountain in mid July. Its not an enormously challenging hike but the less fit need some encouragement going up the steep bits. There are a number of rewards along the way including a beautiful view down onto West Vancouver and across the harbour to UBC. And there is a small lake for a bracing swim. Again I view this as a nice bonding experience for the students and a chance to talk with the other students. This kind of event is very successful.

Research by Undergraduates?

Given the strength and energy of the participants, some fraction get actual research done. I like to spend a lot of time with the students (see them every day for 1/2 to 1 hour; this year several hours but fewer times per week) and have found that helpful. My last 4 students all had research papers; real research that pays my salary. Now I have the advantage that my area inside Discrete Mathematics typically involves fewer tools and instead just requires lots of cleverness. The students had either finished first or second year. They were top students!

Reference letters

For students the big advantage is having letter writers for graduate school that can talk in detail about research readiness.

Recruitment

In general, the competition for these jobs is fierce. I often have students offering to work for free during the summer (I refuse). I go look for top students and recruit them. And sometimes they contact me. This year I identified two top students only to have one of those two students contact me before I had contacted the student.

The students typically have to be given problems. They have great patience for hard computations and writing computer programs. They need help in writing Mathematics but they are sufficiently talented that they learn very quickly. And some need hardly any guidance writing Mathematics; they absorb the style from papers!

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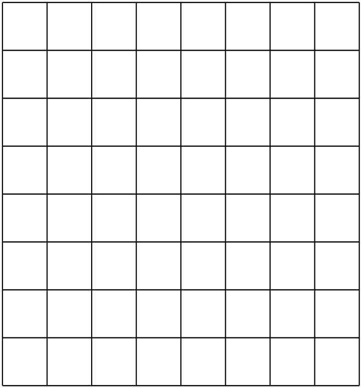
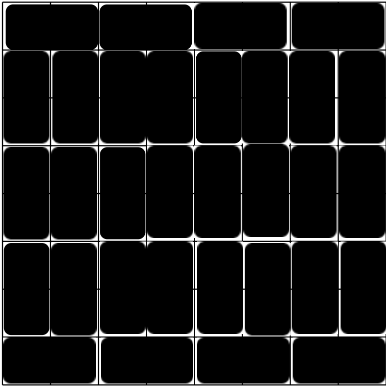
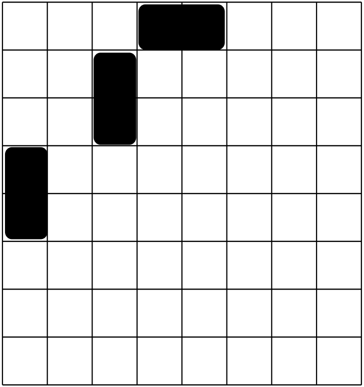
I thought I should show a little easily digestible mathematics arising out of summer projects. I thought I would discuss some matching problems which I'll introduce using Dominoes.



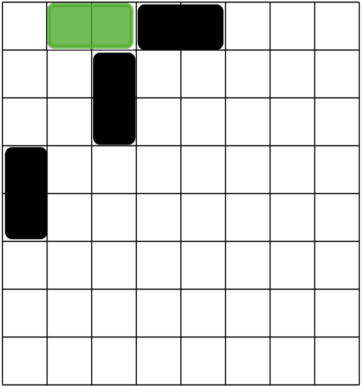
The checkerboard



The checkerboard covered by dominoes



Black dominoes fixed in position. Can you complete?



