MATH 523: Fractional to Integral Strategy for Matching Algorithm. Feb 28, 2011

We start with a graph $G=(V,E)$ with edge multiplicities $\{\lambda(e) : e \in E\}$. We are given a vector $\mathbf{f}=(f(i) : i \in V)$ of vertex degrees. A subgraph of G is a vector $\mathbf{x}=(x(e) : e \in E)$ for which $0 \le x(e) \le \lambda(e)$ and $x(e) \in \mathbf{Z}$. We will be considering *fractional subgraphs* i.e. just a vector $\mathbf{x}=(x(e) : e \in E)$ for which $0 \le x(e) \le \lambda(e)$.

An f-factor is a subgraph $\mathbf{x}=(x(e) : e \in E)$ for which

$$\sum_{e \text{ hits } i} x(e) = f(i)(1)$$

A fractional f-factor is a fractional subgraph $\mathbf{x}=(x(e) : e \in E)$ satisfying (1).

We can search for a fractional f-factor using network flows. Form a digraph $D=(N,A)$ from G as follows. For each vertex i form two nodes i', i''. For each edge (i,j) form two arcs (i',j'') and (j',i''). We may set

$$u(i',j'')=\lambda(i,j)$$

and if there are costs $c(e)$,

$$c(i',j'')=c(i,j).$$

We seek a minimum cost flow (or just a feasible flow) satisfying

$$\sum_{j''} x(i',j'')=f(i), \quad \sum_{j'} x(j',i'')=f(i).$$

We note that if $\mathbf{x}$ is a fractional f-factor then we obtain a feasible flow by setting

$$x(i',j'')=x(j',i'')=x(i,j)$$

And vice versa, we can obtain a fractional f-factor from a feasible flow by setting

$$x(i,j)=\frac{1}{2}(x(i',j'')+x(j',i'')).$$

We verify that since $0 \le x(i',j''), x(j',i'') \le \lambda(i,j)$ then $0 \le \frac{1}{2}(x(i',j'')+x(j',i'')) \le \lambda(i,j)$. Also $\sum_{e \text{ hits } i} x(e) = \frac{1}{2}\sum_{j''}(x(i',j'')+x(j',i''))=f(i)$. We can also verify that if we have a minimum cost flow we obtain a minimum cost fractional f-factor by this trick since

$$\sum_{e\in E} c(e)x(e) = \sum_{a \in A} c(a)(\frac{1}{2}x(a)) = \frac{1}{2}\sum_{a\in A} c(a)x(a).$$

Thus apart from a factor of 2, we are measuring the same cost function. In matrix terms our transformation $x(i,j)=\frac{1}{2}(x(i',j'')+x(j',i''))$ corresponds to a matrix B with ith row and column sums $f(i)$ and form a symmetric matrix $\frac{1}{2}(B+B^T)$.

We note that we may apply our integrality theorem for Network Flows to deduce that we can find an optimal flow for D with $x(i',j'') \in \mathbf{Z}$ for each arc $i',j'') \in A$. Thus if there is any fractional f-factor at all, there will be a fractional f-factor $\mathbf{x}=(x(e) : e \in E)$ with $2 \cdot x(e) \in \mathbf{Z}$ for each $e \in E$.

It is possible to obtain a fractional f-factor with mostly integral entries.

Theorem 0.1 *Assume G has a minimum cost fractional f-factor. Then G has a minimum cost fractional f-factor $\mathbf{x}$ so that either $x(e) \in \mathbf{Z}$ or $2 \cdot x(e) \in \mathbf{Z}$ and the set of edges e for which $x(e)-\lfloor x(e) \rfloor = \frac{1}{2}$ form vertex disjoint odd cycles which are not joined by any edge of G.*

Proof: Let $\mathbf{x}$ be a fractional f-factor with $2 \cdot x(e) \in \mathbf{Z}$. Let $H = (V, E')$ be the graph whose edges correspond to those edged $e \in E$ with $x(e) - \lfloor x(e) \rfloor = \frac{1}{2}$, namely the half edges. Given that $\mathbf{x}$ is an f-factor, we deduce that every degree in H is even. Assume that we can find an even length closed trail (*trail* means no repeated edges) $e_1, e_2, \ldots, e_{2k}$. Then we can obtain a new f-factor by

$$x(e) = \begin{cases} x(e) + \frac{1}{2} & e = e_i \text{ and } i \text{ is odd} \\ x(e) - \frac{1}{2} & e = e_i \text{ and } i \text{ is even} \\ x(e) & \text{otherwise} \end{cases} \quad (2)$$

and hence this reduces the number of edges in H.

Having eliminated all even length closed trails, we deduce that H consist of vertex disjoint odd cycles.

It is useful to note that the dual variables also translate this way.

Now if we are not concerned with costs we can consider two vertex disjoint odd cycles (of H) joined by an edge in G. Let $C_1 = u_1 u_2 \ldots u_{2k+1}$ and $C_2 = v_1 v_2 \ldots v_{2\ell+1}$ with an edge joining u_1 to v_1. There are two cases. If $x(u_1, v_1) < \lambda(u_1, v_1)$ then the following transformation eliminates both odd cycles.

$$x(e) = \begin{cases} x(e) - \frac{1}{2} & e = (u_j, u_{j+1}) \text{ and } j \text{ is odd} \\ x(e) - \frac{1}{2} & e = (v_j, v_{j+1}) \text{ and } j \text{ is odd} \\ x(e) + \frac{1}{2} & e = (u_j, u_{j+1}) \text{ and } j \text{ is even} \\ x(e) + \frac{1}{2} & e = (v_j, v_{j+1}) \text{ and } j \text{ is even} \\ x(e) + 1 & e = (u_1, v_1) \\ x(e) & \text{otherwise} \end{cases} \quad (3)$$

If $x(u_1, v_1) > 0$ then the following transformation eliminates both odd cycles.

$$x(e) = \begin{cases} x(e) + \frac{1}{2} & e = (u_j, u_{j+1}) \text{ and } j \text{ is odd} \\ x(e) + \frac{1}{2} & e = (v_j, v_{j+1}) \text{ and } j \text{ is odd} \\ x(e) - \frac{1}{2} & e = (u_j, u_{j+1}) \text{ and } j \text{ is even} \\ x(e) - \frac{1}{2} & e = (v_j, v_{j+1}) \text{ and } j \text{ is even} \\ x(e) - 1 & e = (u_1, v_1) \\ x(e) & \text{otherwise} \end{cases} \quad (4)$$

■

Theorem 0.2 *Assume G has a minimum cost fractional f-factor. Then G has a minimum cost fractional f-factor $\mathbf{x}$ so that either $x(e) \in \mathbf{Z}$ or $2 \cdot x(e) \in \mathbf{Z}$ and the set of edges e for which $x(e) - \lfloor x(e) \rfloor = \frac{1}{2}$ form vertex disjoint odd cycles.*

Proof: We proceed as above. Notice that changes as proposed in (2) leave the cost unaltered. This may not be immediately obvious since there are two choices (odd versus even) way to eliminate the even closed trails but because both yield fractional f-factors and, because the change in cost is of opposite sign for each of the two possibilities, then both must result in no change in cost otherwise we would be finding a cheaper minimum cost flow. As noted in the above proof, we are left with vertex disjoint cycles in H. ■

The remaining transformation proposed in (3),(4) to eliminate pairs of vertex disjoint odd cycles may have an unpredictable effect on cost. We can't use our previous idea since potentially only one of (3) or (4) are possible but it is true that of both are possible then there is no effect on cost. A case where that is always true is for a perfect matching.