Dominoes and Matchings

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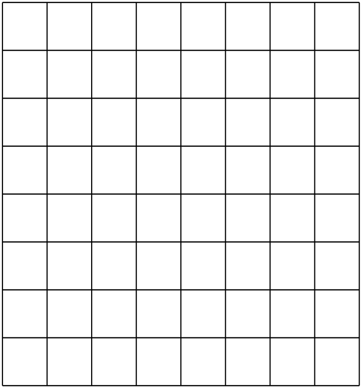
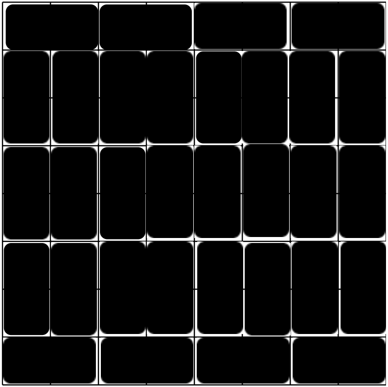
Math 444, March 11, 2021

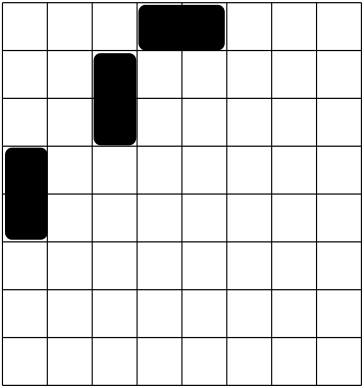
The first set of problems I'd like to mention are really graph theory *Matching Problems* disguised as covering a checkerboard with dominoes. Let me start with the dominoes version



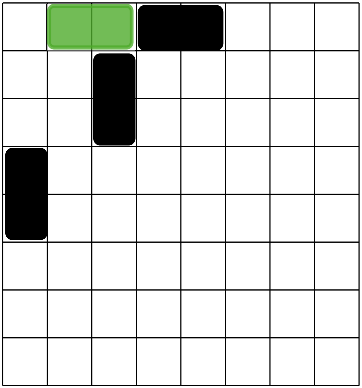
The checkerboard



The checkerboard completely covered by dominoes



Black dominoes fixed in position. Can you complete?



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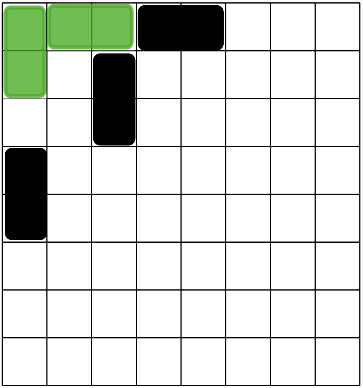

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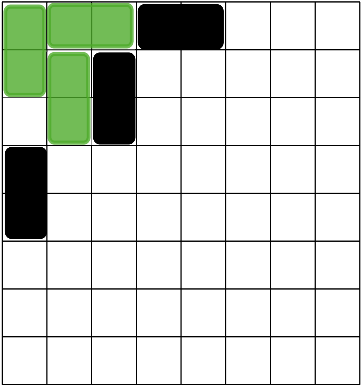
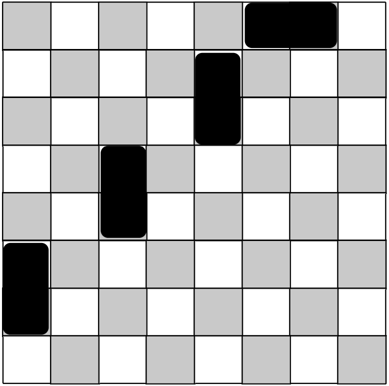


Black dominoes fixed in position. You can't complete.



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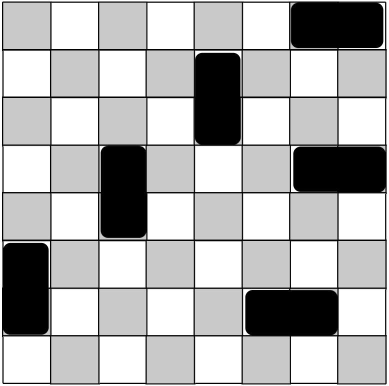

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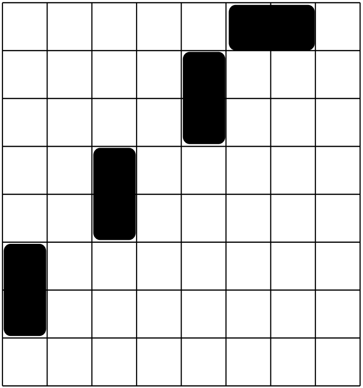

