MATH 523: Primal-Dual Maximum Weight Matching Algorithm

We start with a graph $G = (V, E)$ with edge weights $\{c(e) : e \in E\}$

Primal P:

$$\max \sum\{c(e)x(e) : e \in E\}$$

subject to

$$\sum\{x(e) : e \text{ hits } i\} + y_i = 1 \text{ for all } i \in V$$
$$\sum\{x(e) : \text{ both ends of } e \text{ are in } S_k\} + z_k = s_k \text{ for all } S_k \subseteq V \text{ with } |S_k| = 2s_k + 1$$
$$x(e), y_i, z_k \geq 0$$

Dual D:

$$\min \sum\{\alpha_i : i \in V\} + \sum\{s_k\gamma_k : S_k \subseteq V, |S_k| = 2s_k + 1\}$$

subject to

$$\alpha_i + \alpha_j + \sum\{\gamma_k : i, j \in S_k\} \geq c(e) \text{ for all } e = (i, j) \in E$$
$$\alpha_i, \gamma_k \geq 0$$

Note that the positivity constraints follow from the variables y_i and z_k. Alternatively we could delete these variables y_i, z_k replacing the equalities by inequalities and it will still be true that $\alpha_i, \gamma_k \geq 0$.

An initial solution π for D:

$$\alpha_i = \max\{\{\frac{1}{2}c(e) : e \text{ hits } i\}, 0\} \qquad \text{for all } i \in V$$

$$\gamma_k = 0 \qquad \text{for all } S_k \subseteq V, |S_k| = 2s_k + 1$$

For any solution π to D, we compute $J_e = \{e = (i, j) : \alpha_i + \alpha_j + \sum\{\gamma_k : i, j \in S_k\} = c(e)\}$, $J_m = \{i : \alpha_i = 0\}$, $J_b = \{k : \gamma_k = 0\}$.

Restricted Primal (RP) determined from a solution π to D.

$$\max \quad -\sum\{x_i^a\}$$

subject to

$$\sum\{x(e) : e \text{ hits } i\} + y_i + x_i^a = 1 \quad \text{ for all } i \in V$$
$$\sum\{x(e) : \text{ both ends of } e \text{ are in } S_k\} + z_k = s_k \quad \text{ for all } S_k \subseteq V \text{ with } |S_k| = 2s_k + 1$$
$$x(e), y_i, z_k, x_i^a \geq 0$$
$$e \notin J_e \Rightarrow x(e) = 0, \qquad i \notin J_m \Rightarrow y_i = 0, \qquad k \notin J_b \Rightarrow z_k = 0$$

Dual of Restricted Primal (DRP):

$$\min \sum\{\bar{\alpha}_i : i \in V\} + \sum\{s_k\bar{\gamma}_k : S_k \subseteq V, |S_k| = 2s_k + 1\}$$

subject to

$$\bar{\alpha}_i + \bar{\alpha}_j + \sum\{\bar{\gamma}_k : i, j \in S_k\} \geq 0 \text{ for all } e = (i, j) \in J_e$$
$$\bar{\alpha}_i \geq -1, \bar{\gamma}_k \text{ free}$$
$$i \in J_m \Rightarrow \bar{\alpha}_i \geq 0, \qquad k \in J_b \Rightarrow \bar{\gamma}_k \geq 0$$

We could eliminate y_i and z_k by giving the constraints with cases:

$$\sum\{x(e) : e \text{ hits } i\} + x_i^a \leq 1 \quad \text{ for all } i \in J_m$$
$$\sum\{x(e) : e \text{ hits } i\} + x_i^a = 1 \quad \text{ for all } i \notin J_m$$
$$\sum\{x(e) : \text{ both ends of } e \text{ are in } S_k\} \leq s_k \text{ for all } k \in J_b$$

$$\sum\{x(e): \text{ both ends of } e \text{ are in } S_k\} = s_k \text{ for all } k \notin J_b$$

There are three solution invariants preserved by the algorithm:

invariant a) $x(e) \in \{0,1\}$ i.e. x yields a matching M.

invariant b) $\gamma_k > 0 \Rightarrow G|_{S_k}$ has precisely s_k edges of M.

invariant c) $\gamma_i, \gamma_j > 0 \Rightarrow S_i \cap S_j = \emptyset$ or $S_i \subseteq S_j$ or $S_j \subseteq S_i$.