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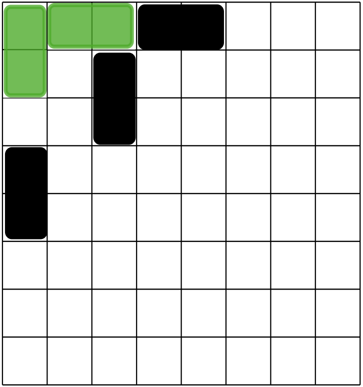

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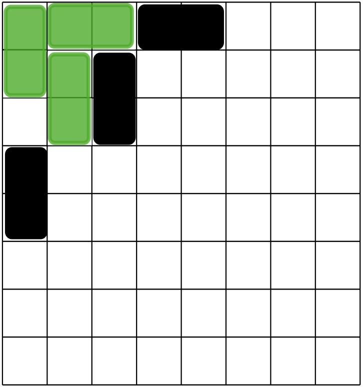
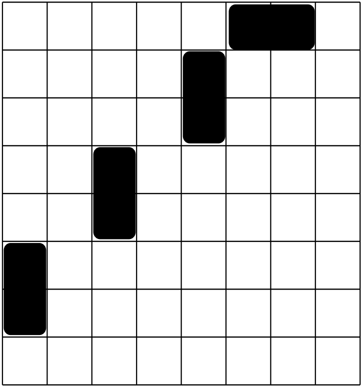


Black dominoes fixed in position. You can't complete.



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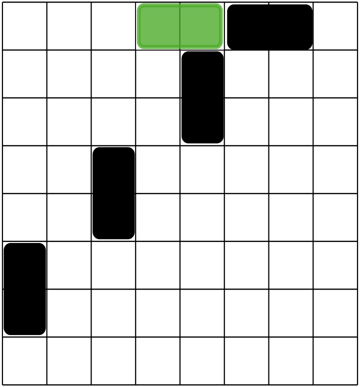

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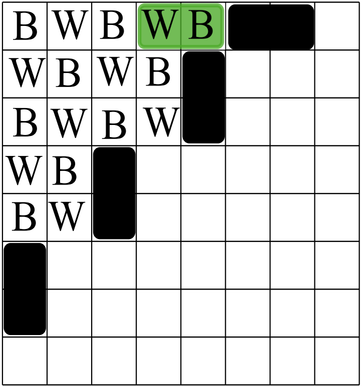
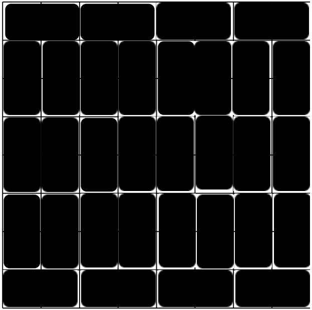
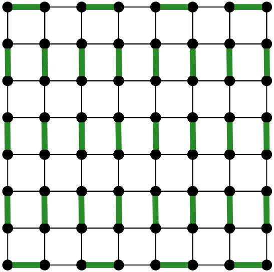


Black dominoes fixed in position. You can't complete.

Covering the checkerboard by dominoes is the same as finding a perfect matching in the associated grid graph.

A perfect matching in a graph is a set M of edges that pair up all the vertices. Necessarily $|M| = |V|/2$.





Our first example considered choosing some edges and asking whether they extend to a perfect matching. I have also considered what happens if you delete some vertices. Some vertex deletions are clearly not possible. Are there some nice conditions on the vertex deletions so that the remaining graph after the vertex deletions still has a perfect matching.

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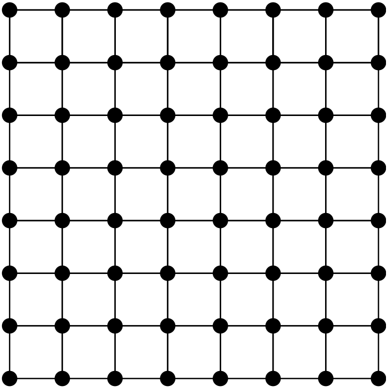
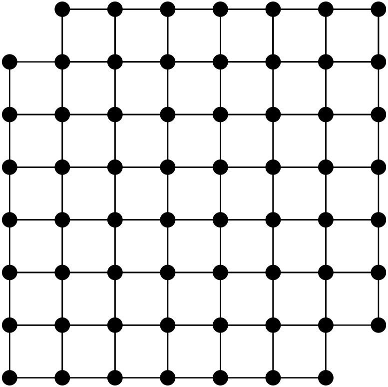
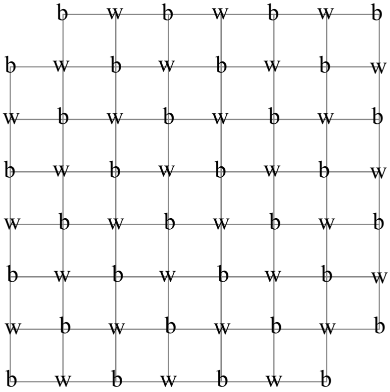
In the checkerboard interpretation we would be deleting some squares from the checkerboard and asking whether the remaining slightly mangled board has a covering by dominoes.