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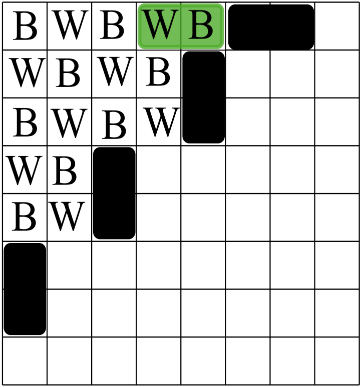
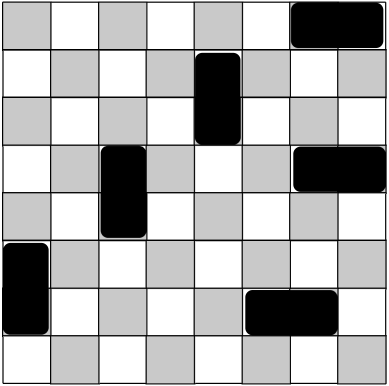


Black dominoes fixed in position. This is why you can't complete!



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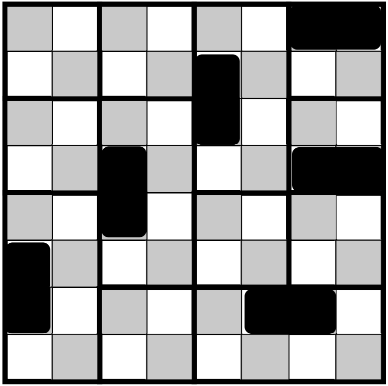
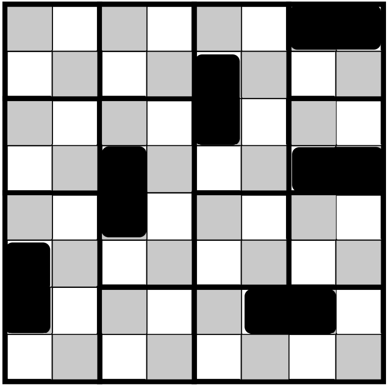


Black dominoes fixed in position but the ends are at distance at least 3 from any other domino. This is why you can complete!



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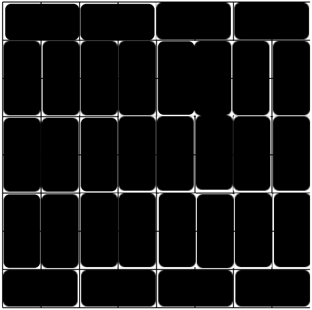
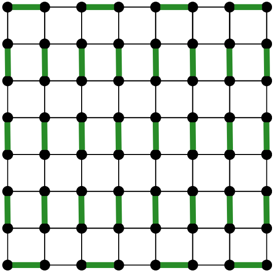
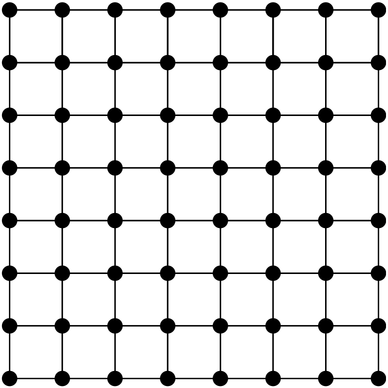
Theorem (A + Tseng 06) Let m be an even integer. Let S be a selection of edges from the $m \times m$ grid G_m^2. Assume for each pair $e, f \in S$, we have $d(e,f) \geq 3$. Then $G_m^2 \backslash S$ has a perfect matching.

Covering the checkerboard by dominoes is the same as finding a perfect matching in the associated grid graph. Each square in the checkerboard becomes a vertex in the graph and two vertices are joined by an edge in the graph if the two associated squares share an edge.

A perfect matching in a graph is a set M of edges that pair up all the vertices. Necessarily $|M| = |V|/2$.







The 8×8 grid.
This graph has many perfect matchings.

Theorem (Temperley and Fisher 1961, Kasteleyn 1961)
The number of perfect of matchings in an $n \times m$ grid is

$$\prod_{i=1}^{n/2} \prod_{j=1}^{m/2} 4\cos^2\left(\frac{\pi i}{n+1}\right) + 4\cos^2\left(\frac{\pi j}{m+1}\right)$$

and for an 8×8 grid the number is 12,988,816.

