Stirling's Approximation

Richard Anstee,
UBC, Vancouver

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When using Sperners theorem, it is important to know that $\binom{2n}{n}$ is a large fraction of all the 2^{2n} sets in $2^{[2n]}$. Stirling's Appromation gives us a good sense of this. We can use integrals to approximate sums. Using the properties of the logarithm

$$\ln(n!) = \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n)$$

$$\approx \int_1^n \ln(x)dx$$

$$= n\ln(n) - n + 1$$

This can be improved (with some work) to have a better error term

$$\ln(n!) \approx n\ln(n) - n + (1/2)\ln(2\pi n).$$

Then

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}$$

$$= \frac{1}{\sqrt{\pi n}} \frac{2^{2n}\left(\frac{n}{e}\right)^{2n}}{\left(\frac{n}{e}\right)^{n}\left(\frac{n}{e}\right)^{n}}$$

$$= \frac{2^{2n}}{\sqrt{\pi n}}$$

Thus we see that there are more subsets of $[2n]$ of size n than you would expect. The trivial estimate would be $\approx \frac{1}{2n}2^{2n}$ based on there being $2n$ choices for set sizes and imagining there are about an equal number of sets of size k for each k. Thus we have shown that this last statement is not true.