In general, our initial dictionary for an LP with slack variables, may not yield a feasible solution.

$$\begin{array}{lrrrrl} \text{Maximize} & -x_1 & +3x_2 & +x_3 & +x_4 & \\ & 2x_1 & +x_2 & & -x_4 & \leq 4 \\ & -2x_1 & & +x_3 & +x_4 & \leq -2 \\ & & 2x_2 & +2x_3 & & \leq 3 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

We now add slack variables

$$\begin{array}{rcrrrrr} x_5 & = & 4 & -2x_1 & -x_2 & & +x_4 \\ x_6 & = & -2 & +2x_1 & & -x_3 & -x_4 \\ x_7 & = & 3 & & -2x_2 & -2x_3 & \\ z & = & & -x_1 & +3x_2 & +x_3 & +x_4 \end{array}$$

A basic solution associated with the dictionary is obtained by setting the non basic variable equal to 0 $x_1 = x_2 = x_3 = x_4 = 0$ yielding $x_5 = 4$, $x_6 = -2$ and $x_7 = 3$. This is not feasible since $x_6 < 0$. So how do we proceed? We add an artificial variable to achieve feasibility and then attempt to drive the artificial variable to 0 using our simple method. This is considered *phase one* of the two phase method.

We use the notation $x_0$ for the artificial variable partly so that when applying 'Anstee's rule' for choosing a leaving variable that $x_0$ would be preferred over other choices that tie $x_0$. We use the objective function maximize $w = -x_0$ to drive $x_0$ to 0.

$$\begin{array}{rcrrrrrr} x_5 & = & 4 & -2x_1 & -x_2 & & +x_4 & +x_0 \\ x_6 & = & -2 & +2x_1 & & -x_3 & -x_4 & +x_0 \\ x_7 & = & 3 & & -2x_2 & -2x_3 & & +x_0 \\ w & = & & & & & & -x_0 \end{array}$$

Now at this point you might say you are done since the coefficients in the $w$ row are all negative but of course we haven't reached feasibility. The idea is we can to a *special pivot to feasibility* that results is a feasible dictionary but not optimal (in terms of minimizing $w = -x_0$).

We choose $x_0$ to enter the basis and choose the leaving variable so we achieve feasibility

$$x_5 = 4 + x_0 \geq 0 \text{ so } x_0 \geq -4$$
$$x_6 = -2 + x_0 \geq 0 \text{ so } x_0 \geq 2$$
$$x_7 = 3 + x_0 \geq 0 \text{ so } x_0 \geq -3$$

We must choose $x_0$ to increase to 2 driving $x_6$ to 0 and so $x_0$ enters the basis and $x_6$ leaves the basis.

$$\begin{array}{rcrrrrrr} x_5 & = & 6 & -4x_1 & -x_2 & +x_3 & +2x_4 & +x_6 \\ x_0 & = & 2 & -2x_1 & & +x_3 & +x_4 & +x_6 \\ x_7 & = & 5 & -2x_1 & -2x_2 & -x_3 & +x_4 & +x_6 \\ w & = & -2 & +2x_1 & & -x_3 & -x_4 & -x_6 \end{array}$$

Now after this *special pivot to feasibility*, we can proceed as before using the simplex method to minimize $w = -x_0$ and hence drive $x_0$ to zero if possible. Note that the clever choice of $x_0$ means that if $x_0$ is driven to zero in a pivot then it will be chosen to leave the basis (using Anstee's Rule) and hence we can say goodbye to $x_0$ at this point.

For our particular dictionary above, we choose $x_1$ to enter and then $x_0$ leaves (oddly quick!)

$$\begin{array}{rclrrrrr} x_5 & = & 2 & +2x_0 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & -\frac{1}{2}x_0 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & +x_0 & -2x_2 & -2x_3 & & \\ w & = & & -x_0 & & & & \end{array}$$

We can now delete $x_0$ and $w$ since they are no longer needed:

$$\begin{array}{rclrrrr} x_5 & = & 2 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & -2x_2 & -2x_3 & & \end{array}$$

We do need $z$ and then to minimize it.

$z = -x_1 + 3x_2 + x_3 + x_4$

This is not so good since $x_1$ is in the basis of our dictionary so we subsititue to eliminate $x_1$ from $z$:

$z = -(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6) + 3x_2 + x_3 + x_4$

$z = -1 + 3x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6$

This yields the dictionary

$$\begin{array}{rclrrrr} x_5 & = & 2 & -x_2 & -x_3 & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_7 & = & 3 & -2x_2 & -2x_3 & & \\ z & = & -1 & +3x_2 & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & -\frac{1}{2}x_6 \end{array}$$

We are now ready to proceed as before to maximize $z$. The computation of the new $z$ row and the subsequent pivots are consiered the *second phase* of the two phase method.

So by our standard pivot process, we choose $x_2$ to enter and $x_7$ to leave.

$$\begin{array}{rclrrrr} x_5 & = & \frac{1}{2} & +\frac{1}{2}x_7 & & & -x_6 \\ x_1 & = & 1 & & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 \\ x_2 & = & \frac{3}{2} & -\frac{1}{2}x_7 & -x_3 & & \\ z & = & \frac{7}{2} & -\frac{3}{2}x_7 & -\frac{5}{2}x_3 & +\frac{1}{2}x_4 & -\frac{1}{2}x_6 \end{array}$$

We repeat our pivot process and choose $x_4$ to enter but there is no leaving variable and so the LP is unbounded. We read off the solutions $(1+\frac{1}{2}t, \frac{3}{2}, 0, t, \frac{1}{2}, 0, 0)^T$ for $t \geq 0$ with $z = \frac{7}{2} + \frac{1}{2}t$. When you have found an LP is unbounded you must give such a parametric solution demonstrating that it is unbounded.