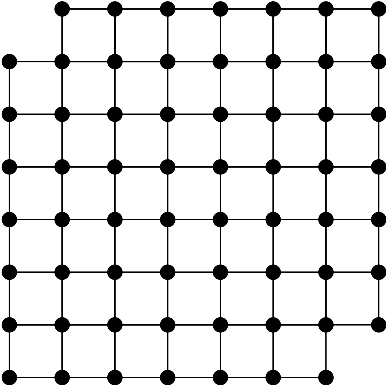
Vertex deletion

Our first example considered choosing some edges and asking whether they extend to a perfect matching. I have also considered what happens if you delete some vertices. Some vertex deletions are clearly not possible. Are there some nice conditions on the vertex deletions so that the remaining graph after the vertex deletions still has a perfect matching?

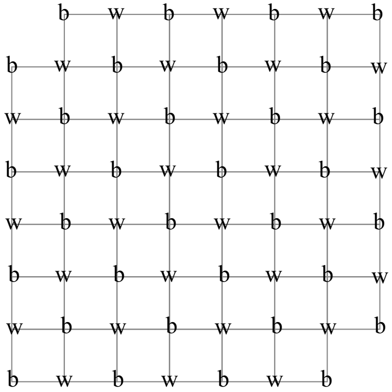
Vertex deletion

Our first example considered choosing some edges and asking whether they extend to a perfect matching. I have also considered what happens if you delete some vertices. Some vertex deletions are clearly not possible. Are there some nice conditions on the vertex deletions so that the remaining graph after the vertex deletions still has a perfect matching?

In the checkerboard interpretation we would be deleting some squares from the checkerboard and asking whether the remaining slightly mangled board has a covering by dominoes.



The 8×8 grid with two deleted vertices.



The black/white colouring revealed:
No perfect matching in the remaining graph.

Deleting Vertices from Grid

Our grid graph (in 2 or in d dimensions) can have its vertices coloured white W or black B so that every edge in the graph joins a white vertex and a black vertex. Graphs G which can be coloured in this way have $V(G) = W \cup B$ and are called bipartite. Bipartite graphs that have a perfect matching must have $|W| = |B|$.

Thus if we wish to delete black vertices B' and white vertices W' from the grid graph, we must delete an equal number of white and black vertices ($|B'| = |W'|$).

Deleting Vertices from Grid

But also you can't do silly things. Consider a corner of the grid with a white vertex. Then if you delete the two adjacent black vertices then there will be no perfect matching. How do you avoid this problem? Our guess was to impose some distance condition on the deleted blacks (and also on the deleted whites).

Hall's Theorem for Bipartite Graphs

Theorem (Hall'sTheorem) Let G be a bipartite graph with the vertices given as $W \cup B$. Then G has a matching hitting all the vertices of W if and only if for every $A \subset W$, the number of vertices in $N(A)$, namely the vertices of B adjacent to at least one vertex of A, is at least equal to the number of vertices in A.

Proof: Hall's Theorem is fairly easy to prove by a variety of techniques. Use the notation $N_G(A)$ to denote the vertics in B adjacent to at least one vertex in A, using edges of G. We will use induction on $|W|$. Define

$$\text{Hall's condition}: \qquad \text{for all } A \subseteq W, \quad |A| \leq |N_G(A)|.$$

We conider a graph G which satisfies Hall's condition. Now delete edges while preserving the validity of Hall's condition (i.e. an edge minimal subgraph of G satisfying Hall's condition). Let the resulting graph be G'.

G' is an edge minimal subgraph of G satisfying Hall's condition

Now delete one more edge $e = (x, y)$ with $x \in W$ and $y \in B$ to obtain a graph $G'' = G' \backslash e$. Thus Hall's condition is no longer satisfied for some $A \subset W$ necessarily having $y \in N_{G''}(A)$.
But now consider a futher graph obtained from G' on vertices $(W \backslash x) \cup (B \backslash y)$ by deleting vertices x, y, say $G^* = G' \backslash \{x, y\}$ where vertices of G^* are $(W \backslash x) \cup (B \backslash y)$. We discover that G^* satsifies Hall's condition since and for any $A \subset W \backslash x$, we have $N_{G^*}(A) = N_{G'}(A)$ and G' satisfies Hall's condition.
Thus by induction on $|W|$, we have that G^* has a matching hitting all vertices of $W \backslash x$) and so by adding in th eedge e we obtain a matching in G hitting all vertices of W. ■

Hall's Theorem for G_m^3

The grid G_m^3 has bipartition $V(G_m^3) = B \cup W$. We consider deleting some black $B' \subset B$ vertices and white $W' \subset W$ vertices. The resulting subgraph has a perfect matching if and only if for each subset $A \subset W - W'$, we have $|A| \leq |N(A) - B'|$ where $N(A)$ consists of vertices in B adjacent to some vertex in A in G_m^3.