We solve RP as a maximum cardinality matching problem, but to preserve invariants b),c) we work in the graph G_J ('admissible' graph) where we include V and only the edges in J_e but then shrink vertex sets S_k with $\gamma_k > 0$ to pseudonodes (where invariant c) makes this well defined). Note that $i \in J_m$ means we can take $x_i^a = 0$ (either i is matched or we can take $y_i = 1$). Also for $k \in J_b$ we either have a set S_k saturated with s_k edges or we can define z_k to take up the slack. For $k \notin J_b$, invariant b) ensures that we may take $z_k = 0$.

We need not match (but can match) the vertices in J_m but otherwise seek a maximum matching in G_J. We do blossom tree growing from unmatched vertices of $V \setminus J_m$. At termination, let G_c be the final graph of trees rooted at unmatched vertices/pseudonodes (more pseudonodes may be created in the tree growing process).

Let a vertex/pseudonode of G_c be *outer* (respectively *inner*) if it belongs to one of the trees and is joined to the root by a path of an even (possibly zero) number of edges (respectively an odd number of edges). Vertices of G inherit the outer/inner designation from the pseudonode containing them (actually the maximal such pseudonode).

An optimal solution $\bar{\pi}$ to DRP can now be determined as follows:

$$\bar{\alpha}_i = \begin{cases} -1 & \text{if } i \text{ is outer} \\ 1 & \text{if } i \text{ is inner} \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\gamma}_k = \begin{cases} 2 & \text{if } S_k \text{ is outer pseudonode of } G_c \\ -2 & \text{if } S_k \text{ is inner pseudonode of } G_c \\ 0 & \text{otherwise} \end{cases}$$

A little work is required for invariant b). Pseudonodes are seen to yield odd sets by checking their inductive buildup.

Now $\bar{\pi}$ is fairly easy to check for feasibility using the fact that G_c is a tree at termination so for example there are no edges of J_e from outer vertices to vertices not in the tree.

Why is $\bar{\pi}$ optimal?

$-(\sum\{\bar{\alpha}_i : i \in V\} + \sum\{s_k\bar{\gamma}_k : S_k \subseteq V\})$

$=$ excess of outer vertices/pseudonodes over inner vertices/pseudonodes of G_c.

$=$ number of unmatched vertices/pseudonodes of G_c apart from J_m.

$=$ number of unmatched vertices of G not in J_m when matching in G_J is mapped to a matching in G preserving invariant b).

$= \sum\{x_i^a : i \in V\}$

Thus the optimality for RP and DRP solutions is verified. What is the θ of the primal dual algorithm? The new solution π^* to D is $\pi^* + \theta\bar{\pi}$. Now let

$$\theta = \min(\delta_1, \delta_2, \delta_3, \delta_4)$$

where

$\delta_1 = \min\{\frac{1}{2}(\alpha_i + \alpha_j - c(e)) : e = (i, j)$ joins two outer vertices in different pseudonodes$\}$

$\delta_2 = \min\{(\alpha_i + \alpha_j - c(e)) : e = (i, j)$ joins outer vertex to vertex not in trees $\}$

$\delta_3 = \min\{\gamma_k/2 : S_k$ inner pseudonode of $G_c\}$

$\delta_4 = \min\{\alpha_i : i$ outer vertex$\}$

Now if

$\theta = \delta_1$ (i, j) will enter J_e forming blossom or extending matching

$\theta = \delta_2$ (i, j) will enter J_e and tree can grow

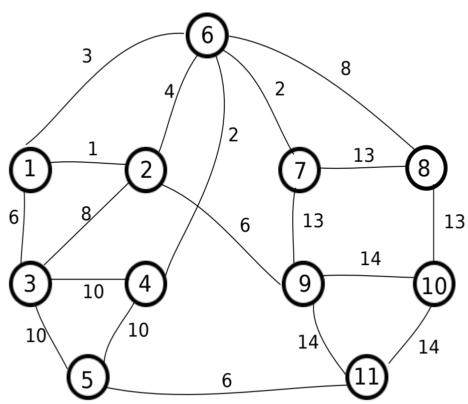
$\theta = \delta_3$ blossom S_k will no longer be shrunk in G_J

