MATH 444 Induction and Inequalities Richard Anstee

Exponentials grow faster than Polynomials

One thing you can prove with some effort in a Calculus course is that exponential functions grow faster than polynomials. This is a nice fact that has some importance when talking about run time of algorithms. We consider an elementary case.

First let us begin with a draft of what we are trying to accomplish

Theorem draft $2^n > n^2$ for $n \in \mathbf{N}$.

Proof: We start by letting $H(n)$ be the statement $2^n > n^2$. We think induction should be useful and so consider the inductive step $H(k) \implies H(k+1)$.

So assume $2^k > k^2$. We start with

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2$$

using induction hypothesis that $H(k)$ is true. Now $2 \cdot k^2 > (k+1)^2$ if $2 > ((k+1)/k)^2$ i.e. $\sqrt{2} > (k+1)/k$. Since

$$\lim_{n\to\infty} \frac{k+1}{k} = 1$$

and $f(x) = (x+1)/x$ is a decreasing function for $x > 1$, then it seems reasonable to discover some k for which the inequality is true. We note $\sqrt{2} \approx 1.4 > 4/3 = (3+1)/4$ and so $H(k) \implies H(k+1)$ for $k \geq 3$.

It is not sufficient to add the hypotheses $n \geq 3$ to the statement of our theorem. We also need base cases for our theorem. We note that $2^2 = 2^2$, $2^3 = 8 < 3^2$, $2^4 = 4^2$ and $2^5 = 32 > 25 = 5^2$. So we can start our proof with $k = 5$. Now we are ready to write the proof.

Theorem 0.1. *$2^n > n^2$ for $n \in \mathbf{N}$ and $n \geq 5$.*

Proof: We use induction on n. We start by letting $H(n)$ be the statement $2^n > n^2$. Our base case is $H(5)$ which we verify $2^5 = 32 > 25 = 5^2$.

We now consider the inductive step $H(k) \implies H(k+1)$. Assume $H(k)$ is true for some $k \geq 5$, namely $2^k > k^2$ for some k with $k \geq 5$. We start with

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2$$

using induction hypothesis that $H(k)$ is true. Now $2 \cdot k^2 > (k+1)^2$ if $2 > ((k+1)/k)^2$ i.e. $\sqrt{2} > (k+1)/k$. Since $\sqrt{2} \approx 1.4 > 6/5 = (5+1)/5$ and $f(x) = (x+1)/x$ is a decreasing function, we have $\sqrt{2} > (k+1)/k$ for $k \geq 5$.

By induction $H(n)$ is true for all $n \geq 5$. ∎

What does this teach? You must be careful. You must check base cases. Can you ask related questions? Is the inequality $2^x > x^2$ valid for all real numbers $x \geq 5$? Can you extend ideas to show that exponentials grow faster than polynomials?