The 8×8 grid.
This graph has many perfect matchings.



The 8×8 grid with two deleted vertices.



The black/white colouring revealed:
No perfect matching in the remaining graph.

Deleting Vertices from Grid

Our grid graph (in 2 or d dimensions) can have its vertices coloured white W or black B so that every edge in the graph joins a white vertex and a black vertex. Graphs G which can be coloured in this way have $V(G) = W \cup B$ and are called bipartite. Bipartite graphs that have a perfect matching must have $|W| = |B|$.

Thus if we wish to delete black vertices B' and white vertices W' from the grid graph, we must delete an equal number of black and white vertices.

Deleting Vertices from Grid

Some conditions on deleted vertices are clear. You must delete and equal number of white and black vertices ($|B'| = |W'|$). But also you can't do silly things. Consider a corner of the grid with a white vertex. Then if you delete the two adjacent black vertices then there will be no perfect matching. How do you avoid this problem? Our guess was to impose some distance condition on the deleted blacks (and also on the deleted whites)

Deleting Vertices from Grid

Theorem (Aldred, A., Locke 07 ($d = 2$),
A., Blackman, Yang 10 ($d \geq 3$)).
Let m, d be given with m even and $d \geq 2$. Then there exist constant c_d (depending only on d) for which we set

$$k = c_d m^{1/d} \quad \left(k \text{ is } \Theta(m^{1/d})\right).$$

Let G_m^d have bipartition $V(G_m^d) = B \cup W$.
Then for $B' \subset B$ and $W' \subset W$ satisfying
i) $|B'| = |W'|$,
ii) For all $x, y \in B'$, $d(x, y) > k$,
iii) For all $x, y \in W'$, $d(x, y) > k$,
we may conclude that $G_m^d \backslash (B' \cup W')$ has a perfect matching.

Hall's Theorem

The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if for each subset $A \subset W - W'$, we have $|A| \leq |N(A)|$ where $N(A)$ consists of vertices in $B - B'$ adjacent to some vertex in A.

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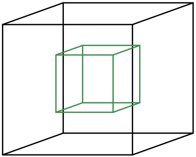

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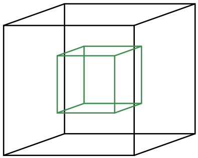

If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.

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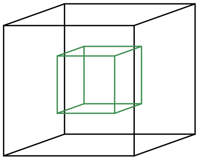

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If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube.

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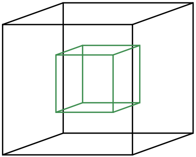

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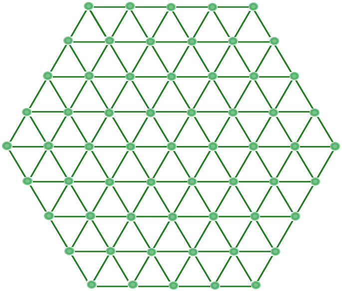
We may choose c small enough so that we cannot find a perfect matching.