$\theta = \delta_4$ node i enters J_m, need not be matched

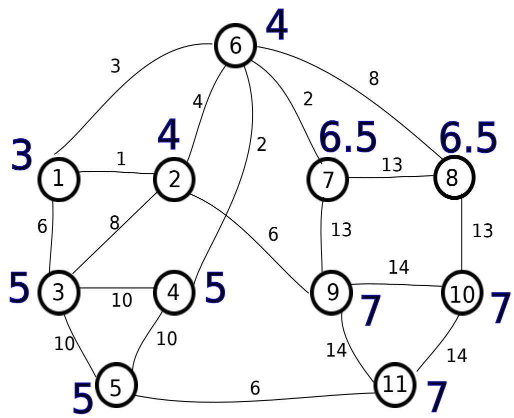
One point that may trouble you is extending the matching in G_J to a matching in G. The pseudonodes of G_J have matchings satisfying invariant b) and the inductive buildup of blossoms ensures that there are alternating walks (paths) from the base to any node of the blossom ending in a matched edge (with the exception of the base). Thus the matched and unmatched edges can be interchanged to make *any* vertex in the pseudonode unmatched.

Example

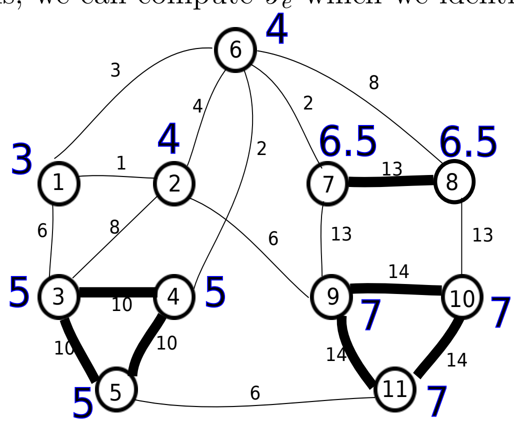
We use the following graph. Note that it has 11 vertices.



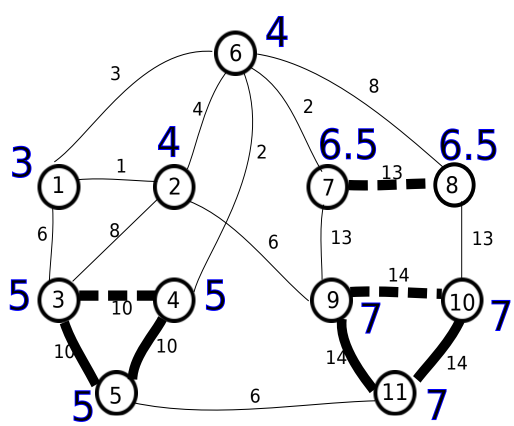
Our first step in the algorithm is to give node labels $\alpha_i = \min_j\{\frac{1}{2}c(i, j) : (i, j) \in E\}$



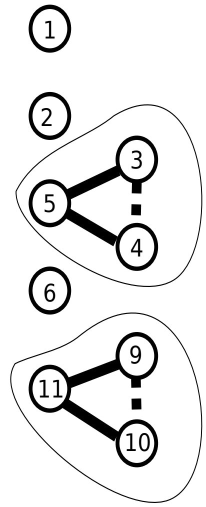
Having given the node labels, we can compute J_e which we identify by bold edges.



When an edge enters the matching it is also bold but dashed. We seek a maximum cardinality matching in the graph with edges J_e. There are a few choices (depending on what unmatched nodes you begin with).



We now grow trees from the unmatched vertices (or pseudonodes but there aren't any yet). As we do so we are able to add $(3,4),(7,8),(9,10)$ TO M. At the end, we have trees rooted at 1,2,5,6,11.

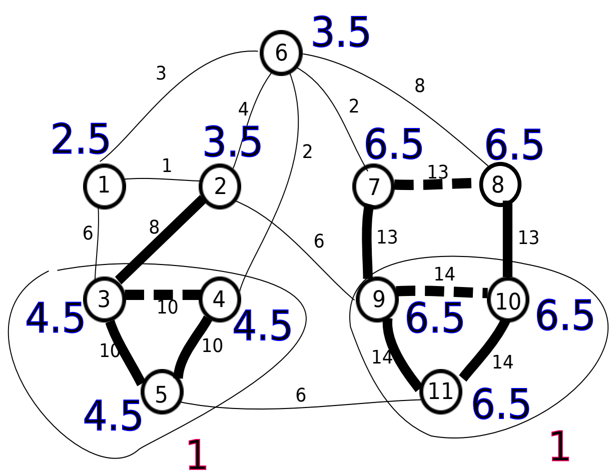


$$\alpha_i = \begin{cases} -1 & i \in \{1,2,3,4,5,6,9,10,11\} \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_k = \begin{cases} 2 & S_k = \{3,4,5\},\{9,10,11\} \\ 0 & \text{otherwise} \end{cases}$$

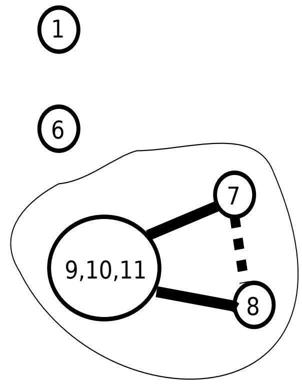
We have

$$\delta_1 = \frac{1}{2}, \quad \delta_2 = \frac{1}{2}, \quad \delta_3 = \infty, \quad \delta_4 = 3 \text{ and so } \theta = \frac{1}{2}$$

We update labels and J_e.



