Recurrences

Richard Anstee,
UBC, Vancouver

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Some notation

We can obtain an integer sequence $a_1, a_2, \ldots, a_n, \ldots$ from recurrence which gives certain initial values explicitly and the remaining values are a function of previous values in the sequences.

Proposition. Assume for each $n > k$, $a_n = f(a_1, a_2, \ldots, a_{n-1})$. Assume $a_1, a_2, \ldots, a_k$ are given. Then this uniquely determines a_n for all $n > 0$.

Examples

Fibonacci numbers
They are determined by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad\qquad f_1 = f_2 = 1$$

There is an explicit formula for f_n

$$f_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Derivation using generating functions

We form a generating function for the fibonacci numbers as follows

$$F(x) = f_1x + f_2x^2 + f_3x^3 + f_4x^4 + \cdots$$

where $f_1 = 1$ and $f_2 = 1$. Thus we have transformed the sequence $f_1, f_2, f_3, \ldots$ into a function(?) $F(x)$.

$$\begin{array}{rclllll} F(x) & = & f_1x & +f_2x^2 & +f_3x^3 & +f_4x^4 & +\cdots \\ xF(x) & = & & +f_1x^2 & +f_2x^3 & +f_3x^4 & +\cdots \\ x^2F(x) & = & & & +f_1x^3 & +f_2x^4 & +\cdots \\ \hline (1-x-x^2)F(x) & = & f_1x & +(f_2-f_1)x^2 & +0x^3 & +0x^4 & +\cdots \end{array}$$

using $f_3 - f_2 - f_1 = 0$, $f_4 - f_3 - f_2 = 0$ etc. We have

$$F(x) = \frac{x}{1-x-x^2}$$

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Then using the method of partial fractions we have

$$F(x) = \frac{A}{(1 - (\frac{1+\sqrt{5}}{2})} + \frac{B}{(1 - (\frac{1-\sqrt{5}}{2})}$$

$$= A\left(1 + (\frac{1+\sqrt{5}}{2})x + (\frac{1+\sqrt{5}}{2}))^2x^2 + \cdots\right)$$

$$+B\left(1 + (\frac{1-\sqrt{5}}{2})x + (\frac{1-\sqrt{5}}{2}))^2x^2 + \cdots\right)$$

Thus

$$f_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$$

Catalan Numbers
They are detemined by the recurrence

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \qquad\qquad C_0 = 1$$

There is an explicit formula for C_n

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We can use recurrences in a variety of ways. The typical way is to obtain a recurrence and then use what we already know to solve the recurrence. Or use generating functions to solve.

Thank you for listening