Two Extremal Set Results

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Introduction

Definition $[m] = \{1, 2, \ldots, m\}$
Definition $2^{[m]} = \{A \,|\, A \subseteq [m]\}$
Definition $\binom{[m]}{k} = \{A \subseteq [m] \,|\, |A| = k\}$
Definition $A^c = [m] \backslash A$

Theorem 0

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Proof: We can partition $2^{[m]}$ into 2^{m-1} pairs of sets A, A^c. At most one of the two sets A, A^c can be in $\mathcal{F}$ since $A \cap A^c = \emptyset$. Thus at most half the sets in $2^{[m]}$ can be in $\mathcal{F}$, proving the bound. ■

Sperner's Theorem

Definition Let $\mathcal{F} \subseteq 2^{[m]}$. We say $\mathcal{F}$ is an antichain if for any pair $A, B \in \mathcal{F}$ neither $A \subset B$ nor $B \subset A$.

Theorem Let $\mathcal{F} \subseteq 2^{[m]}$ and assume $\mathcal{F}$ is an antichain. Then

$$|\mathcal{F}| \leq \binom{m}{\lfloor m/2 \rfloor}.$$

Definition A chain is a sequence $A_1 \subset A_2 \subset \cdots \subset A_k$ of subsets of $[m]$. We say the chain is saturated if $|A_{i+1}| = |A_i| + 1$ for $i = 1, 2, \ldots, k-1$. We say the chain is symmetric if $|A_i| = m - |A_{k-i+1}|$.

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Proof: of Sperner's Theorem. We wish to partition $2^{[m]}$ into $\binom{m}{\lfloor m/2 \rfloor}$ saturated symmetric chains. To be an antichain, at most one element of $\mathcal{F}$ can come from a chain. The chains are saturated and symmetric and hence have at least one set of size $\lfloor m/2 \rfloor$. This yields the bound.

We now seek the partition.

Proof continued

We use induction on m to obtain the partition. Assume we have the appropriate partition for $2^{[m]}$ with symmetric saturated chains $A_1 \subset A_2 \subset \cdots \subset A_k$ and we will obtain the appropriate partition for $2^{[m+1]}$.
We first make the observation that every set in $2^{[m+1]}$ either contains $m+1$ or does not and hence we can obtain $2^{[m+1]}$ from $2^{[m]}$ as follows. For each set $A \in 2^{[m]}$, we form two sets $A, A \cup \{m+1\}$.

Proof continued

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The chain $A_1 \subset A_2 \subset \cdots \subset A_k$ yields the $2k$ sets $A_1, A_2, \ldots, A_k$ and $A_1 \cup \{m+1\}, A_2 \cup \{m+1\}, \ldots, A_k \cup \{m+1\}$. We can readily partition these $2k$ sets into two chains, one of size $k+1$ and one of size $k-1$ as follows: First chain is $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{m+1\}$ and second chain is $A_1 \cup \{m+1\} \subset A_2 \cup \{m+1\} \subset \cdots \subset A_{k-1} \cup \{m+1\}$ which we can verify are saturated chains and given that our original chain is symmetric, our new chain is symmetric with m replaced by $m+1$.

	$A_k \cup \{m+1\}$
A_k	$A_{k-1} \cup \{m+1\}$
A_{k-1}	$A_{k-2} \cup \{m+1\}$
$\vdots$	$\vdots$
A_3	$A_2 \cup \{m+1\}$
A_2	$A_1 \cup \{m+1\}$
A_1	

Thanks for your attention