We now grow trees from the unmatched vertices 1,2,6 and the unmatched pseudonodes $\{3,4,5\}$,$\{9,10,11\}$. So we get trees rooted at 1,6 and $\{9,10,11\}$. As we do so we add the edge joining 2 and the pseudonode $\{3,4,5\}$. and hence add $(2,3)$ to M while adding $(4,5)$ to M and removing $(3,4)$ from M. This last interchange corresponds to the fact that we can move the unmatched vertex of a pseudonode to any vertex we want, in this case we wanted 3 unmatched so we could add $(2,3)$.

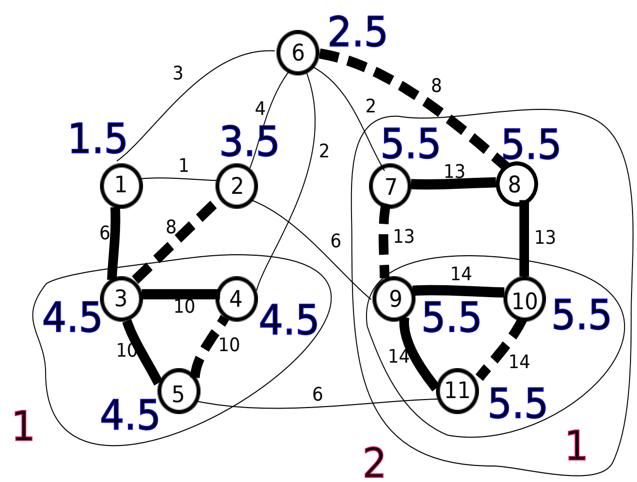


$$\alpha_i = \begin{cases} -1 & i \in \{1,6,7,8,9,10,11\} \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_k = \begin{cases} 2 & S_k = \{7,8,9,10,11\} \\ 0 & \text{otherwise} \end{cases}$$

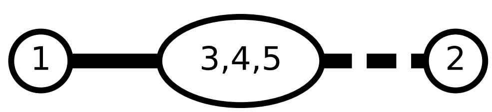
We have

$$\delta_1 = 1, \quad \delta_2 = 1, \quad \delta_3 = \infty, \quad \delta_4 = 2\frac{1}{2} \text{ and so } \theta = 1$$

We update labels and J_e.



We now grow trees from the unmatched vertices 1,6 and the unmatched pseudonode $\{7,8,9,10,11\}$. As we do so the Matching increases by adding the edge $(6,8)$ to M and then we shuffled the edges of M in the pseudonode so that 8 is unmatched by removing $(7,8),(9,10)$ and adding $(7,9),(10,11)$. M now has 5 edges.

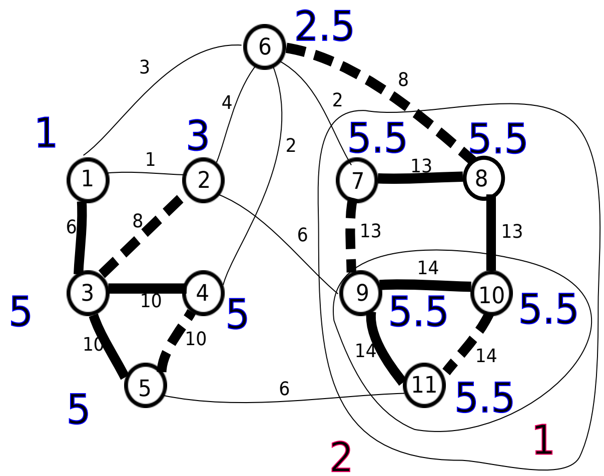


$$\alpha_i = \begin{cases} -1 & i \in \{1,2\} \\ 1 & i \in \{3,4,5\} \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_k = \begin{cases} -2 & S_k = \{3,4,5\} \\ 0 & \text{otherwise} \end{cases}$$

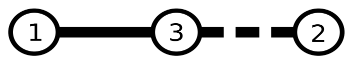
We have

$$\delta_1 = 2, \quad \delta_2 = 1, \quad \delta_3 = \frac{1}{2}, \quad \delta_4 = 1\frac{1}{2} \text{ and so } \theta = \frac{1}{2}$$

We update labels and J_e.