Deleting Vertices from Triangular Grid



A convex portion of the triangular grid

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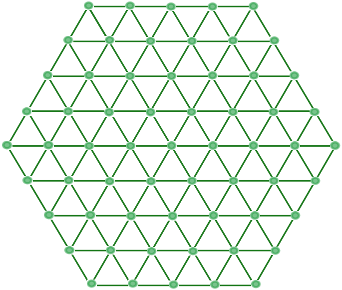

A convex portion of the triangular grid

A *near perfect matching* in a graph is a set of edges such that all but one vertex in the graph is incident with one edge of the matching. Our convex portion of the triangular grid has 61 vertices and many near perfect matchings.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).

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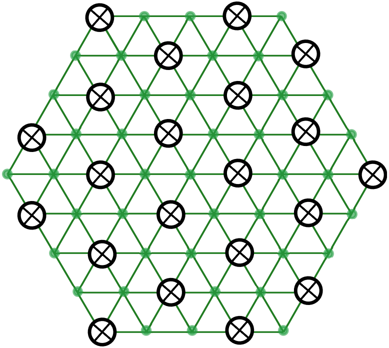

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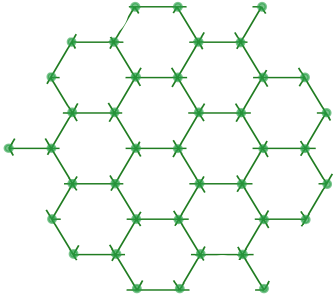

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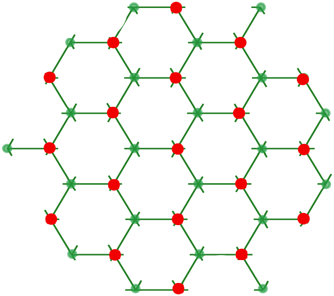

We have chosen 19 red vertices S from the remaining 40 vertices and discover that the other 21 vertices are now all isolated and so the 40 vertex graph has no perfect matching.

One area I work in is the area of Extremal Set Theory. The typical problem asks how many subsets of $[m] = \{1, 2, \ldots, m\}$ can you choose subject to some property? For example: how many subsets of $[m]$ can you choose such that every pair of subsets has a nonempty intersection?

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The answer is seen to be $2^{m-1} = \frac{1}{2}2^m$ by noting that you cannot choose both a set A and its complement $[m]\backslash A$. Easy proof but clever!

I'm going to encode a family of n susbets of $[m] = \{1, 2, \ldots, m\}$ as an $m \times n$ (0,1)-matrix with no repeated columns.

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

i.e. if A is m-rowed and simple then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}$$

We are interested in the maximum n, given m and some property imposed on A.

Theorem (Vapnik and Chervonenkis 71, Perles and Shelah 72, Sauer 72)
Let A be an $m \times n$ simple matrix so that there is no 4×16 submatrix which is a row and column permutation of K_4.

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then

$$n \le \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}.$$

We say forb(m, K_4) is this bound.

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathrm{forb}(m, \mathbf{1}_4) = \mathrm{forb}(m, K_4^3) = \mathrm{forb}(m, K_4^2) = \mathrm{forb}(m, K_4^1)$ $= \mathrm{forb}(m, \mathbf{0}_4) = \mathrm{forb}(m, 2 \cdot \mathbf{1}_3) = \mathrm{forb}(m, 2 \cdot \mathbf{0}_3)$.

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$$K_4 = \left[\begin{array}{ccccccccccccccc|c|} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \color{red}{0} \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \color{red}{0} \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \color{red}{0} \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\color{red}{\mathbf{0}_4}$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
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Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
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$$K_4 = \left[\begin{array}{cccccccccccccc|cc}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

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We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have $\text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m, K_4)$.

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1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have
$\text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m, K_4)$.
We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

Row and Column order matter

$$\text{Let } F = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We were able to show the following row and column ordered result:

Theorem (A., Chen 11) Let m be given. Let A be an $m \times n$ simple matrix. Assume A has no submatrix F. Then $n \leq 2m^2 + m + 1$. In addition there is such an A with $2m^2 - 3m \leq n$.

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More to follow, we are still in June!

Hope you have enjoyed your stay in Vancouver!