**Math 523** Assignment #5 Due Wednesday April 6

1. Let $G = (V, E)$ be a graph with no loops. Arbitrarily orient the edges to obtain a directed graph $D = (V, A)$. Form a node-arc incidence matrix

$$A_D = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1 & \text{head(arc } j) = i \\ -1 & \text{tail(arc } j) = i \\ 0 & \text{otherwise} \end{cases}$$

Verify that a set of columns in $A_D$ that are linearly dependent correspond to a set of edges $E' \subseteq E$ which contain at least one cycle formed by the edges $E'$.

2. A *branching* is a (out)directed tree rooted at a node $r$. That is to say it contains directed paths from the root $r$ to each node in the tree. Explain how to create two matroids $M_1, M_2$ from a directed graph $D = (V, A)$ such that any common independent set of size $|V| - 1$ is a branching rooted at $r$. Now assume $D$ has no branching. How can we use the matroid intersection theorem ?

Theorem. $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq E} (r_1(U) + r_2(E - U))$

Relate it to the standard graph theory result about reachablity that follows from considering all the vertices reachable by directed paths from $r$ in $D$.

3. Let $G = (V, E)$ be a graph. Let $\mathcal{M}$ be the set of all matchings.

a) Let

$$\mathcal{I} = \{I \subseteq V \ : \ \text{there exists a matching } M \in \mathcal{M} \text{ with } I \subseteq V(M)\}$$

Show that $(E, \mathcal{I})$ is a matroid. Berge's theorem?

b) Let $G = (V, E)$ have vertex weights $w : V \to \mathcal{Z}^+$. Indicate how to find a set of vertices $U \subseteq V$ that maximizes $\sum_{v \in U} w(v)$ over all sets of vertices $U$ that are *matching covered*; namely there is a matching $M \in \mathcal{M}$ with $U \subseteq V(M)$.

4. The following two graphs on the same set $E$ of edges yields a pair of graphic matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. You may assume $I = \{e2, e7, e9\}$ is $w$-maximal in $\mathcal{I}_1^3 \cap \mathcal{I}_2^3$. Recall our definition of $D_{M_1,M_2}(I)$ to be the directed graph on $E$ such that for $x \in I$ and $y \in E \backslash I$, there is an edge $x \to y$ if $I \backslash x + y \in \mathcal{I}_1$ and an edge $x \leftarrow y$ if $I \backslash x + y \in \mathcal{I}_2$. We have $X_1 = \{e \in E \ : \ I + e \in \mathcal{I}_1\}$ and $X_2 = \{e \in E \ : \ I + e \in \mathcal{I}_2\}$. Using our auxiliary graph $D_{M_1,M_2}(I)$, find a $w$-maximal common independent set $I' \in \mathcal{I}_1^4 \cap \mathcal{I}_2^4$ with weights as below.

| $e$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $w(e)$ | 0 | 7 | 8 | 0 | 0 | 5 | 9 | 6 | 10 |