We now grow a tree from the unmatched vertex 1.

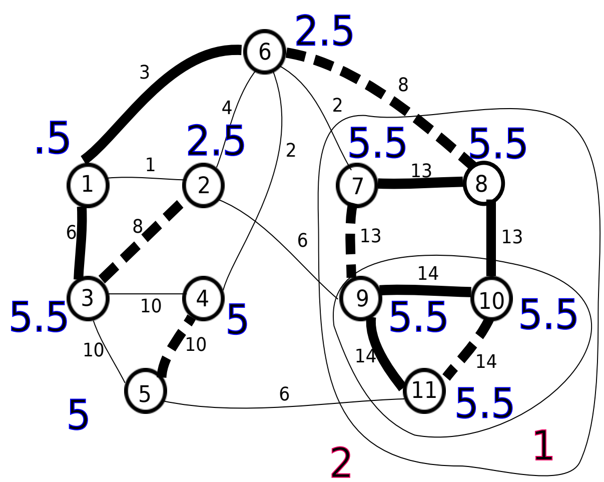


$$\alpha_i = \begin{cases} -1 & i \in \{1,2\} \\ 1 & i \in \{3\} \\ 0 & \text{otherwise} \end{cases}$$

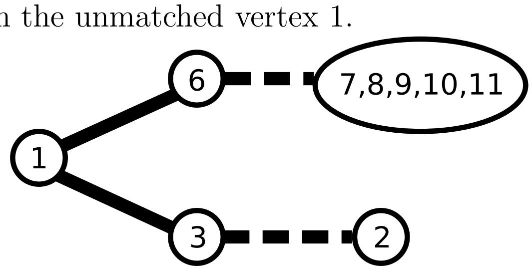
We have

$$\delta_1 = \frac{3}{2}, \quad \delta_2 = \frac{1}{2}, \quad \delta_3 = \infty, \quad \delta_4 = 1 \text{ and so } \theta = \frac{1}{2}$$

We update labels and J_e.



We again grow a tree from the unmatched vertex 1.

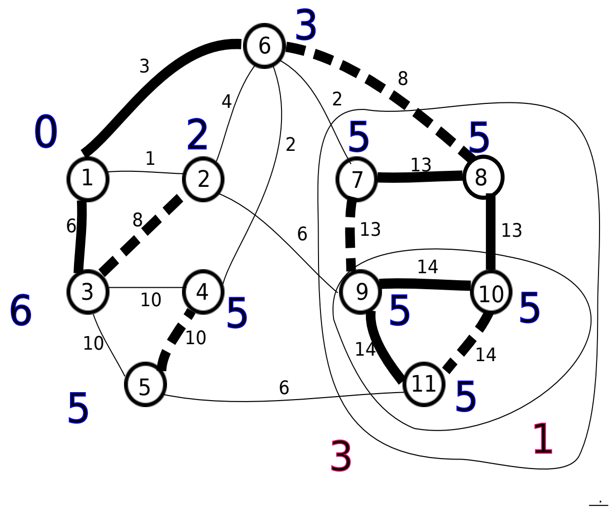


$$\alpha_i = \begin{cases} -1 & i \in \{1,2,7,8,9,10,11\} \\ 1 & i \in \{3,6\} \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_k = \begin{cases} 2 & S_k = \{7,8,9,10,11\} \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\delta_1 = 1, \quad \delta_2 = 3\frac{1}{2}, \quad \delta_3 = \infty, \quad \delta_4 = \frac{1}{2} \text{ and so } \theta = \frac{1}{2}$$

We update labels and J_e.



There is no unmatched vertex with $\alpha > 0$. The edges of J_e are no longer crucial, just the five edges of M. We verify that we have an optimal solution to the Primal.

$$\text{Weight of matching } = 8 + 10 + 8 + 13 + 14 = 53$$

$$\sum \alpha_i + \sum s_k \gamma_k = 0 + 2 + 6 + 5 + 5 + 3 + 5 + 5 + 5 + 5 + 5 + 1 \cdot 1 + 2 \cdot 3 = 53$$

Thus the α_i's and γ_k's provide a certificate of the optimality of the final matching M.