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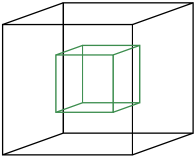

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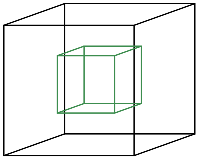

If we let A be the white vertices in the green cube, then $|N(A)| - |A|$ is about $6 \times \frac{1}{2}(\frac{1}{2}m)^2$.

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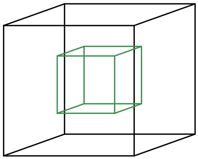

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If the deleted blacks are about $cm^{1/3}$ apart then we can fit about $(\frac{1}{2c}m^{2/3})^3$ inside the small green cube $\frac{1}{2}m \times \frac{1}{2}m \times \frac{1}{2}m$.

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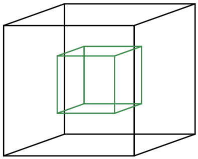

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We may choose c small enough so that we cannot find a perfect matching.

Deleting Vertices from Grid

Theorem (Aldred, A., Locke 07 ($d = 2$),
A., Blackman, Yang 10 ($d \geq 3$)).
Let m, d be given with m even and $d \geq 2$. Then there exist constant c_d (depending only on d) for which we set

$$k = c_d m^{1/d} \quad \left(k \text{ is } \Theta(m^{1/d})\right).$$

Let G_m^d have bipartition $V(G_m^d) = B \cup W$.
Then for $B' \subset B$ and $W' \subset W$ satisfying
i) $|B'| = |W'|$,
ii) For all $x, y \in B'$, $d(x, y) > k$,
iii) For all $x, y \in W'$, $d(x, y) > k$,
we may conclude that $G_m^d \backslash (B' \cup W')$ has a perfect matching.

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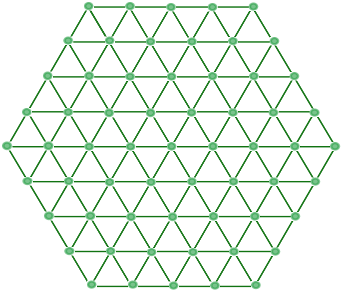
Jonathan Blackman on left

Deleting Vertices from Triangular Grid



A convex portion of the triangular grid

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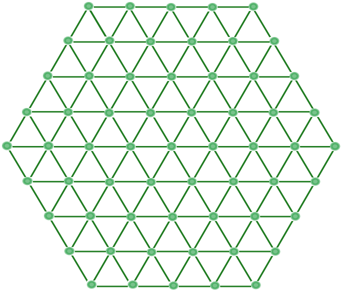

A convex portion of the triangular grid

A *near perfect matching* in a graph is a set of edges such that all but one vertex in the graph is incident with one edge of the matching. Our convex portion of the triangular grid has 61 vertices and many near perfect matchings.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).

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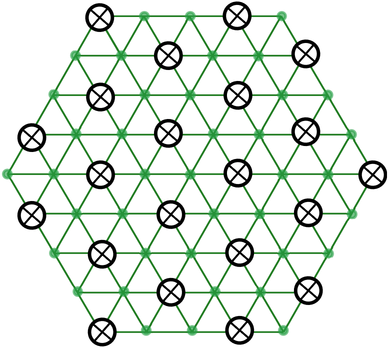
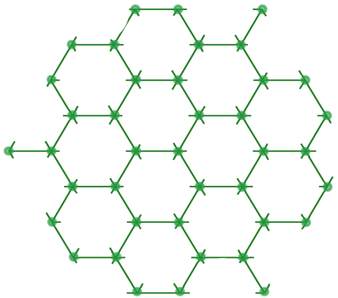


We have deleted 21 vertices from the 61 vertex graph, many at distance 2.

Theorem (A., Tseng 06) Let $T = (V, E)$ be a convex portion of the triangular grid and let $X \subseteq V$ be a set of vertices at mutual distance at least 3. Then $T \backslash X$ has either a perfect matching (if $|V| - |X|$ is even) or a near perfect matching (if $|V| - |X|$ is odd).



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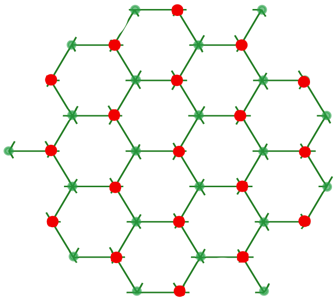

We have chosen 19 red vertices S from the remaining 40 vertices and discover that there are 21 other vertices joined only to red vertices and so the 40 vertex graph has no perfect matching.